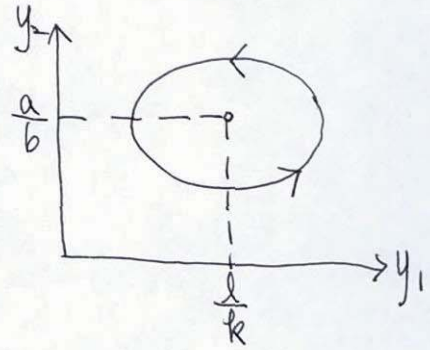
$$\begin{cases} \tilde{y}_1' = -\dfrac{lb}{k}\tilde{y}_2 \\[2ex] \tilde{y}_2' = \dfrac{ak}{b}\tilde{y}_1 \end{cases}$$

$$\frac{ak}{b}\tilde{y}_1\tilde{y}_1' = -\frac{lb}{k}\tilde{y}_2\tilde{y}_2'$$

$$\frac{ak}{b}\tilde{y}_1^2 = -\frac{lb}{k}\tilde{y}_2^2 + c$$

$$\frac{ak}{b}\tilde{y}_1^2 + \frac{lb}{k}\tilde{y}_2^2 = c$$



center

~~3~~4.6. Nonhomogeneous Linear Systems

$$\bar{y}' = A\bar{y} + \bar{g}$$

Gen. sol.: $\bar{y} = \bar{y}^{(h)} + \bar{y}^{(p)}$

$$\bar{y}^{(h)\prime} = A\bar{y}^{(h)}$$

* Method of Undetermined Coefficients

Ex. 1). $\bar{y}' = A\bar{y} + \bar{g} = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\bar{y} + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix}$

(1) $\bar{y}^{(h)\prime} = A\bar{y}^{(h)}$

$\bar{y}^{(h)} = \bar{x}e^{\lambda t}$. $\quad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -4 \\ 1 & -3-\lambda \end{vmatrix} = (\lambda+3)(\lambda-2) + 4$

$= \lambda^2 + \lambda - 2$

$= (\lambda+2)(\lambda-1) = 0$

$\lambda_1 = -2$: $\begin{bmatrix} 2-\lambda & -4 \\ 1 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 \\ x_1 - x_2 \end{bmatrix} = 0$. $\quad x_1 = x_2 \quad \bar{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1 \quad \begin{bmatrix} x_1 - 4x_2 \\ x_1 - 4x_2 \end{bmatrix} = 0$. $\quad x_1 = 4x_2$. $\quad \bar{x}^{(2)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$\bar{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{t}$$

(2) $\bar{y}^{(p)} = \bar{u} + \bar{v}t + \bar{w}t^2$

substituting,

$$\bar{v} + 2\bar{w}t = A\bar{u} + A\bar{v}t + A\bar{w}t^2 + \bar{g}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 2w_1 t \\ 2w_2 t \end{bmatrix} = \begin{bmatrix} 2u_1 - 4u_2 \\ u_1 - 3u_2 \end{bmatrix} + \begin{bmatrix} (2v_1 - 4v_2)t \\ (v_1 - 3v_2)t \end{bmatrix} + \begin{bmatrix} (2w_1 - 4w_2)t^2 \\ (w_1 - 3w_2)t^2 \end{bmatrix}$$

$$+ \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix}$$

t^2-terms: $0 = (2w_1 - 4w_2 + 2)t^2$. $\quad w_1 - 2w_2 + 1 = 0$ $\quad \begin{pmatrix} w_2 = 0 \\ w_1 = -1 \end{pmatrix}$

$0 = (w_1 - 3w_2 + 1)t^2$. $\quad w_1 - 3w_2 + 1 = 0$.

t-terms: $2w_1 = 2v_1 - 4v_2 + 10$. $\quad -1 = v_1 - 2v_2 + 5$

$0 = v_1 - 3v_2 + 9$ $\quad -)\ 0 = v_1 - 3v_2 + 9$

$-1 = v_2 - 4$. $\quad [v_2 = 3.\ \ v_1 = 0]$

t^0-terms: $v_2 = u_1 - 3u_2 + 3$

$\cancel{v_1}^{0} = 2u_1 - 4u_2$ $\quad \begin{pmatrix} u_2 = 0 \\ u_1 = 0 \end{pmatrix}$

$-)\ 2v_2 = 2u_1 - 6u_2 + 6$

$-6 = 2u_2 - 6$.

