$$\bar{N} = \bar{r}_u \times \bar{r}_v$$

$$\hat{n} = \frac{1}{|\bar{N}|}\bar{N} = \frac{1}{|\bar{r}_u \times \bar{r}_v|}\,\bar{r}_u \times \bar{r}_v$$

Recall. $S:\ g(x,y,z) = 0.$

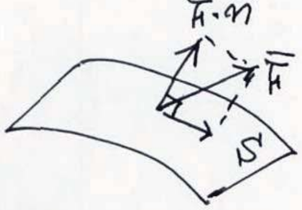
$$\hat{n} = \frac{1}{|\nabla g|}\nabla g$$

ps 9.5 # 9, 13, 29

~~9~~10.6. Surface Integrals

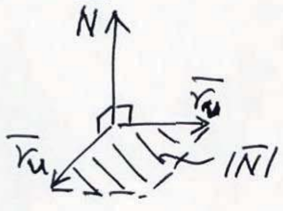
$$I = \iint_S \bar{F}\cdot\hat{n}\, dA = \iint_R \bar{F}(\bar{r}(u,v))\cdot \bar{N}(u,v)\, du\,dv$$

$$\bar{N} = \bar{r}_u \times \bar{r}_v$$



I = "flux" integral

$$\hat{n}\,dA = \bar{N}\,du\,dv \ ?$$



$$\Delta A = \Delta r_1 \Delta r_2 \sin\theta$$

$$= \frac{\partial r}{\partial u}\Delta u \cdot \frac{\partial r}{\partial v}\Delta v \cdot \sin\theta$$

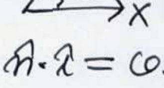
$$dA = |\bar{r}_u \times \bar{r}_v|\, du\,dv$$

$$\therefore\ \hat{n}\,dA = \hat{n}|\bar{N}|\,du\,dv$$

$$= \bar{N}\,du\,dv$$

In terms of direction cosines:

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$



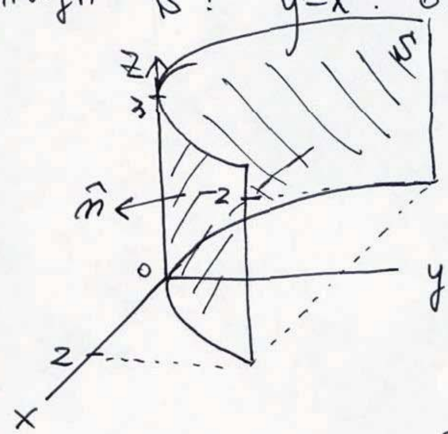
$$\hat{n}\cdot\hat{i} = \cos\alpha$$

$$I = \iint_S \bar{F}\cdot\hat{n}\,dA = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dA$$

$$= \iint_R (F_1N_1 + F_2N_2 + F_3N_3)\,du\,dv$$

Ex. 1. flux of water $\bar{v} = [3z^2, 6, 6xz]$

through S : $y = x^2$. $0 \le x \le 2$, $0 \le z \le 3$



$$\dot{m} = \iint_S \rho\bar{v}\cdot\hat{n}\,dA$$

$$Q = \iint_S \bar{v}\cdot\hat{n}\,dA$$

$$S:\ \bar{r} = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

$$= u\hat{i} + u^2\hat{j} + v\hat{k}$$

$$\begin{cases} 0 \le u \le 2 \\ 0 \le v \le 3 \end{cases}$$

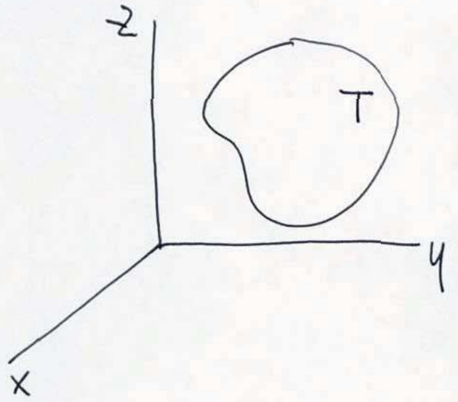
$$\bar{N} = \frac{\partial\bar{r}}{\partial u} \times \frac{\partial\bar{r}}{\partial v} = (\hat{i} + 2u\hat{j}) \times (\hat{k}) = 2u\hat{i} - \hat{j}$$

$$Q = \int_0^3\int_0^2 [3v^2, 6, 6uv]\cdot[2u, -1, 0]\,du\,dv$$

$$= \int_0^3\int_0^2 (6uv^2 - 6)\,du\,dv = 72\ m^3/s$$

PS 9.6. # 1, 5, 7, 9

~~9~~10.7. Triple Integrals



$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_T f(x,y,z)\,dV$$

Theorem 1. Divergence Theorem of Gauss

(volume integral $\leftrightarrow$ surface integral)

