PS 4.4 (#2~15.)

3, 7, 15

#5. Bessel's equation

$$\boxed{x^2y'' + xy' + (x^2-\nu^2)y = 0} \quad (\nu = \text{real} > 0)$$

cylindrical coordinates.

$$y'' + \frac{1}{x}y' + \frac{(x^2-\nu^2)}{x^2}y = 0$$

→ Frobenius method.

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$$
y', y'' substituting.

lowest power (x^r): $(r+\nu)(r-\nu) = 0$. : indicial eq

$r_1 = \nu$, $r_2 = -\nu$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)a_m x^{m+r}$$
$$+ \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$m+2=s$

$$\sum_{s=2}^{\infty} a_{s-2}x^{s+r}$$

x^{1+r}: $(r+1)r a_1 + (r+1)a_1 - \nu^2 a_1 = 0$

$$(\nu^2+\nu+\nu+1-\nu^2)\,a_1 = (2\nu+1)\,a_1 = 0.$$

$$\boxed{a_1 = 0.}$$

x^{s+r}: $(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \nu^2 a_s = 0$

$(s = 2, 3, \cdots)$

$$(s+r+\nu)(s+r-\nu)a_s + a_{s-2} = 0.$$

For $r = \nu$

$$(s+2\nu)s\,a_s + a_{s-2} = 0$$

$$\cancel{a_{s-2} = -s(s+2\nu)a_s} \quad ; \quad a_s = -\frac{1}{s(s+2\nu)}a_{s-2}$$

$s=3$: $a_3 = -\dfrac{1}{3(3+2\nu)}a_1 = 0$

$a_5 = \cdots a_3 = 0$

$a_7 = \cdots a_5 = 0$

$s=2$: $a_2 = -\dfrac{1}{2^2(\nu+1)}a_0$

$s=4$: $a_4 = -\dfrac{1}{4(4+2\nu)}a_2$

$\vdots \quad = -\dfrac{1}{2^2 \cdot 2(2+\nu)}a_2$

$s=2m$: $a_{2m} = -\dfrac{1}{2^2 m(\nu+m)}a_{2m-2}$

$$\therefore a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(\nu+1)(\nu+2)\cdots(\nu+m)}, \quad m=1,2,\cdots$$

Remember $\quad y_1(x) = \sum_{m=0}^{\infty} a_m x^{m+\nu} = \sum_{m=0}^{\infty} a_{2m} x^{2m+\nu}$

* Bessel Function $J_n(x)$ for integer $\nu = n$

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} \cdot m!(n+1)(n+2)\cdots(n+m)}, \quad m=1,2,\cdots$$

$$a_0 : \text{arbitrary} = \frac{1}{2^n \cdot n!}$$

$$a_{2m} = \frac{(-1)^m}{2^{2m+n} \cdot m!(n+m)!}$$

$$\boxed{J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} \cdot m!(n+m)!}}$$

Bessel function of the first kind of order n

$$R \to \infty$$

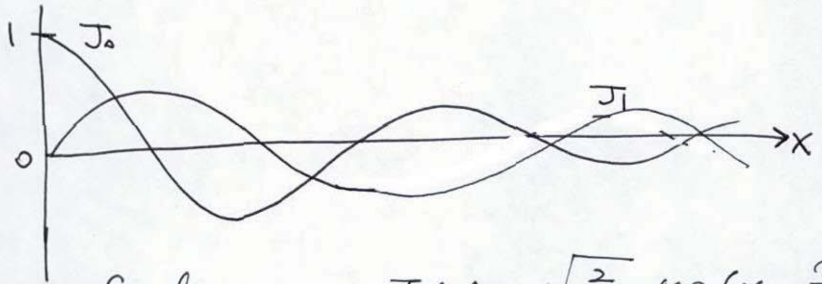
Ex. 1 Bessel functions $J_0(x)$, $J_1(x)$

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + - \cdots$$

compare with $\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}$

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!\,(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 \cdot 1! \, 2!} + \frac{x^5}{2^5 \cdot 2! \, 3!} - + \cdots$$

compare with $\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$



for large x, $J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$

※ Bessel functions $J_\nu(x)$ for any $\nu \geq 0$.