$$\bar{y}^{(p)} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} t^2 = \begin{bmatrix} -t^2 \\ 3t \end{bmatrix}$$

G.S. $\bar{y} = \bar{y}^{(h)} + \bar{y}^{(p)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} -t^2 \\ 3t \end{bmatrix}$

Modification

$$\bar{g} \underset{\text{contains}}{:} e^{\lambda t} \Rightarrow \bar{y}^{(p)} = \bar{u}te^{\lambda t} + \bar{v}e^{\lambda_1 t}$$

Ex. 1 $\bar{y}' = A\bar{y} + \bar{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\bar{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$

(1) $\bar{y}^{(h)'} = A\bar{y}^{(h)}$. $\quad \bar{y}^{(h)} = \bar{x}e^{\lambda t}$

$$\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = (\lambda+3)^2 - 1 = \lambda^2 + 6\lambda + 8$$
$$= (\lambda+2)(\lambda+4) = 0$$

$\lambda_1 = -2$. $\begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = 0$. $\quad x_1 = x_2$. $\quad \bar{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -4$ $\quad \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = 0 \quad \bar{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\bar{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

(2) $\bar{y}^{(p)} = \bar{u}te^{-2t} + \bar{v}e^{-2t}$

$$\bar{u}e^{-2t} - 2\bar{u}te^{-2t} - 2\bar{v}e^{-2t} = A\bar{u}te^{-2t} + A\bar{v}e^{-2t} + \bar{g}$$

te^{-2t} : $\quad -2\bar{u} = A\bar{u} \quad \Rightarrow \quad \bar{u} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\quad (a \neq 0)$.

e^{-2t} : $\quad \bar{u} - 2\bar{v} = A\bar{v} + \bar{g}$

$$\begin{bmatrix} u_1 - 2v_1 \\ u_2 - 2v_2 \end{bmatrix} = \begin{bmatrix} -3v_1 + v_2 \\ v_1 - 3v_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$a - 2v_1 = -3v_1 + v_2 - 6 \qquad v_1 - v_2 = -a - 6$

$a - 2v_2 = v_1 - 3v_2 + 2. \qquad v_1 - v_2 = a - 2$

$0 = -2a - 4. \qquad a = -2.$

$v_1 - v_2 = -4. \qquad v_1 = v_2 - 4 \qquad \begin{bmatrix} v_1 = k \\ v_2 = k+4 \end{bmatrix}$

$$\bar{v} = \begin{bmatrix} k \\ k+4 \end{bmatrix} = \underbrace{k\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{unnecessary}} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

* Method of Variation of parameters.

$$\bar{y}' = A\bar{y} + \bar{g} \qquad A : [n \times n].$$

$$\bar{y}^{(h)} = c_1 \bar{y}^{(1)} + \cdots + c_n \bar{y}^{(n)}$$

Recall "simple y"

$$y_p = y_1 \int \frac{W_1}{W} r\,dx + y_2 \int \frac{W_2}{W} r\,dx + \cdots + y_n \int \frac{W_n}{W} r\,dx$$

$$\bar{y}^{(h)} = \begin{bmatrix} c_1 y_1^{(1)} + \cdots + c_n y_1^{(n)} \\ \vdots \qquad \vdots \\ c_1 y_n^{(1)} + \cdots + c_n y_n^{(n)} \end{bmatrix}$$

$$= \begin{bmatrix} y_1^{(1)} & \cdots & y_1^{(n)} \\ \vdots & & \\ y_n^{(1)} & \cdots & y_n^{(n)} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbb{Y}(t) \cdot \bar{c}$$

$\mathbb{Y}(t)$: fundamental matrix

Let $\bar{y}^{(p)} = \mathbb{Y}(t)\,\bar{u}(t) \qquad \bar{u}(t) = ?$

substituting,

$$\mathbb{Y}'\bar{u} + \mathbb{Y}\bar{u}' = A\mathbb{Y}\bar{u} + \bar{g}$$

Recall that

$$\left.\begin{aligned} \bar{y}^{(1)\prime} &= A\bar{y}^{(1)} \\ \bar{y}^{(2)\prime} &= A\bar{y}^{(2)} \\ &\vdots \\ \bar{y}^{(n)\prime} &= A\bar{y}^{(n)} \end{aligned}\right\} \Rightarrow \mathbb{Y}' = A\mathbb{Y}$$

$$\mathbb{Y}\bar{u}' = \bar{g} \quad . \quad \bar{u}' = \mathbb{Y}^{-1}\bar{g}$$

$$\bar{u} = \int \mathbb{Y}^{-1}(t)\,\bar{g}(t)\,dt$$

$$\bar{y}_p = \mathbb{Y}\int \mathbb{Y}^{-1}\bar{g}\,dt$$

$\therefore$ G.S $\bar{y} = \mathbb{Y}\bar{c} + \mathbb{Y}\int \mathbb{Y}^{-1}\bar{g}\,dt$

