· Real analytic function

$f(x)$ is called analytic at a point $x = x_0$

if $f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$ with $R > 0$

↳ power series

<u>Theorem</u>

p, q, r : analytic at $x = x_0$ $\Rightarrow$ y_{sol} : analytic

→ power series

$\left.\begin{array}{l}\tilde{h}, \tilde{p}, \tilde{q}, \tilde{r} : \text{analytic at } x = x_0 \\ \tilde{h}(x_0) \neq 0\end{array}\right) \Rightarrow$ "

4. 3. Legendre's Equation

$$\boxed{(1-x^2)\, y'' - 2x y' + n(n+1)\, y = 0}$$: spherical coordinates

solution : Legendre function

$$y'' - \underbrace{\frac{2x}{(1-x^2)}}_{\text{analytic at } x=0} y' + \underbrace{\frac{n(n+1)}{(1-x^2)}}_{\text{analytic at } x=0} y = 0$$

$\therefore$ assume $y = \sum_{m=0}^{\infty} a_m x^m$

$$\underbrace{(1-x^2) \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}}_{\begin{array}{c}\| \\ \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} \\ - \sum_{m=2}^{\infty} m(m-1) a_m x^{m}\end{array}} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + k \sum_{m=0}^{\infty} a_m x^m = 0$$

$k = n(n+1)$

Coefficients of x^0 : $2a_2 + n(n+1)\, a_0 = 0$

" x^1 : $6a_3 + [-2 + n(n+1)]\, a_1 = 0$

$\vdots$

$$(s+2)(s+1)\, a_{s+2} + [\underbrace{-s(s-1) - 2s + n(n+1)}_{\substack{\parallel \\ -s^2 - s + n^2 + n \\ \cancel{= -s(s+1) + n(n+1)} \\ = (n+s)(n-s) + (n-s) \\ = (n-s)(n+s+1)}}]\, a_s = 0$$

$$\therefore\ a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}\, a_s \qquad (s = 0, 1, \cdots)$$

: recursion formula, recurrence relation

$$a_2 = -\frac{n(n+1)}{2!}\, a_0 \qquad \Big| \qquad a_3 = -\frac{(n-1)(n+2)}{3!}\, a_1$$

$$a_4 = \frac{(n-2)n(n+1)(n+3)}{4!}\, a_0 \qquad \Big| \qquad a_5 = -\frac{(n-3)(n+4)}{5 \cdot 4}\, a_3$$

$$\Big| \qquad = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}\, a_1$$

a_0, a_1 : arbitrary const.

$$y(x) = a_0\, y_1(x) + a_1\, y_2(x)$$

- $$y_1(x) = 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 - + \cdots$$

- $$y_2(x) = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - + \cdots$$

$\dfrac{y_1}{y_2} \neq$ const. not proportional $\therefore$ linearly independent

G. S. $y(x) = a_0\, y_1(x) + a_1\, y_2(x)$

Radius of convergence

$$y_1(x) = \sum_{m=0}^{\infty} a_m x^m \underset{\uparrow}{=} \sum_{m=0}^{\infty} a_{2m} x^{2m} \underset{\substack{\uparrow \\ x^2=t}}{=} \sum_{m=0}^{\infty} a_{2m} t^m \underset{\substack{\uparrow \\ a_{2m}=b_m}}{=} \sum_{m=0}^{\infty} b_m t^m$$

$$R = \frac{1}{\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|}. \qquad \frac{b_{m+1}}{b_m} = \frac{a_{2m+2}}{a_{2m}} = -\frac{(n-2m)(n+2m+1)}{(2m+2)(2m+1)} \to 1 \text{ as } m\to\infty$$

$R=1$. $\quad |t|<1$. $\quad x^2<1$. $\quad |x|<1$. $\quad \underline{-1<x<1}$ //

The same as $y_2(x)$

or y_1/y_2 may terminate when $n =$ nonnegative integer

§ Legendre Polynomials. $P_n(x)$

If $n =$ nonnegative integer

a_0

a_1

$a_2 = (n-0)\cdots a_0$

$a_3 = (n-1)\cdots a_1$

$a_4 = (n-2)\cdots a_0$

$a_5 = (n-3)\cdots a_1$

$\vdots$

$s=n$: $a_n = a_s = (n-s)\cdots = 0$.

a_{n+1}

$a_{n+2} = 0$

$a_{n+4} = 0$

$n=$ even : $y_1(x) = a_0 + a_2x^2 + a_4x^4 + \cdots + a_nx^n + \cancel{a_{n+2}x^{n+2}} \to 0 + \cdots$

$n=$ odd : $y_2(x) = a_1x + a_3x^3 + \cdots + a_nx^n + \cancel{a_{n+2}x^{n+2}} \to 0 + \cdots$

