Engineering Mathematics I
- Chapter 4. Systems of ODEs

민기복

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Ch.4 Systems of ODEs. Phase Plane. Qualitative Methods

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- Basics of Matrices and Vectors
- Systems of ODEs as Models
- Basic Theory of Systems of ODEs
- Constant-Coefficient Systems. Phase Plane Method
- Criteria for Critical Points. Stability
- Qualitative Methods for Nonlinear Systems
- Nonhomogeneous Linear Systems of ODEs

Basics of Matrices and Vectors

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- Systems of Differential Equations
  - more than one dependent variable and more than one equation

$$y_1' = a_{11}y_1 + a_{12}y_2,$$
$$y_2' = a_{21}y_1 + a_{22}y_2,$$

$$\cdots$$

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n, \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n, \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n, \end{aligned}$$

- Differentiation
  - The derivative of a matrix with variable entries is obtained by differentiating each entry.

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \quad \Rightarrow \quad \mathbf{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix}$$

Basics of Matrices and Vectors

Eivenvalues (고유치) and Eivenvectors (고유벡터)

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- Let $\mathbf{A} = [a_{jk}]$ be an *n x n* matrix. Consider the equation $\mathbf{Ax} = \lambda \mathbf{x}$ where λ is a scalar and **x** is a vector to be determined.
  - A scalar λ such that the equation $\mathbf{Ax} = \lambda \mathbf{x}$ holds for some vector $\mathbf{x} \neq \mathbf{0}$ is called an eigenvalue of **A**,
  - And this vector is called an eigenvector of **A** corresponding to this eigenvalue λ.

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Basics of Matrices and Vectors

Eivenvalues (고유치) and Eivenvectors (고유벡터)

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$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \longrightarrow \mathbf{A}\mathbf{x} - \lambda\mathbf{I}\mathbf{x} = 0 \longrightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

- *n* linear algebraic equations in the *n* unknowns $x_1, \cdots, x_n$ (the components of **x**).
- The determinant of the coefficient matrix $\mathbf{A} - \lambda\mathbf{I}$ must be zero in order to have a solution x ≠ 0.

- Characteristic Equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

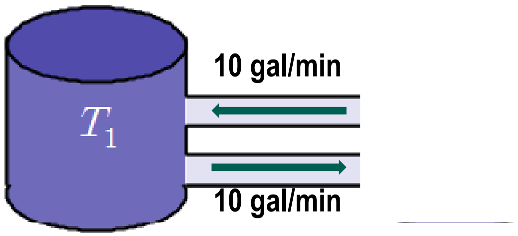
  - Determine λ_1 and λ_2
  - Determination of eigenvector corresponding to λ_1 and λ_2

- Example 1. Find the eigenvalues and eigenvectors

$$\mathbf{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}$$

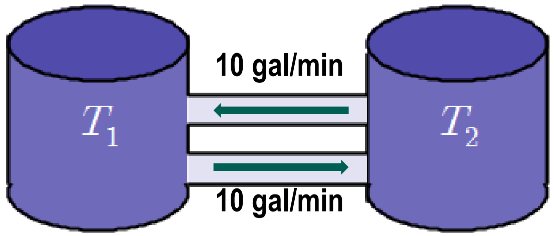
- If **x** is an eigenvector, so is k**x**

Systems of ODEs as Models



Single ODE

VS.



Systems of ODEs

Systems of ODEs as Models
Example 3. Mixing problem (Sec 1.3)

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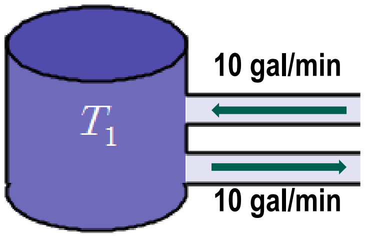
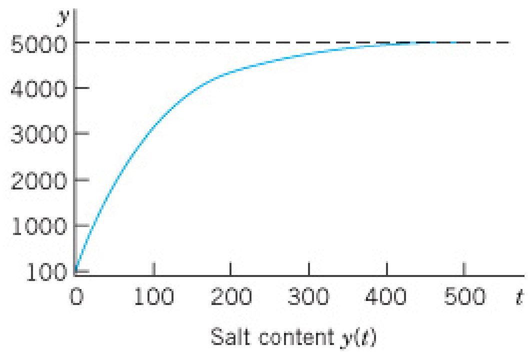
- Initial Condition: 1000 gal of water, 100 lb salt, initially brine runs in 10 gal/min, 5 lb/gal, stirring all the time, brine runs out at 10 gal/min
- Amount of salt at t?



