



## 15 Solving PDEs

### 15.1 Separation of Variables. Use of Fourier Series.

- Governing equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (1)$$

- Boundary conditions

$$u(0, t) = 0, \quad u(L, t) = 0 \quad \text{for all } t \quad (2)$$

- Initial conditions

$$u(x, 0) = f(x) \quad (3)$$

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x) \quad (4)$$

- Solution procedures

**Step I:** Method of separating variables or product method → two ordinary differential equations

**Step II:** Determination of the solutions of those two equations satisfying the boundary conditions (2)

**Step III:** Using Fourier series, acquirement of a solution of the wave equation (1) satisfying the initial conditions (3) and (4).

#### 15.1.1 Step I : Two Ordinary Differential Equations

- Method of separating variables

$$u(x, t) = F(x) \cdot G(t) \quad (5)$$

$$\frac{\partial^2 u}{\partial t^2} = F \ddot{G} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F'' G$$

- By inserting this into (1),

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow F \ddot{G} = c^2 F'' G$$

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k = \text{constant}$$

$$F'' - kF = 0 \quad (6)$$

$$\ddot{G} - c^2 k G = 0 \quad (7)$$

### 15.1.2 Step II: Second Step. Satisfying the Boundary Conditions (2)

$$u(0, t) = F(0)G(t) = 0, \quad u(L, t) = F(L)G(t) = 0 \quad \text{for all } t$$

#### Solving (6)

- Since  $G \neq 0$ ,

$$(a) F(0) = 0, \quad (b) F(L) = 0. \quad (8)$$

- For  $k = 0$ ,  $F(x) = ax + b$ , and from (8)  $a = b = 0$ .

- For positive  $k = \mu^2$ ,

$$F = A \cdot e^{\mu x} + B \cdot e^{-\mu x}$$

from (8),  $F \equiv 0$ .

- For negative  $k = -p^2$ ,

$$F'' + p^2 F = 0$$

$$F(x) = A \cos px + B \sin px$$

$$F(0) = A = 0, \quad F(L) = B \sin pL = 0$$

-  $\sin pL = 0$ .

$$pL = n\pi \quad p = \frac{n\pi}{L} \quad (n : \text{ integer}) \quad (9)$$

- Setting  $B = 1$ ,

$$F_n(x) = \sin \frac{n\pi}{L} x \quad (n = 1, 2, \dots). \quad (10)$$

#### Solving (7)

$$k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$$

$$\ddot{G} + \lambda_n^2 G = 0 \quad \text{where} \quad \lambda_n = \frac{cn\pi}{L}$$

$$G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t$$

$$u_n(x, t) = F_n(x)G_n(t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x \quad (n = 1, 2, \dots) \quad (11)$$

- The function in (11) are the eigenfunctions or characteristic functions.  $\lambda_n = cn\pi/L$  are the eigenvalues, or characteristic values. The set  $\{\lambda_1, \lambda_2, \dots\}$  is the spectrum.

### 15.1.3 Step III: Solution of the Entire Problem. Fourier Series

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x \quad (12)$$

*Satisfying Initial Condition (3) (given Initial Displacement)*

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x) \quad (13)$$

- By using the orthogonality of trigonometric functions

$$\begin{aligned}
B_n \int_0^L \sin^2 \frac{n\pi x}{L} dx &= \int_0^L f(x) \sin \frac{n\pi x}{L} dx \\
B_n \int_0^L \frac{1}{2} \left( 1 - \cos \frac{2n\pi x}{L} \right) dx &= \int_0^L f(x) \sin \frac{n\pi x}{L} dx \\
B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, \dots
\end{aligned} \tag{14}$$

*Satisfying Initial Condition (4) (given Initial Velocity)*

$$\begin{aligned}
\frac{\partial u}{\partial t} \Big|_{t=0} &= \left[ \sum_{n=1}^{\infty} (-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t) \sin \frac{n\pi x}{L} \right]_{t=0} \\
&= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L} = g(x) \\
B_n^* \lambda_n &= \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx
\end{aligned}$$

- Since  $\lambda_n = cn\pi/L$ ,

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \cdot \sin \frac{n\pi x}{L} dx \quad n = 1, 2, \dots \tag{15}$$

- Solution (12) is established!