: Legendre polynomial

Let $a_n = \frac{(2n)!}{2^n(n!)^2} = \frac{1\cdot3\cdot5\cdots(2n-1)}{n!}$, $n = 1, 2, \ldots$

$a_n = 1$ when $n = 0$

$\because P(x=1) = 1$

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n$$

$$= -\frac{n(n-1)(2n)!}{2(2n-1)2^n(n!)^2}$$

$$= -\frac{\cancel{n(n-1)}\,\cancel{2n}\,\cancel{(2n-1)}\,(2n-2)!}{\cancel{2}\cancel{(2n-1)}\,2^n\,\cancel{n}(n-1)!\,\cancel{n(n-1)}(n-2)!}$$

$$= -\frac{(2n-2)!}{2^n(n-1)!(n-2)!}$$

$$a_{n-4} = \frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!}$$

$$\vdots$$

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n\cdot m!(n-m)!(n-2m)!} \qquad (n-2m \geq 0)$$

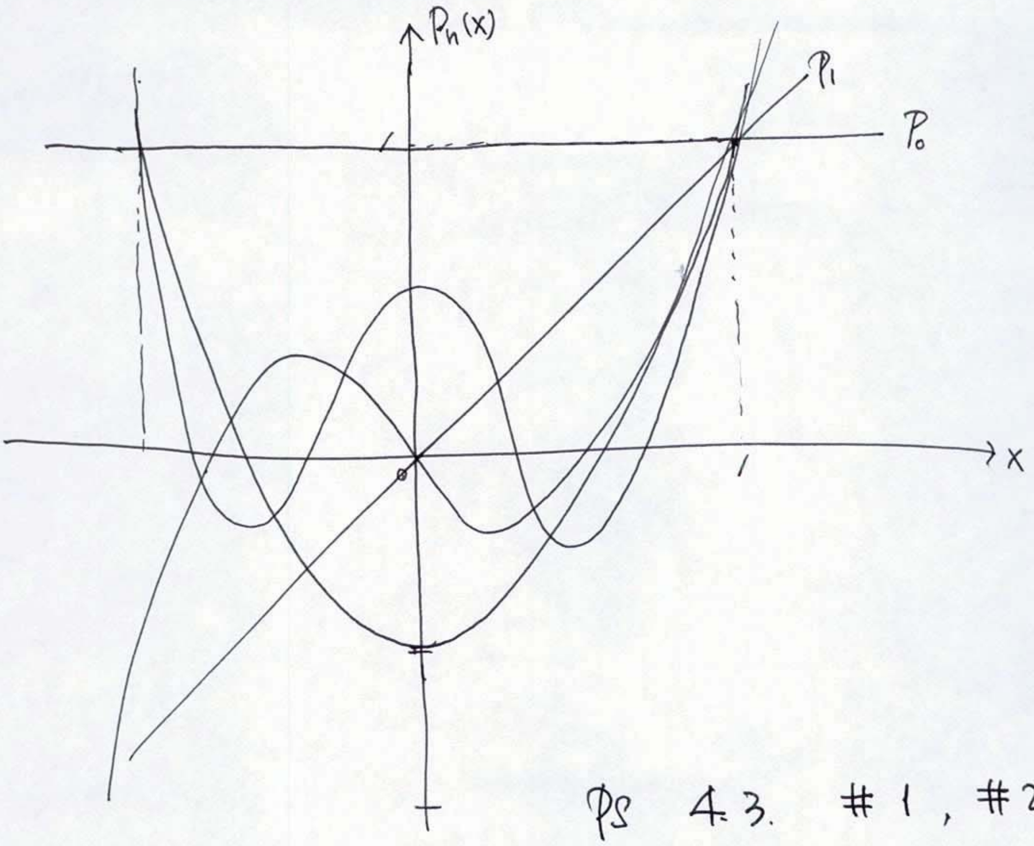
Legendre polynomial of degree n $\quad P_n(x)$

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n\cdot m!(n-m)!(n-2m)!} x^{n-2m}$$

$$= \frac{(2n)!}{2^n(n!)} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + - \cdots$$

where $M = \frac{n}{2}$ or $\frac{(n-1)}{2}$, whichever is an integer.