?



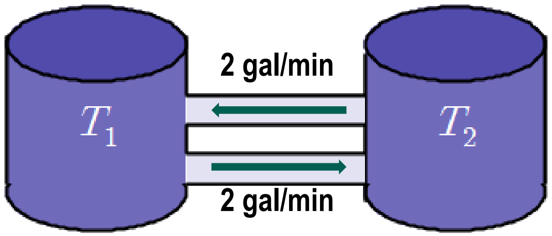
Systems of ODEs as Models

– Tank T_1 and T_2 contain initially 100 gal of water each.

– In T_1 the water is pure, whereas 150 lb of fertilizer are dissolved in T_2.

– By circulating liquid at a rate of 2 gal/min and stirring the amounts of fertilizer $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time *t*.

– Model Set Up



$$y_1' = \text{Inflow / min} - \text{Outflow / min} = \frac{2}{100}y_2 - \frac{2}{100}y_1 \quad (\text{Tank } T_1) \quad \Rightarrow y_1' = -0.02y_1 + 0.02y_2$$

$$y_2' = \text{Inflow / min} - \text{Outflow / min} = \frac{2}{100}y_1 - \frac{2}{100}y_2 \quad (\text{Tank } T_2) \quad \Rightarrow y_2' = 0.02y_1 - 0.02y_2$$

$$\therefore \quad \mathbf{y}' = \mathbf{A}\mathbf{y}, \qquad \mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

Systems of ODEs as Models

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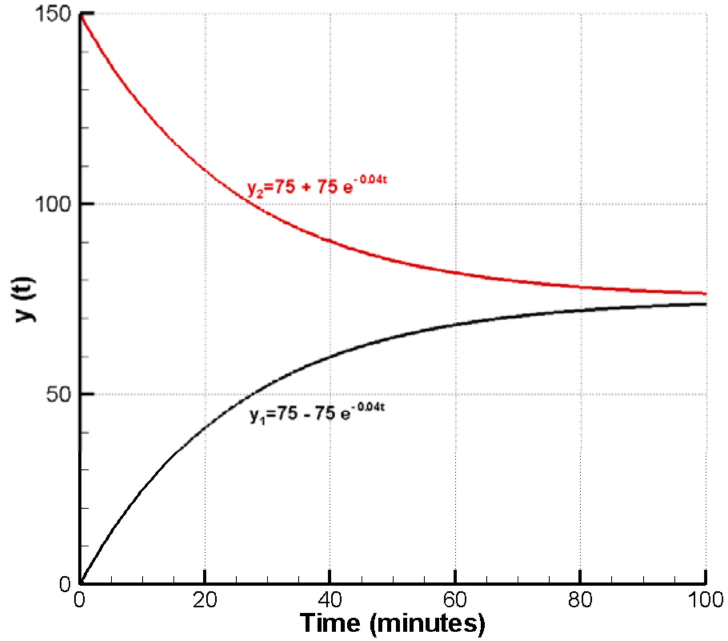

