8.6. Curves in Mechanics

$r(t)$: position vector

$\bar{v} = \bar{r}' = \frac{d\bar{r}}{dt}$: velocity

$\bar{a} = \bar{v}' = \bar{r}''$: acceleration

Ex.1. centripetal acceleration

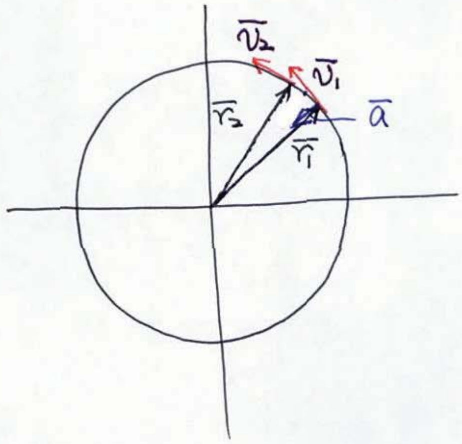
$\bar{r}(t) = R\cos\omega t\,\hat{i} + R\sin\omega t\,\hat{j}$

$\bar{v} = \bar{r}'$,

$|\bar{v}| = \sqrt{\bar{r}' \cdot \bar{r}'} = R\omega$

$\bar{a} = \bar{v}' = -\omega^2 \bar{r}$

$|\bar{a}| = \omega^2 R$



· Tangential / Normal Acceleration — reading

8.7. Curvature and Torsion of a curve

curvature : $\kappa(s) = |\bar{u}'(s)| = |\bar{r}''(s)|$: deviation of C from tangent

$\bar{u}(s) = \bar{r}'(s)$: unit tangent vector

for general parameter t

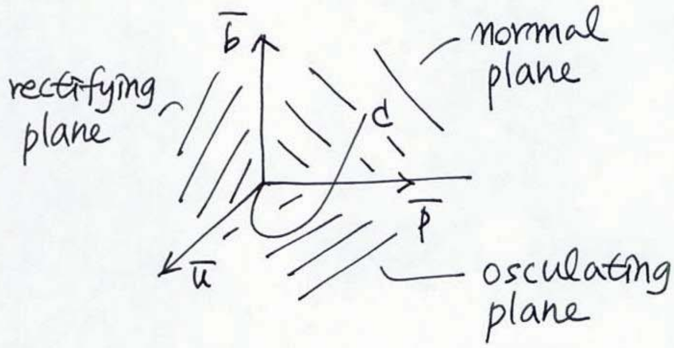
$$\kappa(t) = \frac{\sqrt{(\bar{r}' \cdot \bar{r}')(\bar{r}'' \cdot \bar{r}'') - (\bar{r}' \cdot \bar{r}'')^2}}{(\bar{r}' \cdot \bar{r}')^{3/2}}$$

· unit principal normal vector

$$\bar{p} = \frac{1}{|\bar{u}'|}\bar{u}' = \frac{1}{\kappa}\bar{u}' \quad : \text{recall } \bar{u}\cdot\bar{u}' = 0 \quad (\because |\bar{u}|^2 = 1)$$

· unit binormal vector

$$\bar{b} = \bar{u} \times \bar{p}$$



* Torsion of a curve

$$\tau(s) = -\bar{p}(s)\cdot\bar{b}'(s)$$

: deviation of C from the osculating plane

PS 8.7 # 1, 5

~~8.9~~ 9.7 Gradient of a Scalar field

· Definition

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) f$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

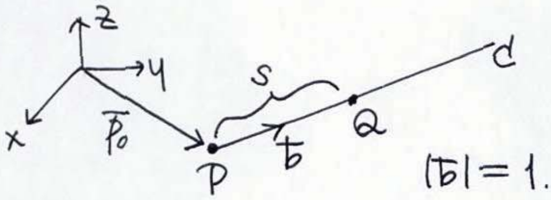
(nabla, del)

e.g. $f = 2x + yz - 3y^2$

$\nabla f = 2\hat{i} + (z - 6y)\hat{j} + y\hat{k}$

* Directional Derivative

· rate of change of f at P in a direction ($\bar{b}$)