$P_0(x) = 1$	$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2-1)$	$P_3(x) = \frac{1}{2}(5x^3-3x)$
$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$	$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$
$\vdots$	$\vdots$



PS 4.3. #1, #2

§4.4 Frobenius Method

Theorem 1

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

$b(x), c(x)$: analytic at $x=0$

$\Rightarrow$ at least one solution :

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \quad (a_0 \neq 0)$$

r : any real or complex number

· Regular and singular points

$$y'' + p(x) y' + q(x) y = 0$$

$x = x_0$ is a regular pt if p & q are analytic at $x = x_0$

$\Rightarrow y(x)$: power series

p, q : not analytic $\Rightarrow x = x_0$: singular

$\tilde{h}(x) y'' + \tilde{p}(x) y' + \tilde{q}(x) y = 0$

$x = x_0$ regular : $\tilde{h}, \tilde{p}, \tilde{q}$ analytic & $\tilde{h}(x_0) \neq 0$

$\leftrightarrow$ singular

§ Indicial equation

$x^2 \times$ original eq

$x^2 y'' + x b(x) y' + c(x) y = 0$

$$\left[\begin{array}{l} b(x) = b_0 + b_1 x + b_2 x^2 + \cdots \\ c(x) = c_0 + c_1 x + c_2 x^2 + \cdots \end{array} \right.$$

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$y'(x) = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} = x^{r-1} [r a_0 + (r+1) a_1 x + \cdots]$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

$$= x^{r-2} [r(r-1) a_0 + (r+1) r a_1 x + \cdots]$$

substituting.

$$x^r [r(r-1) a_0 + \cdots] + (b_0 + b_1 x + \cdots) x^r (r a_0 + \cdots)$$

$$+ (c_0 + c_1 x + \cdots) x^r (a_0 + a_1 x + \cdots) = 0$$

$$\underset{0}{\underset{\parallel}{(\cdots)}} x^r + \underset{0}{\underset{\parallel}{(\cdots)}} x^{r+1} + \underset{0}{\underset{\parallel}{(\cdots)}} x^{r+2} + \cdots = 0$$

coeff. of x^r $\hookrightarrow$ $[r(r-1) + b_0 r + c_0] a_0 = 0$

smallest power

$a_0 \neq 0$

$$\boxed{r(r-1) + b_0 r + c_0 = 0}$$

: indicial eq

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

$$y_1(x) = x^r \sum_{m=0}^{\infty} a_m x^m \quad : \quad r \in \text{indicial eq}$$

$y_2 = ?$ three cases of r

- case 1 : distinct ~~real~~ roots not differing by an integer 1. 2. 3. ...
- case 2 : A double root
- case 3 : roots differing by an integer 1. 2. 3. ...

Theorem 2

Case 1. r_1, r_2 . $r_1 - r_2 \neq$ integer.

$$y_1(x) = x^{r_1} (a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$y_2(x) = x^{r_2} (A_0 + A_1 x + A_2 x^2 + \cdots)$$

Case 2. $r_1 = r_2 = r$

$$y_1(x) = x^r (a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$y_2(x) = y_1(x) \ln x + x^r (A_1 x + A_2 x^2 + \cdots) \qquad (x > 0)$$

Case 3. $r_1 - r_2 =$ integer > 0

$$y_1(x) = x^{r_1} (a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$y_2(x) = k y_1(x) \ln x + x^{r_2} (A_0 + A_1 x + A_2 x^2 + \cdots)$$

(k may be zero)

Ex. 1 Euler-Cauchy equation

$$x^2 y'' + b_0 x y' + c_0 y = 0$$

$$y = x^r$$

$$r(r-1) + b_0 r + c_0 = 0$$

review : $r_1 \neq r_2$. $y = c_1 x^{r_1} + c_2 x^{r_2}$: case 1

$r_1 = r_2$ $y = (c_1 + c_2 \ln x)\, x^r$: case 2

Ex. 2. $$x(x-1)\, y'' + (3x-1)\, y' + y = 0$$

$$y'' + \frac{(3x-1)}{x(x-1)}\, y' + \frac{1}{x(x-1)}\, y = 0$$

$$b(x) = \frac{(3x-1)/(x-1)}{x} \qquad c(x) = \frac{x/(x-1)}{x^2}$$

b, c : analytic at $x=0$.

$$y = x^r \sum_{m=0}^{\infty} a_m x^m.$$

$$x^2 y'' - x y'' + 3x y' - y' + y = 0$$

$$\Sigma \quad \Sigma \quad \Sigma \quad \Sigma \quad \Sigma$$

the smallest power : x^{r-1}

coeff : $[-r(r-1) - r]\, a_0 = -r^2 a_0 = 0$

$r = 0$: double root : Case 2

· First solution : $y = \sum_{m=0}^{\infty} a_m x^m$

we find $a_{s+1} = a_s$

$$a_0 = a_1 = a_2 = \cdots$$

$$y_1(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x} \qquad |x| < 1$$

· Second solution : Reduction of order

$$y_2 = u(x)\, y_1(x)$$

$$y_2 = y_1 \int U\,dx$$

$$U = \frac{1}{y_1^2} \exp\left[-\int p(x)\,dx\right] \quad \Leftarrow y'' + py' + qy = 0$$

$$-\int p\,dx = -\int \frac{3x-1}{x(x-1)}\,dx = -\int \left(\frac{2}{x-1} + \frac{1}{x}\right) dx$$

partial fractions

$$= -2\ln|x-1| - \ln x = \ln \frac{1}{(x-1)^2 x}$$

$$U = (1-x)^2 \exp\left[\ln \frac{1}{(x-1)^2 x}\right]$$

$$= \frac{1}{x}$$

$$y_2 = \frac{1}{1-x} \int \frac{1}{x}\,dx = \frac{\ln x}{1-x} \qquad 0 < x < 1$$