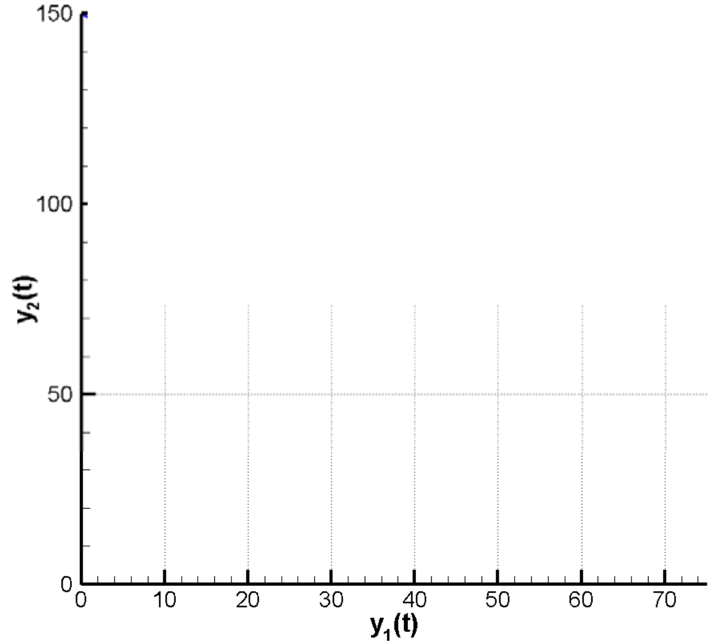

y_1-y_2 plane - phase plane (상평면)

Trajectory (궤적) :

- 상평면에서의 곡선
- 전체 해집합의 일반적인 양상을 잘 표현함.

Phase portrait (상투영): trajectories in phase plane

Systems of ODEs as Models

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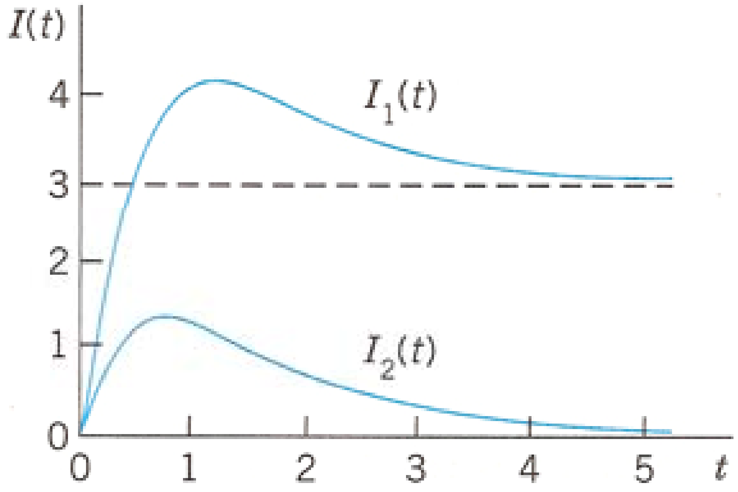
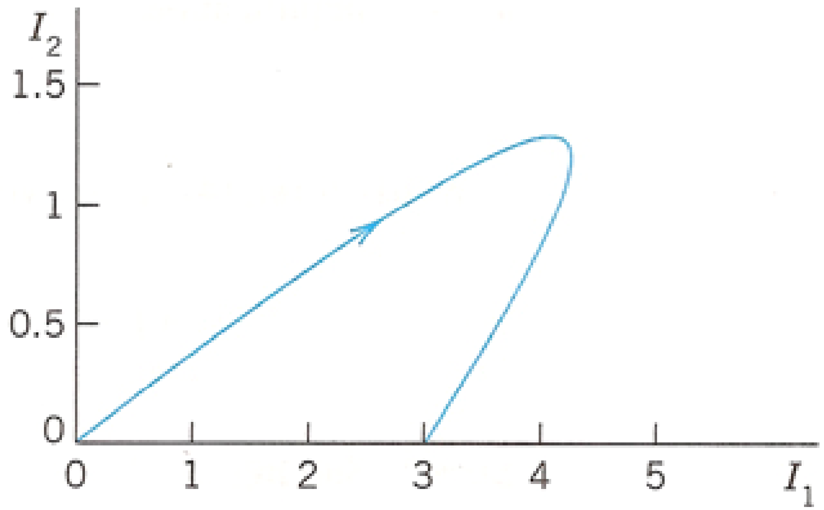
- Example 2. Electrical Network





y_1-y_2 plane - phase plane (상평면)
Trajectory (궤적) :
- 상평면에서의 곡선
- 전체 해집합의 일반적인 양상을 잘 표현함.