$|\bar{b}| = 1.$

$$C: \bar{r}(s) = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$$
$$= \bar{p}_0 + s\bar{b}$$

$$D_{\bar{b}} f = \frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s}$$

$$\frac{df}{ds} = \frac{d}{ds} f(x(s), y(s), z(s))$$

$\downarrow$ chain rule

$$= \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

$$= \underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)}_{\nabla f} \cdot \underbrace{(x'\hat{i} + y'\hat{j} + z'\hat{k})}_{\frac{d\bar{r}}{ds} = \bar{b}}$$

$$\therefore \boxed{\frac{df}{ds} = \bar{b} \cdot \nabla f}$$

In general

$$D_{\bar{a}} f = \frac{df}{ds} = \frac{1}{|\bar{a}|}\bar{a} \cdot \nabla f$$

Ex. 1. $f(x, y, z) = 2x^2 + 3y^2 + z^2.$ $D_{\bar{a}} f = ?$ $\bar{a} = \hat{i} - 2\hat{k}.$
$P(2, 1, 3)$

$$D_{\bar{a}} f = \frac{1}{|\bar{a}|}\bar{a} \cdot \nabla f.$$

$$\nabla f = 4x\hat{i} + 6y\hat{j} + 2z\hat{k}$$

$$|\bar{a}| = \sqrt{1+4} = \sqrt{5}$$

$$D_{\bar{a}} f = \frac{1}{\sqrt{5}}(\hat{i} - 2\hat{k}) \cdot (8\hat{i} + 6\hat{j} + 6\hat{k}) = -\frac{4}{\sqrt{5}}$$

Theorem 1.

$f(P) = f(x, y, z)$: scalar fn having continuous first partial derivative

$\Rightarrow \nabla f$
- exists / length & direction independent of coordinate system.
- * direction of maximum increase of f at P.

Proof. $D_{\bar{b}} f = \bar{b} \cdot \nabla f$

$= |\bar{b}|\,|\nabla f| \cos\gamma$

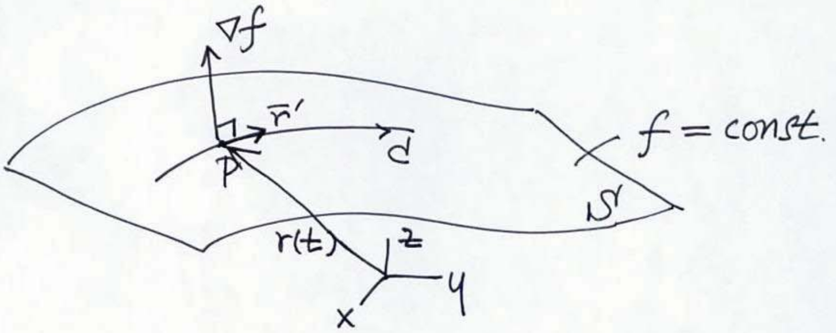
maximum rate of change at $\gamma = 0$. : $\bar{b} \parallel \nabla f$

Theorem 2.

surface S : $f(x, y, z) = \text{const}$

$\nabla f|_P$: normal vector of S at P.

Proof.



curve C : $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \in S$.

$$f(x(t), y(t), z(t)) = c \quad \cdots (\S)$$

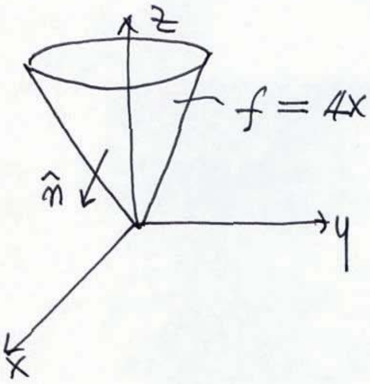
tangent vector of C : $\bar{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$

$$\frac{d}{dt}(\S) \Rightarrow \frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial z} z' = 0.$$

$$\nabla f \cdot \bar{r}' = 0 \qquad \nabla f \perp \bar{r}'$$

Ex. 2. Cone of revolution : $z^2 = 4(x^2+y^2)$.

unit normal vector at $P(1, 0, 2)$



$$\hat{n} = \frac{\nabla f}{|\nabla f|}\Big|_P.$$

$$\nabla f = 8x\,\hat{i} + 8y\,\hat{j} - 2z\,\hat{k}$$

$$\nabla f|_P = 8\hat{i} - 4\hat{k} = 4(2\hat{i} - \hat{k})$$

$$|\nabla f| = \sqrt{64+16} = 4\sqrt{5}$$

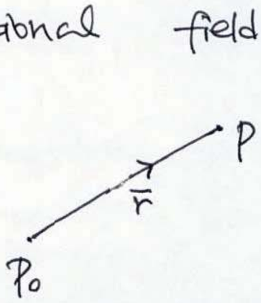
$$\hat{n} = \frac{2}{\sqrt{5}}\hat{i} - \frac{1}{\sqrt{5}}\hat{k}. \quad \text{or } -\hat{n}.$$