Ex. 3. $\quad (x^2 - x)y'' - xy' + y = 0$

$$y'' - \frac{x}{x(x-1)} y' + \frac{1}{x(x-1)} y = 0$$

$$\begin{cases} b(x) = -\dfrac{x}{x-1} \\ c(x) = \dfrac{x}{x-1} \end{cases} \quad \text{analytic at } x=0.$$

$y = x^r \sum_{m=0}^{\infty} a_m x^m$, substituting

$$\sum_{m=0}^{\infty} \cdots x^{m+r} - \sum_{m=0}^{\infty} \cdots x^{m+r-1} = 0$$

$$\Big\downarrow m-1 = s$$

$$\sum_{s=-1}^{\infty} \cdots x^{s+r}$$

lowest power : x^{r-1} : indicial eq : $r(r-1) = 0$

$r_1 = 1, \quad r_2 = 0 \quad . \quad r_1 - r_2 = 1 \quad$: Case 3

First solution : $r_1 = 1 = r$

$$\sum_{s=0}^{\infty} \left[s^2 a_s - (s+2)(s+1) a_{s+1} \right] x^{s+1} = 0$$

$$a_{s+1} = \frac{s^2}{(s+2)(s+1)} a_s$$

$s=0$: $a_1 = 0 \cdot a_0 = 0$

$s=1$: $a_2 = \frac{1}{3 \cdot 2} a_1 = 0$

$a_3 = 0$

$\vdots$

$$\therefore y_1 = x^{r_1} (a_0 + \cancel{a_1 x + \cdots})^{0} = x$$

Second solution : reduction of order

$$y_2 = x\,u(x)$$

$$y_2' = u + xu'$$

$$y_2'' = 2u' + xu''$$

$$(x^2 - x)(2u' + xu'') - x(u + xu') + xu = 0$$

$$(x-1)(2u' + xu'') - (u + xu') + u = 0$$

$$(x^2 - x) u'' + (x-2) u' = 0$$

$$u' = U$$

$$(x^2 - x) \frac{dU}{dx} = -(x-2) U \quad : \text{separable}$$

$$\frac{dU}{U} = -\frac{(x-2)}{x(x-1)} dx = \left(-\frac{2}{x} + \frac{1}{x-1} \right) dx$$

$$\ln U = -2 \ln x + \ln (x-1)$$

$$= \ln \frac{(x-1)}{x^2}$$

$$u' = U = \frac{(x-1)}{x^2} = \frac{1}{x} - \frac{1}{x^2}$$

logarithmic term !

$$u = \int \left(\frac{1}{x} - \frac{1}{x^2} \right) dx = \ln x + \frac{1}{x} \qquad \therefore y_2 = x \ln x + 1$$

$$\therefore y_{gen} = c_1 x + c_2 (x \ln x + 1)$$

PS 4.4 (#2~15.)

3, 7, 15

#5. Bessel's equation

$$\boxed{x^2y'' + xy' + (x^2 - \nu^2) y = 0.} \quad (\nu = \text{real} > 0)$$

cylindrical coordinates.

$$y'' + \frac{1}{x} y' + \frac{(x^2-\nu^2)}{x^2} y = 0$$

→ frobenius method.

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$$
y', y'') substituting.

lowest power (x^r): $(r+\nu)(r-\nu) = 0$. ∴ indicial eq

$$r_1 = \nu, \quad r_2 = -\nu$$

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r}$$

$$+ \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$m+2 = s$

$$\sum_{s=2}^{\infty} a_{s-2} x^{s+r}$$

x^{1+r}: $(r+1) r a_1 + (r+1) a_1 - \nu^2 a_1 = 0$

$$(\nu^2 + \nu + \nu + 1 - \nu^2)\, a_1 = (2\nu + 1)\, a_1 = 0.$$

$$\boxed{a_1 = 0.}$$

x^{s+r}: $(s+r)(s+r-1) a_s + (s+r) a_s + a_{s-2} - \nu^2 a_s = 0$

$(s = 2, 3, \cdots)$

$$(s+r+\nu)(s+r-\nu) a_s + a_{s-2} = 0.$$