Phase portrait (상투영): trajectories in phase plane

Systems of ODEs as Models
Conversion of an nth-order ODE to a system

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- 2nd order to a system of ODE – Example 3. Mass on a Spring

$$my''+cy'+ky=0$$

$$\xrightarrow{y_1 = y,\ y_2 = y'} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{k}{m}y_1 - \frac{c}{m}y_2 \end{aligned} \quad \xrightarrow{\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \quad \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m}-\lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \quad \Rightarrow$$

Calculation same as before!

$$Ex)\ y''+2y'+0.75y=0$$

Systems of ODEs as Models

Conversion of an nth-order ODE to a system

Conversion of an ODE

An nth-order ODE

(8) $$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)})$$

can be converted to a system of n first-order ODEs by setting

(9) $$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \cdots, y_n = y^{(n-1)}.$$

This system is of the form

(10) $$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, \cdots, y_n). \end{aligned}$$

– solve single ODE by methods for systems

– includes higher order ODEs into …first order ODE

Basic Theory of Systems of ODEs
Concepts (analogy to single ODE)

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- First-Order Systems

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

$$\begin{aligned} y_1' &= f_1(t, y_1, \cdots, y_n) \\ y_2' &= f_2(t, y_1, \cdots, y_n) \\ &\vdots \\ y_n' &= f_n(t, y_1, \cdots, y_n) \end{aligned} \quad \longrightarrow \quad \mathbf{y}' = \mathbf{f}(t, \mathbf{y})$$

- Solution on some interval a<t<b
  - A set of n differentiable functions $y_1 = h_1(t), \quad \cdots, \quad y_n = h_n(t)$

- Initial Condition: $y_1(t_0) = K_1, \quad y_2(t_0) = K_2, \quad \cdots, \quad y_n(t_0) = K_n$

Basic Theory of Systems of ODEs
Existence and Uniqueness

- Theorem 1. Existence and Uniqueness Theorem

  – Let $f_1, \ \cdots, \ f_n$ be continuous functions having continuous partial derivatives $\partial f_1/\partial y_1, \ \cdots, \ \partial f_1/\partial y_n, \ \cdots, \ \partial f_{\mathbf{n}}/\partial y_n$ in some domain *R* of $ty_1y_2\cdots y_n$ -space containing the point $(t_0, \ K_1, \ \cdots, K_n)$. Then the first-order system has a solution on some interval $t_0 - \alpha < t < t_0 + \alpha$ satisfying the initial condition , and this solution is unique.

Basic Theory of Systems of ODEs
Homogeneous vs. Nonhomogeneous

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- Linear Systems

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \ \mathbf{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_2 \end{bmatrix}$$

$$\begin{gathered} y_1' = a_{11}(t)\,y_1 + \cdots + a_{1n}(t)\,y_n + g_1(t) \\ \vdots \\ y_n' = a_{n1}(t)\,y_1 + \cdots + a_{nn}(t)\,y_n + g_n(t) \end{gathered} \longrightarrow \mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$$

  - Homogeneous: $\mathbf{y}' = \mathbf{A}\mathbf{y}$
  - Nonhomogeneous: $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}, \quad \mathbf{g} \neq \mathbf{0}$

Basic Theory of Systems of ODEs
Concepts (analogy to single ODE)

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- Theorem 2 (special case of Theorem 1)

Existence and Uniqueness in the Linear Case

Let the a_{jk}'s and g_j's in (3) be continuous functions of t on an open interval $\alpha < t < \beta$ containing the point $t = t_0$. Then (3) has a solution $\mathbf{y}(t)$ on this interval satisfying (2), and this solution is unique.

- Theorem 3

Superposition Principle or Linearity Principle

*If $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ are solutions of the **homogeneous linear** system (4) on some interval, so is any linear combination $\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}$.*

$$\mathbf{y}' = \left[c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}\right]' = c_1\mathbf{y}^{(1)\prime} + c_2\mathbf{y}^{(2)\prime} = c_1\mathbf{A}\mathbf{y}^{(1)} + c_2\mathbf{A}\mathbf{y}^{(2)}$$

$$= \mathbf{A}\left(c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}\right) = \mathbf{A}\mathbf{y}$$

Basic Theory of Systems of ODEs
General solution

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- Basis, General solution, Wronskian

미분 아님!!