* Potential

$$\nabla f = \bar{v}$$

(f : potential of $\bar{v}$
(vector field ⇒ conservative (no energy lost)

Ex. 3. Gravitational field



$$\bar{P}\ (\text{force of attraction}) = -\frac{c}{r^2}\frac{\bar{r}}{|\bar{r}|} = -\frac{c}{r^3}\bar{r}$$

$$\nabla f = \bar{P}$$

$f(x,y,z) = \frac{c}{r}$: potential of the gravitational field

* Laplace's equation

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

$$= \nabla\cdot\nabla = \nabla^2 \quad : \text{Laplacian}$$

$$\nabla^2 f = 0.$$

PS 8.9 # 5, 11, 14 $(6\hat{i}-5\hat{j})$, 19, 21, 25, 29

#28: $\nabla(fg) = f\nabla g + g\nabla f$. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

8.10. Divergence

$$\bar{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

$$\operatorname{div}\bar{v} = \nabla\cdot\bar{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

e.g. $\bar{v} = 3xz\hat{i} + 2xy\hat{j} - yz^2\hat{k}$

$$\operatorname{div}\bar{v} = \nabla\cdot\bar{v} = 3z + 2x - 2yz$$

Note: $\boxed{\nabla^2 f = \nabla\cdot(\nabla f)}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

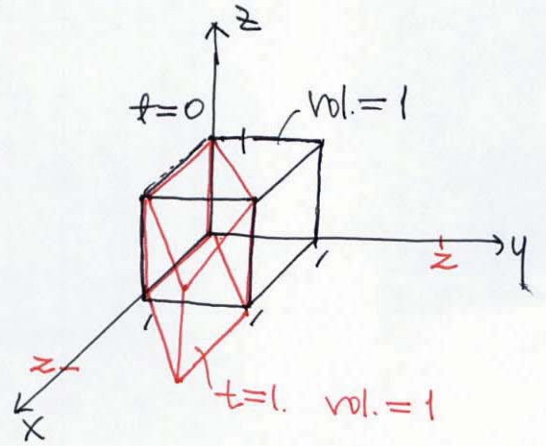
$$= \operatorname{div}(\operatorname{grad} f)$$

PS 8.10. # 7

11. $\overline{v} = y\hat{i}$: flow velocity

$$\nabla\cdot\overline{v} = \frac{\partial y}{\partial x} = 0.$$

$\nabla\cdot\overline{v} = 0$: incompressible



13.

$$\nabla\cdot(k\overline{v}) = k\,\nabla\cdot\overline{v}$$

$$\nabla\cdot(f\overline{v}) = f\,\nabla\cdot\overline{v} + \overline{v}\cdot\nabla f$$

$$\nabla\cdot(f\,\nabla g) = f\,\nabla^2 g + \nabla f\cdot\nabla g$$

8.11. Curl

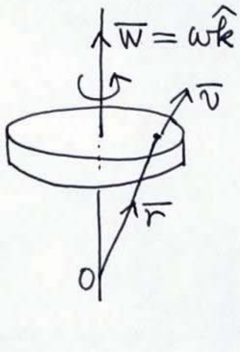
$$\overline{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

$$\text{curl}\,\overline{v} = \nabla\times\overline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Ex. 1. $\overline{v} = yz\hat{i} + 3zx\hat{i} + z\hat{k}$

$$\text{curl}\,\overline{v} = \nabla\times\overline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} = \hat{i}\,(-3x) + \hat{j}\,(+y) + \hat{k}\,(3z - z)$$

Ex. 2. Rotation of a rigid body



$$\overline{v} = \overline{\omega}\times\overline{r}$$

$$= \omega\hat{k}\times(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \omega(-y\hat{i} + x\hat{j})$$

$$\text{curl}\,\overline{v} = \nabla\times\overline{v} = 2\omega\hat{k} = 2\overline{w}$$

* $\text{curl}(\text{grad} f) = \nabla \times (\nabla f)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = 0.$$

* $\text{div}(\text{curl}\, \overline{v}) = \nabla \cdot (\nabla \times \overline{v}) = 0.$

PS 8.11 # 7

#14. $\nabla \times (\overline{u} + \overline{v}) = \nabla \times \overline{u} + \nabla \times \overline{v}$

$\nabla \times (f\overline{v}) = (\nabla f) \times \overline{v} + f(\nabla \times \overline{v})$

$\nabla \cdot (\overline{u} \times \overline{v}) = \overline{v} \cdot \nabla \times \overline{u} - \overline{u} \cdot \nabla \times \overline{v}$