Chap. ~~4~~ 5 Series Solutions of ~~Differential Equations~~ ODEs

§~~4~~5.1. Power Series Method

- power series :

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

a_0, a_1, a_2 : coefficients (const.)

x_0 : center of the series

if $x_0 = 0$

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$$

ex) Maclaurin series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots$$

$$e^x = \sum_{m=0}^{\infty} \frac{1}{m!} x^m = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{1}{2!}x^2 + \frac{x^4}{4!} - + \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots$$

§ Idea of the Power Series Method

$$y'' + p(x) y' + q(x) y = 0$$

(1) $p(x), q(x) \leftrightarrow$ power series.

(2) Assume $y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$

then $y' = \sum_{m=1}^{\infty} a_m \cdot m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 3\cdot 2 a_3 x + 4\cdot 3 a_4 x^2 + \cdots$$

$$y'' + p(x)y' + q(x)y = \underbrace{(\quad)}_{=0}\cdot x^0 + \underbrace{(\quad)}_{=0}\cdot x^1 + \underbrace{(\quad)}_{=0} x^2 + \cdots$$

Ex. ①* $\quad y' - y = 0$

Assume $\quad y = \sum_{m=0}^{\infty} a_m x^m$

$$y' - y = (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots \quad = 0$$

$$a_1 = a_0$$

$$a_2 = \frac{a_1}{2} = \frac{a_0}{2}$$

$$a_3 = \frac{a_2}{3} = \frac{a_0}{3\cdot 2} = \frac{a_0}{3!}$$

$$y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots$$

$$= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= a_0 \cdot e^x$$

Ex. ~~2~~1 $\quad y' = 2xy \quad\quad y = \sum_{m=0}^{\infty} a_m x^m$

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots = 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + \cdots$$

$$a_1 = 0$$

$$a_2 = a_0$$

$$a_3 = 0$$

$$a_4 = \frac{a_2}{2} = \frac{a_0}{2}$$

$$a_5 = 0$$

$$a_6 = \frac{a_4}{3} = \frac{a_0}{3!}$$

$$\vdots$$

$$y = a_0 + a_0 x^2 + \frac{a_0}{2!}x^4 + \frac{a_0}{3!}x^6 + \cdots$$

$$= a_0\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right)$$

$$= a_0 e^{x^2}$$

Ex. 2 $\quad y'' + y = 0$

Assume $\quad y = \sum_{m=0}^{\infty} a_m x^m$

$$(2a_2 + 3\cdot 2\, a_3 x + 4\cdot 3\, a_4 x^2 + \cdots) + (a_0 + a_1 x + a_2 x^2 + \cdots) = 0$$

$$(2a_2 + a_0) + (3\cdot 2 a_3 + a_1) x + (4\cdot 3 a_4 + a_2) x^2 + \cdots = 0$$

$$a_2 = -\frac{a_0}{2}$$

$$a_3 = -\frac{a_1}{3\cdot 2}$$

$$a_4 = -\frac{a_2}{4\cdot 3} = +\frac{a_0}{4\cdot 3\cdot 2}$$

$$a_5 = -\frac{a_3}{5\cdot 4} = \frac{a_1}{5!}$$

$$y = \left(a_0 - \frac{a_0}{2}x^2 + \frac{a_0}{4!}x^4 - + \cdots\right) + \left(a_1 x - \frac{a_1}{3!}x^3 + \frac{a_1}{5!}x^5 - + \cdots\right)$$

$$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots\right)$$

$$= a_1 \cos x + a_1 \sin x$$

2. Theory of the Power Series Method

· power series

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0) + \cdots$$

· n th partial sum

$$S_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n.$$

Remainder : $R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \cdots$

· $\lim_{n\to\infty} S_n(x_1) = S(x_1) \quad$: convergent

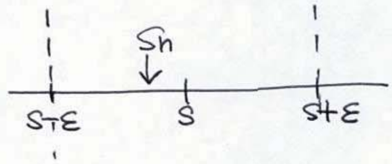
$$= \sum_{m=0}^{\infty} a_m (x_1 - x_0)^m$$

$$S(x_1) = S_n(x_1) + R_n(x_1)$$

· Convergence :

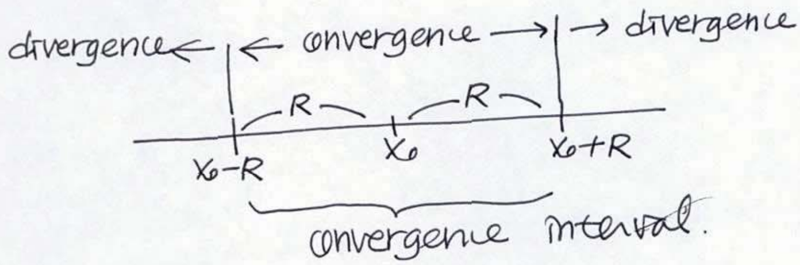
$\exists N$

$$|R_n(x_1)| = |s(x_1) - s_n(x_1)| < \varepsilon \quad \text{for all } n > N$$



* Convergence interval, Radius of convergence

1. The series always converges at $x = x_0$
2. Convergence interval



for all x such that $|x - x_0| < R$, the series converges
" $> R$ " diverges

$\Rightarrow$ R : radius of convergence

$$R = \frac{1}{\lim_{m\to\infty} \sqrt[m]{|a_m|}} \quad \text{or} \quad R = \frac{1}{\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|}.$$