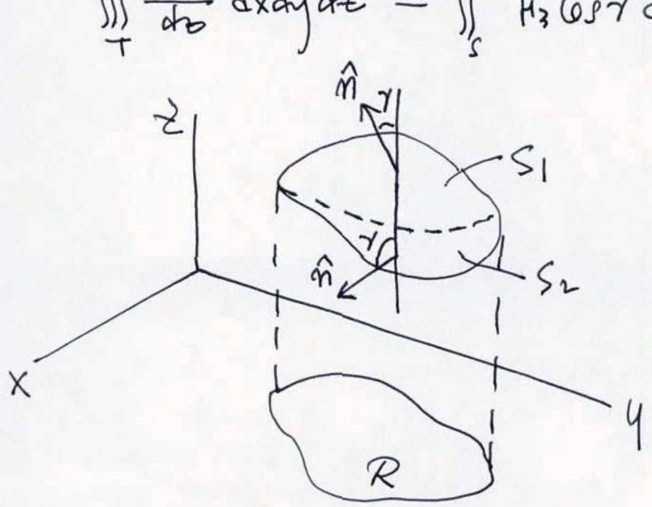
$$\boxed{\iiint_T \nabla\cdot\bar{F}\,dV = \iint_S \bar{F}\cdot\hat{n}\,dA}$$

if $\hat{n} = [\cos\alpha, \cos\beta, \cos\gamma]$

$$\iiint_T \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dx\,dy\,dz = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dA$$

$$= \iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy)$$

Proof) $\displaystyle\iiint_T \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_S F_3\cos\gamma\,dA$



$$\underset{S_1}{g(x,y)} \le z \le \underset{S_2}{h(x,y)}$$

$$(x,y)\in R$$

$$LHS = \iiint_T \frac{\partial F_3}{\partial z}\, dx\,dy\,dz = \iint_R \Big[\underbrace{\int_{g(x,y)}^{h(x,y)} \frac{\partial F_3}{dz}\, dz}_{F_3(x,y,h(x,y)) - F_3(x,y,g(x,y))} \Big] dx\,dy$$

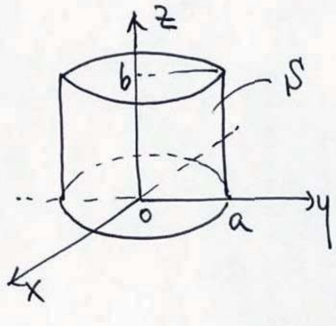
$$= \iint_R F_3(x,y,h(x,y))\, dx\,dy - \iint_R F_3(x,y,g(x,y))\, dx\,dy$$

$$RHS = \iint_S F_3 \cos\gamma\, dA = \iint_S F_3\, dx\,dy$$

$$= + \iint_R F_3(x,y,h(x,y))\, dx\,dy - \iint_R F_3(x,y,g(x,y))\, dx\,dy$$

$$\therefore\ LHS = RHS$$

Ex. 1. $\quad I = \iint_S (x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy)$



$x^2+y^2=a^2. \quad 0 \le z \le b.$

$$F_1 = x^3, \quad F_2 = x^2y, \quad F_3 = x^2z$$

$$\nabla \cdot F = 3x^2 + x^2 + x^2 = 5x^2$$

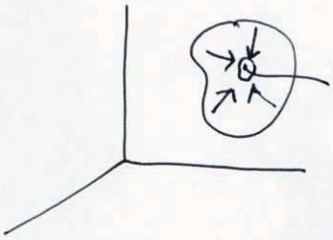
$$I = \iiint_T 5x^2\, dx\,dy\,dz = 5 \int_{z=0}^{b} \int_{r=0}^{a} \int_{\theta=0}^{2\pi} r^2\cos^2\theta\, r\, dr\, d\theta\, dz$$

$$= \frac{5}{4}\pi b a^4$$

PS 9.7 # 1, 7, 15.