  - Basis: A linearly independent set of *n* solutions $\mathbf{y}^{(1)},\ \cdots,\ \mathbf{y}^{(\mathbf{n})}$ of the homogeneous system on that interval
  - General Solution: A corresponding linear combination

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + \cdots + c_n\,\mathbf{y}^{(\mathbf{n})} \quad (c_1,\ \cdots,\ c_n \text{ arbitrary})$$

  - Fundamental Matrix : An *n* x *n* matrix whose columns are *n* solutions $\mathbf{y}^{(1)},\ \cdots,\ \mathbf{y}^{(\mathbf{n})}$ $\quad \mathbf{Y} = \left[\mathbf{y}^{(1)},\ \cdots,\ \mathbf{y}^{(\mathbf{n})}\right]$ ⟶ $\mathbf{y} = \mathbf{Yc}$
  - Wronskian of $\mathbf{y}^{(1)},\ \cdots,\ \mathbf{y}^{(\mathbf{n})}$: The determinant of **Y**

$$W\left(\mathbf{y}^{(1)},\ \cdots,\ \mathbf{y}^{(\mathbf{n})}\right) = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ y_2^{(1)} & y_2^{(2)} & \cdots & y_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{vmatrix}$$

ex)

$$\mathbf{y} = Yc = \begin{pmatrix} y_1^{(1)} & y_1^{(2)} \\ y_2^{(1)} & y_2^{(2)} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2e^{-0.5t} & e^{-1.5t} \\ e^{-0.5t} & -1.5e^{-1.5t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-0.5t} + c_2\begin{pmatrix} 1 \\ -1.5 \end{pmatrix}e^{-1.5t}$$

Constant Coefficient Systems.

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

- Homogeneous linear system with constant coefficients
  - Where *n* x *n* matrix $\mathbf{A} = [a_{jk}]$ has entries not depending on *t*
  - Try $\mathbf{y} = \mathbf{x}e^{\lambda t}$

$$\Rightarrow \quad \mathbf{y}' = \lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{x}e^{\lambda t} \quad \Rightarrow \quad \mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

- Theorem 1. General Solution
  - The constant matrix **A** in the homogeneous linear system has a linearly independent set of *n* eigenvectors, then the corresponding solutions $\mathbf{y}^{(1)}, \cdots, \mathbf{y}^{(\mathbf{n})}$ form a basis of solutions, and the corresponding general solution is $\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + \cdots + c_n \mathbf{x}^{(n)} e^{\lambda_n t}$

Constant Coefficient Systems.

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- Example 1. Improper node (비고유마디점)

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\mathbf{y}$$

$$y_1' = -3y_1 + y_2$$
$$y_2' = y_1 - 3y_2$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-4t}$$

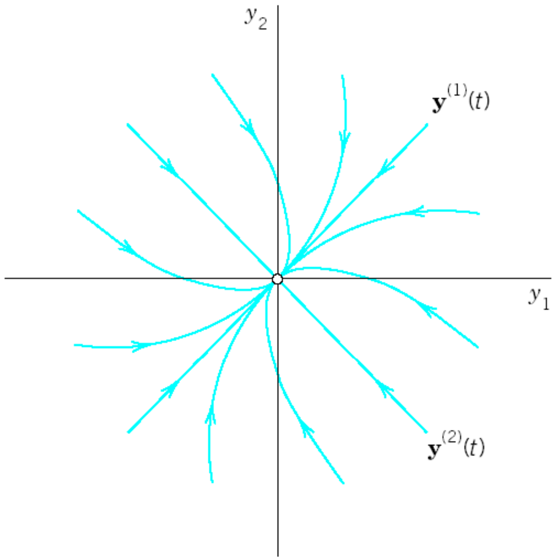

Fig. 81. Trajectories of the system (8) (Improper node)

Constant Coefficient Systems.

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- Example 2. Proper node (고유마디점)

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{y} \qquad \begin{aligned} y_1' &= y_1 \\ y_2' &= y_2 \end{aligned}$$

$$\mathbf{y} = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}e^t + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}e^t \quad \Rightarrow \quad \begin{aligned} y_1 &= c_1 e^t \\ y_2 &= c_2 e^t \end{aligned}$$