Gamma function $\Gamma(\nu) \equiv \int_0^\infty e^{-t} t^{\nu-1}\,dt$

– properties of Gamma function

$$\Gamma(\nu+1) = \int_0^\infty \underbrace{e^{-t}}_{v'} \underbrace{t^\nu}_{u}\,dt = \underbrace{-e^{-t}t^\nu \Big|_0^\infty}_{0} + \nu \int_0^\infty e^{-t} t^{\nu-1}\,dt$$

$$= \nu\,\Gamma(\nu)$$

$$\therefore \boxed{\Gamma(\nu+1) = \nu\,\Gamma(\nu)}$$

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = -\left[e^{-t}\right]_0^\infty = -(0-1) = 1$$

$$\Gamma(2) = 1 \cdot 1$$

$$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2!$$

$$\vdots$$

$$\boxed{\Gamma(n+1) = n!} \qquad n = 0, 1, 2, \ldots$$

Recall that when $\nu = n$, $a_0 = \frac{1}{2^n \cdot n!} = \frac{1}{2^n \Gamma(n+1)}$

→ for any ν, $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$.

$$a_{2m} = \frac{(-1)^m}{2^{2m+\nu} m! \underbrace{(\nu+1)(\nu+2)\cdots(\nu+m)\Gamma(\nu+1)}_{\Gamma(\nu+m+1)}}.$$

$$= \frac{(-1)^m}{2^{2m+\nu} m! \, \Gamma(\nu+m+1)}.$$

$$\boxed{J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m! \, \Gamma(\nu+m+1)}} \quad \Leftarrow x^\nu \sum_{m=0}^{\infty} a_{2m} x^{2m}$$

Bessel function of the first kind of order ν

$R \to \infty$ converges for all x.

For $r = -\nu$

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m! \, \Gamma(m-\nu+1)} \qquad (x \neq 0).$$

$\nu \neq$ integer $\rightarrow$ J_ν & $J_{-\nu}$: linearly independent

Theorem 1.

If $\nu \neq$ integer. general sol. of Bessel's eq for all $x \neq 0$

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

Theorem 2

For $\nu = n$ (integer)

$$J_{-n}(x) = (-1)^n J_n(x) \qquad \text{: linearly dep.}$$

pf.) $$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m! \, \Gamma(m-n+1)}$$

$m=0$: $\Gamma(-n+1) \to \infty$

$\vdots$ see Fig. ~~517~~ 552 ~~p. A54~~ in App. A3.1

$m=n-1$: $\Gamma(0) \to \infty$

$$\frac{x^{2m-n}}{\infty} \to 0$$

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m!\,(m-n)!}$$

let $m-n=s$

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} x^{2s+n}}{2^{2s+n}(n+s)!\,s!} = (-1)^n J_n(x)$$

Properties of Bessel functions.

$$\frac{d}{dx}[x^{\nu} J_{\nu}(x)] = x^{\nu} J_{\nu-1}(x)$$

$$\frac{d}{dx}[x^{-\nu} J_{\nu}(x)] = -x^{-\nu} J_{\nu+1}(x) \quad \cdots \ (*)$$

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x) \quad \cdots \ (\S)$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_{\nu}'(x)$$

pf: practice!

Ex. 2 $\quad I = \int_1^2 x^{-3} J_4(x)\,dx$. integrate using J_0, J_1 (table A1)

$(*) \rightarrow \quad x^{-3} J_4(x) = -\frac{d}{dx}[x^{-\nu} J_{\nu}(x)] \quad \nu=3$

$$I = \Big[-x^{-3} J_3(x)\Big]_1^2$$

$(\S) \rightarrow \ \nu=2: \quad J_1(x) + J_3(x) = \frac{4}{x} J_2(x)$

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

$\nu=1: \quad J_0(x) + J_2(x) = \frac{2}{x} J_1(x)$

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

$$\therefore J_3(x) = \frac{4}{x}\left(\frac{2}{x} J_1 - J_0\right) - J_1 = \left(\frac{8}{x^2} - 1\right) J_1 - \frac{4}{x} J_0.$$

$$I = J_3(1) - \frac{1}{8} J_3(2) = 0.0038$$

§ $J_\nu(x)$ with $\nu = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \dots$ are elementary