3. if $R \to \infty$, the series converges for all x

Ex. 1 radius of convergence $\sum_{m=0}^{\infty} m!\, x^m$: $a_m = m!$

$$\frac{a_{m+1}}{a_m} = \frac{(m+1)!}{m!} = (m+1) \to \infty \quad \text{as } m \to \infty$$

$\therefore R \to 0$.

the series converges only at $x = 0$: useless

Ex. 2 $\quad \frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots$

$a_m = 1$

$$R = \frac{1}{\lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right|} = 1$$

The series (geometric series) $\therefore$ converges when $|x| < 1$ "R"

Ex. 3. $\quad e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}. \qquad a_m = \frac{1}{m!}$

$$\frac{a_{m+1}}{a_m} = \frac{m!}{(m+1)!} = \frac{1}{m+1} \to 0 \quad \text{as} \quad m \to \infty$$

$R \to \infty.$

The series converges for all x

Ex. 4. Convergence interval

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^{3m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} t^m. \qquad t = x^3.$$

$$a_m = \frac{(-1)^m}{8^m}$$

$$\left| \frac{a_{m+1}}{a_m} \right| = \left| \frac{8^m}{8^{m+1}} \right| = \frac{1}{8} \qquad |t| < 8$$

$$\therefore |x| < 2$$

· Operations on Power Series

* Termwise Differentiation

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m \qquad \text{converges for } |x - x_0| < R$$

$$y'(x) = \sum_{m=1}^{\infty} m a_m (x - x_0)^{m-1} \qquad |x - x_0| < R$$

$$y''(x) = \sum_{m=2}^{\infty} m(m-1) a_m (x - x_0)^{m-2}. \qquad |x - x_0| < R$$

* Termwise Addition

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = f(x)$$

$$\sum_{m=0}^{\infty} b_m (x-x_0)^m = g(x)$$

$$f(x) + g(x) = \sum_{m=0}^{\infty} (a_m + b_m)(x-x_0)^m$$

* Termwise Multiplication

$$f(x)g(x) = \sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} + \cdots + a_m b_0)(x-x_0)^m$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)(x-x_0)$$

$$+ (a_0 b_2 + a_1 b_1 + a_2 b_0)(x-x_0)^2 + \cdots$$

* shifting summation indices

e.g $\quad x^2 \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + 2 \sum_{m=1}^{\infty} m a_m x^{m-1}$

compare

$$= \sum_{m=2}^{\infty} m(m-1) a_m x^{m} + 2 \sum_{m=1}^{\infty} m a_m x^{m-1}$$

let $m-1 = s$. $\quad m = s+1$

$$= \sum_{\substack{m=2 \\ \hookrightarrow 0}}^{\infty} m(m-1) a_m x^m + \sum_{s=0}^{\infty} 2(s+1) a_{s+1} x^s$$

$$= \sum_{m=0}^{\infty} \left[m(m-1) a_m + 2(m+1) a_{m+1} \right] x^m$$

§ Existence of Power Series Solutions

$$y'' + p(x) y' + q(x) y = r(x)$$

$$\tilde{h}(x) y'' + \tilde{p}(x) y' + \tilde{q}(x) y = \tilde{r}(x)$$

$$? \quad y = \sum_{m=0}^{\infty} a_m x^m \quad ?$$

· Real analytic function

$f(x)$ is called analytic at a point $x = x_0$

if $f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$ with $R > 0$

↳ power series

Theorem

p, q, r : analytic at $x = x_0$ $\Rightarrow$ y_{sol} : analytic
→ power series

$\tilde{h}, \tilde{p}, \hat{q}, \tilde{r}$: analytic at $x = x_0$, $\tilde{h}(x_0) \neq 0$) $\Rightarrow$ "

5.3. Legendre's Equation

$$\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$$: spherical coordinates

solution : Legendre function

$$\underbrace{y'' - \frac{2x}{(1-x^2)} y' + \frac{n(n+1)}{(1-x^2)} y = 0}_{\text{analytic at } x=0}$$

∴ assume $y = \sum_{m=0}^{\infty} a_m x^m$

$$\underbrace{(1-x^2) \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + k \sum_{m=0}^{\infty} a_m x^m = 0$$

$k = n(n+1)$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$- \sum_{m=2}^{\infty} m(m-1) a_m x^m$$