$(y_m, y_n) = \delta_{mn}$: Kronecker delta

$$\delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

norm $\|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b \cancel{p}\, r(x)\, y_m^2(x)\, dx}$

* Orthogonal expansion

$$\boxed{f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots}$$

$y_0, y_1, \cdots$: orthogonal set w.r.t $\cancel{p}\, r(x)$ on $[a, b]$

~ generalized Fourier series

* How to obtain a_m ? — use orthogonality

$$(f, y_n) = \int_a^b \cancel{p}\, r\, y_n \underbrace{\sum_{m=0}^{\infty} a_m y_m(x)}_{\parallel \atop a_0 y_0 + a_1 y_1 + \cdots + a_n y_n + \cdots} dx$$

$$= \int_a^b \cancel{p}\, r\, a_n y_n^2\, dx = a_n \|y_n\|^2$$

$$\boxed{a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b \cancel{p}\, r(x)\, f(x)\, y_m(x)\, dx} \qquad (m = 0, 1, \cdots)$$

Ex. 1. Fourier Series

$$y'' + \lambda y = 0.$$

$y(\pi) = y(-\pi)$, $y'(\pi) = y'(-\pi)$: periodic B.C.

$y = e^{\tilde{k}x}$. $\tilde{k}^2 + \lambda = 0$ $\tilde{k} = \pm i\sqrt{\lambda}$

$y = A\cos kx + B\sin kx$ $(k = \sqrt{\lambda})$.

B.C. $\Rightarrow$ $A\cos k\pi + B\sin k\pi = A\cos k\pi - B\sin k\pi$

$B\sin k\pi = 0$

$-Ak\sin k\pi + Bk\cos k\pi = Ak\sin k\pi + Bk\cos k\pi$

$Ak\sin k\pi = 0$

For nontrivial solutions,

$\sin k\pi = 0$ $k = 0, \pm 1, \pm 2, \cdots$

eigenvalue: $\lambda = k^2 = 0, 1, 4, \cdots$

eigenfunctions: $1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots$

: orthogonal w.r.t. $p(x) = 1$

: norm $\|1\| = \sqrt{2\pi}$, $\|\sin mx\| = \|\cos mx\| = \sqrt{\pi}$

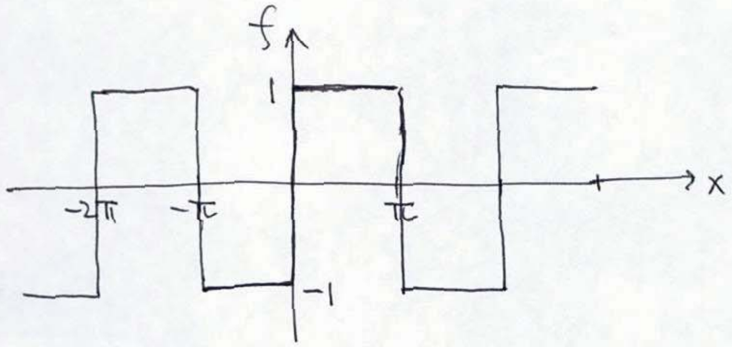
$\rightarrow$ orthogonal series

Fourier Series of $f(x)$

$$f(x) = a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$$

$$\begin{cases} a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx \\ a_m = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos mx\,dx & (m = 1, 2, \cdots) \\ b_m = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin mx\,dx & (m = 1, 2, \cdots) \end{cases}$$

e.g. $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi. \end{cases}$ and $f(x+2\pi) = f(x)$



: periodic square wave

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_m = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-1)\cos mx\,dx + \int_{0}^{\pi} \cos mx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\frac{1}{m}\left[-\sin mx\right]_{-\pi}^{0} + \frac{1}{m}\left[\sin mx\right]_{0}^{\pi}\right\} = 0$$

$$b_m = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-1)\sin mx\,dx + \int_{0}^{\pi} \sin mx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\frac{1}{m}\left[\cos mx\right]_{-\pi}^{0} + \frac{-1}{a}\left[\cos mx\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\frac{1}{m}(1-\cos m\pi) - \frac{1}{m}(\cos m\pi - 1)\right]$$

$$= \frac{1}{m\pi}(2 - 2\cos m\pi)$$

$$= \frac{2}{m\pi}(1-\cos m\pi) = \begin{cases} \frac{4}{m\pi} & (m = 1, 3, \cdots) \\ 0 & (m = 2, 4, \cdots) \end{cases}$$

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

$$= \frac{4}{\pi}\sum_{m=1}^{\infty}\left[\frac{1}{(2m-1)}\sin(2m-1)x\right]$$

Ex. 2 Fourier-Legendre Series

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 P_0 + a_1 P_1(x) + \cdots$$
$$= a_0 + a_1 x + a_2\left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right) + \cdots$$

$P_m(x)$: Legendre's eq S.L. $r(x) = 1 - x^2$, $p = 1$ " $-1 \le x \le 1$ " → orthogonal

$$a_m = \frac{1}{\|y_m\|^2} \int_a^b p f y_m \, dx$$

$$\|y_m\| = \|P_m\| = \left[\int_{-1}^{1} P_m^2(x)\, dx\right]^{1/2} = \sqrt{\frac{2}{2m+1}} \quad m = 0, 1, 2, \cdots$$

accept w/o proof

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx$$

Ex. 3 Fourier-Bessel series

$$f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{mn} x) = a_1 J_n(k_{1n} x) + a_2 J_n(k_{2n} x) + \cdots$$

$J_n(k_{mn} x)$: Bessel's eq. S.L. $r(x) = x$, $p(x) = x$

$0 \le x \le R$ any fixed positive #

$\xrightarrow{\text{orthog}} J_n(k_{mn} R) = 0.$

$$\|J_n(k_{mn} x)\|^2 = \int_0^R x J_n^2(k_{mn} x)\, dx$$
$$= \frac{R^2}{2} J_{n+1}^2(k_{mn} R)$$

$$a_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{mn})} \int_0^R x f(x) J_n(k_{mn} x)\, dx. \quad m = 1, 2, \cdots$$

* Completeness of orthonormal sets

$$\text{orthonormal set } (y_0, y_1, y_2, \cdots) \xrightarrow[\text{orthogonal expansion}]{\text{generalized Fourier series}} \text{a set of functions } S$$

on $a \leq x \leq b$.