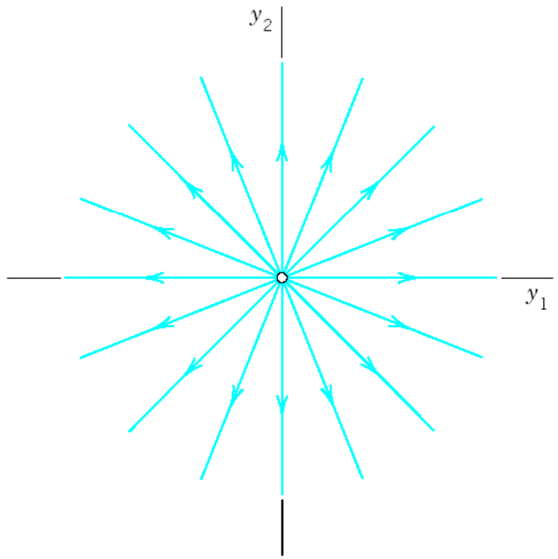

Fig. 82. Trajectories of the system (10) (Proper node)

Constant Coefficient Systems.

- Example 3. Saddle point (안장점)

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{y} \qquad \begin{aligned} y_1' &= y_1 \\ y_2' &= -y_2 \end{aligned}$$

$$\mathbf{y} = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{-t} \quad \Rightarrow \quad \begin{aligned} y_1 &= c_1 e^{t} \\ y_2 &= c_2 e^{-t} \end{aligned}$$

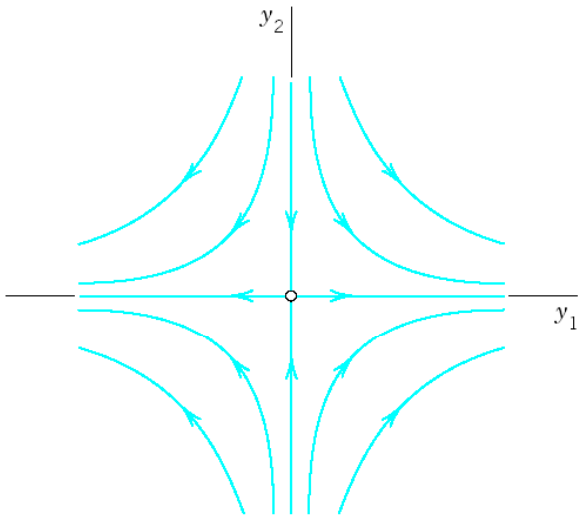

Fig. 83. Trajectories of the system (11) (Saddle point)

Constant Coefficient Systems.

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- Example 4. Center (중심)

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\mathbf{y} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= -4y_1 \end{aligned}$$

$$\mathbf{y} = c_1\begin{bmatrix} 1 \\ 2i \end{bmatrix}e^{2it} + c_2\begin{bmatrix} 1 \\ -2i \end{bmatrix}e^{-2it} \quad \Rightarrow \quad \begin{aligned} y_1 &= c_1e^{2it} + c_2e^{-2it} \\ y_2 &= 2ic_1e^{2it} - 2ic_2e^{-2it} \end{aligned}$$

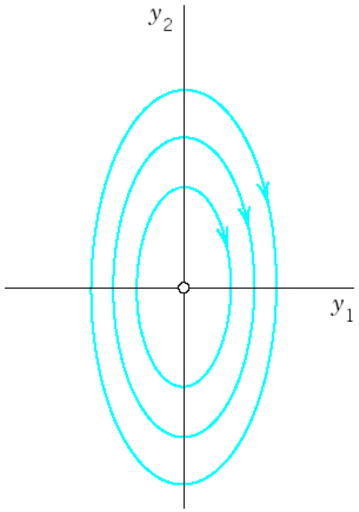

Fig. 84. Trajectories of the system (12) (Center)

Constant Coefficient Systems.

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- Example 5. Spiral Point (나선점)