- When  $g(x) = 0$ ,  $B_n^* = 0$ .

$$\begin{aligned}
u(x, t) &= \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi x}{L}, \quad \lambda_n = \frac{cn\pi}{L} \\
\cos \frac{cn\pi t}{L} \sin \frac{n\pi x}{L} &= \frac{1}{2} \left[ \sin \left\{ \frac{n\pi}{L}(x - ct) \right\} + \sin \left\{ \frac{n\pi}{L}(x + ct) \right\} \right] \\
u(x, t) &= \frac{1}{2} \sum_{n=1}^{\infty} B_n \sin \left\{ \frac{n\pi}{L}(x - ct) \right\} + \frac{1}{2} \sum_{n=1}^{\infty} B_n \sin \left\{ \frac{n\pi}{L}(x + ct) \right\}
\end{aligned} \tag{16}$$

These two series are those obtained by substituting  $x - ct$  and  $x + ct$ , respectively, for the variable  $x$  in the Fourier sine series (13) for  $f(x)$ .

$$u(x, t) = \frac{1}{2} [f^*(x - ct) + f^*(x + ct)] \tag{17}$$

where  $f^*$  is the odd periodic extension of  $f$  with the period  $2L$ . *Physical Interpretation of the Solution*

$f^*(x - ct)$ : a wave traveling to the right as  $t$  increases.

$f^*(x + ct)$ : a wave traveling to the left as  $t$  increases.

$u(x, t)$  is the superposition of these two waves.

**Example 1.** Vibrating string if the initial deflection is triangular

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L - x) & \text{if } \frac{L}{2} < x < L \end{cases}$$

- Initial velocity

$$u_t(x, 0) = 0$$

**Solution.**

$$g(x) \equiv 0 \implies B_n^* \equiv 0$$

- From Example 1. in sec. 11. 1

$$B_n = \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2}$$

$$\therefore u(x, t) = \frac{8k}{\pi^2} \left[ \frac{1}{1^2} \sin \frac{\pi x}{L} \cdot \cos \frac{\pi c t}{L} - \frac{1}{3^2} \sin \frac{3\pi x}{L} \cdot \cos \frac{3\pi c t}{L} \dots \right]$$

## 15.2 D'Alembert's Solution of the Wave Equation

- Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho}$$

-  $(x, t) \rightarrow (v, z)$  transformation

$$v = x + ct, \quad z = x - ct$$

-  $v_x = 1$  and  $z_x = 1$ .

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} = u_v \cdot (1) + u_z \cdot (1) = u_v + u_z \\ u_{xx} &= \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right)_v v_x + \left( \frac{\partial u}{\partial x} \right)_z z_x \\ &= (u_v + u_z)_v \cdot v_x + (u_v + u_z)_z \cdot z_x = u_{vv} + 2u_{vz} + u_{zz} \end{aligned}$$

-  $v_t = c$  and  $z_t = -c$ .

$$\begin{aligned} u_t &= \frac{\partial u}{\partial t} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial t} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial t} = c \frac{\partial u}{\partial v} - c \frac{\partial u}{\partial z} = c(u_v - u_z) \\ u_{tt} &= c(u_v - u_z)_v \cdot v_t + c(u_v - u_z)_z \cdot z_t \\ &= c^2(u_{vv} - u_{vz} - u_{vz} + u_{zz}) = c^2(u_{vv} - 2u_{vz} + u_{zz}) \end{aligned}$$

$$u_{tt} = c^2 u_{xx} \implies c^2(u_{vv} - 2u_{vz} + u_{zz}) = c^2(u_{vv} + 2u_{vz} + u_{zz})$$

$$u_{vz} = \frac{\partial^2 u}{\partial v \partial z} = 0 \tag{18}$$

- By integrating (18) with respect to  $z$ ,

$$\frac{\partial u}{\partial v} = h(v)$$

$$u = \int h(v)dv + \psi(z)$$

- D'Alembert's solution of the wave equation

$$u(x, t) = \phi(v) + \psi(z) = \phi(x + ct) + \psi(x - ct) \quad (19)$$

*D'Alembert's Solution Satisfying the Initial Conditions*

$$\begin{aligned} u(x, 0) &= f(x) \\ u_t(x, 0) &= g(x) \end{aligned}$$

- By differentiating (19),

$$u_t(x, t) = c\phi'(x + ct) - c\psi'(x - ct)$$

$$u(x, 0) = \phi(x) + \psi(x) = f(x) \quad (20)$$

$$u_t(x, 0) = c\phi'(x) - c\psi'(x) = g(x) \quad (21)$$

- Dividing (20) by  $c$  and integrating with respect to  $x$ ,

$$\phi(x) - \psi(x) = k(x_0) + \frac{1}{c} \int_{x_0}^x g(s)ds, \quad k(x_0) = \phi(x_0) - \psi(x_0) \quad (22)$$

- [(20)+(22)]/2:

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c} \int_{x_0}^x g(s)ds + \frac{1}{2}k(x_0) \quad (23)$$

- [(20)-(22)]/2:

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c} \int_{x_0}^x g(s)ds - \frac{1}{2}k(x_0) \quad (24)$$

- (23)+(24):

$$u(x, t) = \phi(x + ct) + \psi(x - ct)$$

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)ds \quad (25)$$

- If  $g(x) = 0$ ,

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] \quad (26)$$