Ex. ~~3~~ 2 $\quad \bar{y}' = A\bar{y}+\bar{g} = \begin{bmatrix}-3 & 1\\ 1 & -3\end{bmatrix}\bar{y} + \begin{bmatrix}-6\\ 2\end{bmatrix}e^{-2t}$

$\bar{y}^{(h)} \Leftarrow$ Ex. ~~x~~ 1. $\quad \bar{y}^{(h)} = c_1 \underbrace{\begin{bmatrix}1\\ 1\end{bmatrix}e^{-2t}}_{=\bar{y}^{(1)}} + c_2 \underbrace{\begin{bmatrix}1\\ -1\end{bmatrix}e^{-4t}}_{=\bar{y}^{(2)}}$

$$\mathbb{Y} = [\bar{y}^{(1)}\ \ \bar{y}^{(2)}] = \begin{bmatrix}e^{-2t} & e^{-4t}\\ e^{-2t} & -e^{-4t}\end{bmatrix}$$

$$\mathbb{Y}^{-1} = \frac{1}{-2e^{-6t}}\begin{bmatrix}-e^{-4t} & -e^{-4t}\\ -e^{-2t} & e^{-2t}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}e^{2t} & e^{2t}\\ e^{4t} & -e^{4t}\end{bmatrix}$$

$$\bar{u}' = \mathbb{Y}^{-1}\bar{g} = \frac{1}{2}\begin{bmatrix}e^{2t} & e^{2t}\\ e^{4t} & -e^{4t}\end{bmatrix}\begin{bmatrix}-6e^{-2t}\\ 2e^{-2t}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-4\\ -8e^{2t}\end{bmatrix}$$

$$= \begin{bmatrix}-2\\ -4e^{+2t}\end{bmatrix}$$

$$\bar{u} = \int \begin{bmatrix}-2\\ -4e^{2t}\end{bmatrix}dt = \begin{bmatrix}-2t\\ -2e^{2t}\end{bmatrix}$$

$$\bar{y} = c_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}1\\ -1\end{bmatrix}e^{-4t} + \begin{bmatrix}e^{-2t} & e^{-4t}\\ e^{-2t} & -e^{-4t}\end{bmatrix}\begin{bmatrix}-2t\\ -2e^{2t}\end{bmatrix}$$

$$= c_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}1\\ -1\end{bmatrix}e^{-4t} + \begin{bmatrix}-2te^{-2t} - 2e^{-2t}\\ -2te^{-2t} + 2e^{-2t}\end{bmatrix}$$

$$= c_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}1\\ -1\end{bmatrix}e^{-4t} - 2t\begin{bmatrix}1\\ 1\end{bmatrix}e^{-2t} + \begin{bmatrix}-2\\ 2\end{bmatrix}e^{-2t}$$

* Method of Diagonalization

$$\bar{y}' = A\bar{y} + \bar{g} \quad \rightarrow \quad \begin{cases} y_1' = ay_1 + g_1(t) \\ y_2' = ay_2 + g_2(t) \\ \vdots \end{cases}$$

- $$D = X^{-1} A X = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$: diagonal matrix

  . λ: eigenvalues

$$X = [X^{(1)} \cdots X^{(n)}]$$

Define $\bar{z} = X^{-1}\bar{y}$. $\quad \bar{y} = X\bar{z}$

$$X\bar{z}' = AX\bar{z} + \bar{g}$$

$$\bar{z}' = \underbrace{X^{-1}AX}_{=D}\bar{z} + \bar{h} \qquad \text{where } \bar{h} = X^{-1}\bar{g}$$

$$\bar{z}' = D\bar{z} + \bar{h}$$

n components : $z_j' = \lambda_j z_j + h_j$

$$z_j' - \lambda_j z_j = h_j \qquad \text{Bernoulli eq}$$

$$z_j = c_j e^{\lambda_j t} + e^{\lambda_j t} \int e^{-\lambda_j t} h_j(t)\,dt.$$