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{y}$$

$$y_1' = -y_1 + y_2$$
$$y_2' = -y_1 - y_2$$

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}$$

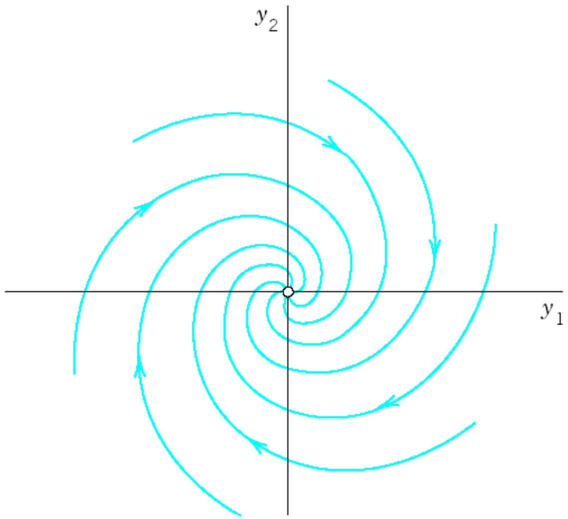

Fig. 85. Trajectories of the system (13) (Spiral point)

Constant Coefficient Systems.

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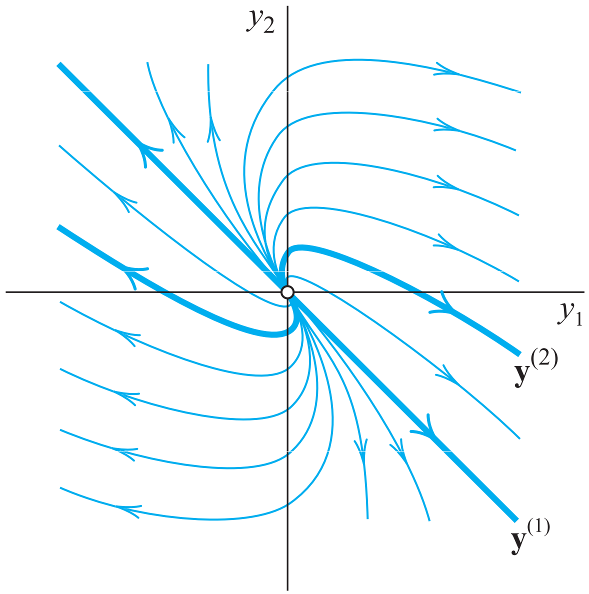
- Example 6. Degenerate Node (퇴화마디점):
  - 고유벡터가 기저를 형성하지 않는 경우: 행렬이 대칭이거나 반대칭(skew-symmetric)인 경우 퇴화마디점은 생길 수가 없음.

$$\mathbf{y}' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{y}$$

$$\mathbf{y}^{(1)} = \mathbf{x} e^{\lambda t}$$

$$\mathbf{y}^{(2)} = \mathbf{x} t e^{\lambda t} + \mathbf{u} e^{\lambda t}$$

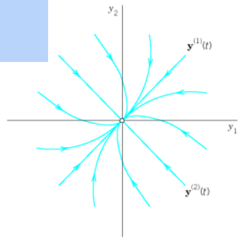
$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t}$$



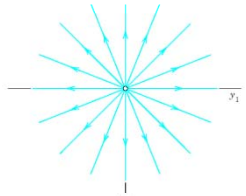
Constant Coefficient Systems.
Summary

Δ is discriminant (판별식) of determinant of (A-λI)

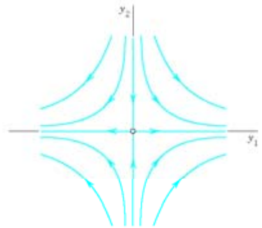
- Improper node (비고유마디점): $\Delta > 0,\ \lambda_1\lambda_2 > 0$
  - Real and distinct eigenvalues of the same sign



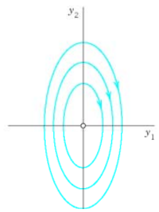
- Proper node (고유마디점): $\Delta = 0,\ \lambda_1 = \lambda_2$ *
  - Real and equal eigenvalues



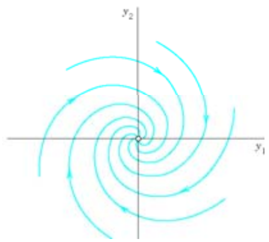
- Saddle point (안장점): $\Delta > 0,\ \lambda_1\lambda_2 < 0$
  - Real eigenvalues of opposite sign