Using $\boxed{\Gamma(\frac{1}{2}) = \sqrt{\pi}}$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}}_{\text{Maclaurin series of } \sin x}$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$(\S) \rightarrow \quad J_{\frac{3}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$$

$$\vdots$$

PS 4.5 #1, 7, 9, 12, 25

§4.6. Bessel functions of the second kind $Y_\nu(x)$

$\boxed{n = 0}$

Bessel's eq: $xy'' + y' + xy = 0$

indicial eq $\rightarrow$ $r = 0$ (double root)

first solution: $J_0(x)$

second solution: $y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} A_m x^m$

$$y_2' = J_0' \ln x + J_0 \frac{1}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}$$

$$y_2'' = J_0'' \ln x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-2}$$

$$xy'' + y' + xy = (\overbrace{xJ_0'' + J_0' + J_0}^{0}) \ln x + 2J_0' - \cancel{\frac{J_0}{x}} + \cancel{\frac{J_0}{x}}$$
$$+ \sum_{m=1}^{\infty} m(m-1) A_m x^{m-1} + \sum_{m=1}^{\infty} m A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1}$$

$$= 0$$

$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!(m-1)!}$$

$$2J_0'(x) + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

$1 \cdot A_1 x^0 = 0 \quad \therefore \quad A_1 = 0$

even powers x^{2s}: 2nd series: $m-1 = 2s$: $(2s+1)^2 A_{2s+1} x^{2s}$

3rd series: $m+1 = 2s$: $A_{2s-1} x^{2s}$. $s = 1, 2, \dots$

$$(2s+1)^2 A_{2s+1} + A_{2s-1} = 0.$$

$$A_{2s+1} = -\frac{1}{(2s+1)^2} A_{2s-1}$$

$A_1 = 0$

$A_3 = 0$

$\vdots$

odd powers x^{2s+1}: x^1: $\frac{-1}{1!0!} + A_2 \cdot 4 = 0. \qquad A_2 = \frac{1}{4}$

$$\frac{(-1)^{s+1}}{2^{2s}(s+1)!s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0$$

$$A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2} \underbrace{\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}\right)}_{= h_m}$$

$$\therefore \; y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m}$$

OR $\tilde{y}_2 = a(y_2 + bJ_0) \quad (a \neq 0)$

$a = \frac{2}{\pi}, \quad b = \gamma - \ln 2.$

$$\gamma = 0.577\dots = \lim_{s \to \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{s} - \ln s\right)$$

: Eule constant

$$\Rightarrow \quad Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln \frac{x}{2} + \gamma\right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m} \cdot (m!)^2} x^{2m}\right]$$

: Bessel function of the second kind of order zero

* Bessel function of the second kind of order ν

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}\left[J_\nu(x) \cos \nu\pi - J_{-\nu}(x) \right]$$

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$

J_ν & Y_ν are linearly independent for all ν.

For integer n,

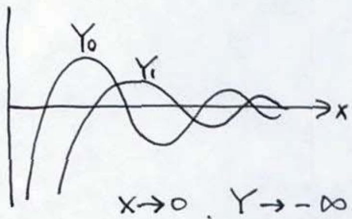
$$Y_n(x) = \frac{2}{\pi} J_n(x) \left(\ln \frac{x}{2} + \gamma \right) + \frac{x^n}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m-1}(h_m + h_{m+n})}{2^{2m+n}\, m!\,(m+n)!} x^{2m}$$

$$- \frac{x^{-n}}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{2^{2m-n}\, m!} x^{2m}$$

$x > 0$. $n = 0, 1, \cdots$

$$h_0 = 0, \quad h_s = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{s} \qquad (s = 1, 2, \cdots)$$

note $\quad Y_{-n}(x) = (-1)^n Y_n(x)$



$x \to 0$. $Y \to -\infty$ $(\sim \ln x)$

Theorem 1

General Sol of Bessel's equation for all ν:

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

complex solutions:

$$H_\nu^{(1)}(x) = J_\nu(x) + i Y_\nu(x)$$
$$H_\nu^{(2)}(x) = J_\nu(x) - i Y_\nu(x)$$

] linearly independent

: Bessel functions of the third kind of order ν

Hankel functions of order ν

PS 4.6. #7, 9.