~~9~~ 10.8. Divergence Theorem : Further Applications

Ex. 1.



for infinitesimal volume δV

$$\iiint_T f(x, y, z)\, dV = f(x_0, y_0, z_0)\, \delta V$$

$$\nabla \cdot \bar{F}(x_1, y_1, z_1) = \lim_{V \to 0} \frac{1}{V} \iint_{S(T)} \bar{F} \cdot \hat{n}\, dA$$

Ex. 4

$$\iiint_T \nabla^2 f\, dV = \iiint_T \nabla \cdot (\nabla f)\, dV$$

$$= \iint_S \nabla f \cdot \hat{n}\, dA$$

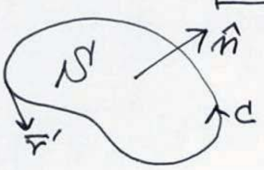
$$= \iint_S \frac{\partial f}{\partial n}\, dA$$

PS 9.8 # 3, 7, 10.

~~9~~ 10.9. Stokes' Theorem

$\uparrow$ generalized

Green's theorem



$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_C (F_1\, dx + F_2\, dy)$$

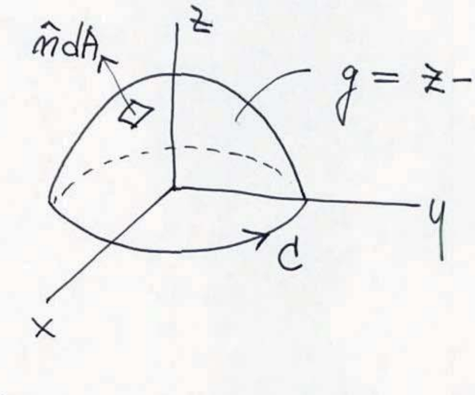
Theorem 1. Stokes' theorem (surface int $\leftrightarrow$ line integral)

$$\boxed{\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, dA = \oint_C \bar{F} \cdot \frac{d\bar{r}}{ds}\, ds}$$

$\frac{d\bar{r}}{ds}$: unit tangent vector

Ex. 1. $\bar{F} = [y, z, x]$

S (paraboloid) : $z = f(x, y) = 1 - (x^2+y^2)$. $z \geq 0$



$g = z - 1 + (x^2+y^2) = 0$

$\nabla g = \bar{N} = 2x\hat{i} + 2y\hat{j} + \hat{k}$

(i) $\iint_S (\nabla\times\bar{F})\cdot\hat{n}\,dA = ?$

$$\begin{cases} \nabla\times\bar{F} = -\hat{i} - \hat{j} - \hat{k} \\ \hat{n} = \frac{1}{|\nabla g|}\nabla g = \frac{1}{|\bar{N}|}\bar{N} \qquad \hat{n}dA = \bar{N}\,du\,dv \end{cases}$$

$$\iint_S (\nabla\times\bar{F})\cdot\hat{n}\,dA = \iint_R (-\hat{i}-\hat{j}-\hat{k})\cdot\nabla g\,dx\,dy = -\pi$$

(ii) $\oint_C \bar{F}\cdot\bar{r}'\,ds = \oint_C \bar{F}\cdot\frac{d\bar{r}}{ds}\,ds = \int_0^{2\pi}(-\sin^2 s)\,ds = -\pi$

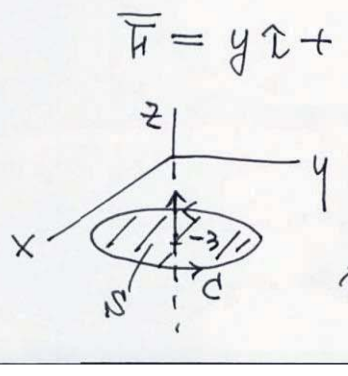
$$\begin{cases} \bar{r}(s) = \cos s\,\hat{i} + \sin s\,\hat{j} + 0 \\ \frac{d\bar{r}}{ds} = -\sin s\,\hat{i} + \cos s\,\hat{j} \end{cases}$$

$\bar{F} = y\hat{i} + z\hat{j} + x\hat{k}$

$= \sin s\,\hat{i} + \cos s\,\hat{k}$ on C

Ex. 3. $I = \int_C \bar{F}\cdot\bar{r}'\,ds = ?$ C : $x^2+y^2 = 4$. $z = -3$.

$\bar{F} = y\hat{i} + xz^3\hat{j} - zy^3\hat{k}$



$\hat{n} = \hat{k}$

$I = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,dA$

$= \iint_S \underbrace{(\nabla\times\bar{F})\cdot\hat{k}}_{=-28}\,dx\,dy = -28\,(4\pi)$

$= -112\pi$

PS 9.9. # 1, 5, 7, 11