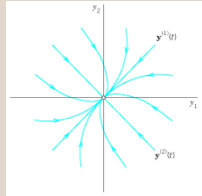
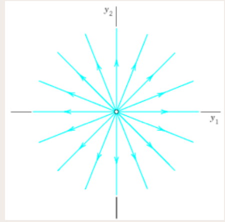
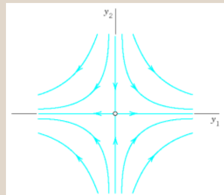
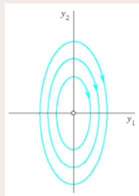
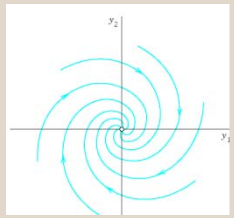
- Center (중심): $\Delta < 0,\ \textit{pure imaginary}$
  - Pure imaginary eigenvalues



- Spiral point (나선점): $\Delta < 0,\ \textit{complex}$
  - Complex conjugates eigenvalues with nonzero real part



*** : when two linearly independent eigenvectors exist. Otherwise, degenerate node**

Name	Δ	Eigenvalue	Trajectories
Improper node (비고유마디점)	$\Delta > 0$	$\lambda_1 \lambda_2 > 0$	
Proper node (고유마디점)	$\Delta = 0$	$\lambda_1 = \lambda_2$	
Saddle Point (안장점)	$\Delta > 0$	$\lambda_1 \lambda_2 < 0$	
Center (중심)	$\Delta < 0$	Pure imaginary e.g., $\pm i$	
Spiral Point (나선점)		Complex number e.g., $1 \pm i$	

Criteria for Critical Points. Stability

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- Critical Points of the system

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \quad \Rightarrow \quad \frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$$

– dy_2/dy_1 A unique tangent direction of the trajectory passing through $P:(y_1, y_2)$, except for the point $P = P_0 : (0,0)$

– Critical Points: The point at which dy_2/dy_1 becomes undetermined, 0/0

Criteria for Critical Points. Stability

- Five Types of Critical Points
  - Depending on the geometric shape of the trajectories near them
    - Improper Node
    - Proper Node
    - Saddle Point
    - Center
    - Spiral Point

Criteria for Critical Points. Stability

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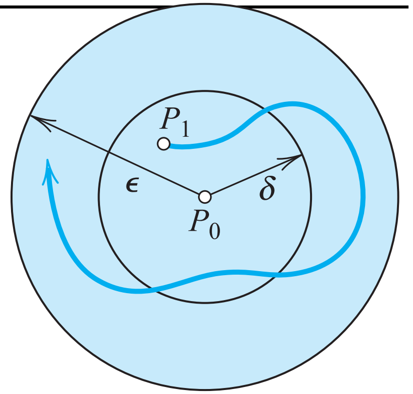
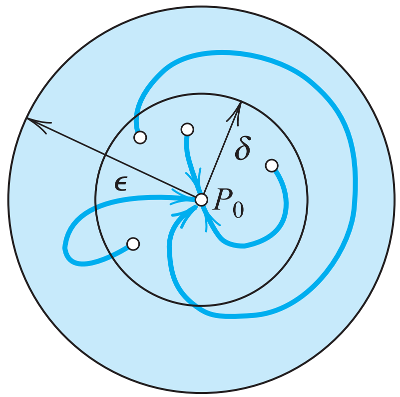
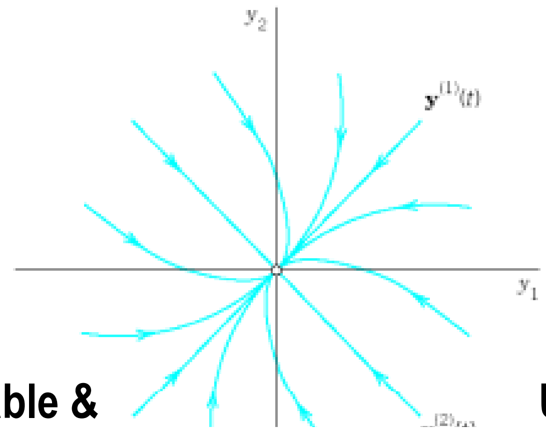
- **Stable Critical point (안정적 임계점):**
  - 어떤 순간 $t = t_0$ 에서 임계점에 아주 가깝게 접근한 모든 궤적이 이후의 시간에서도 임계점에 아주 가까이 접근한 상태로 남아 있는 경우.



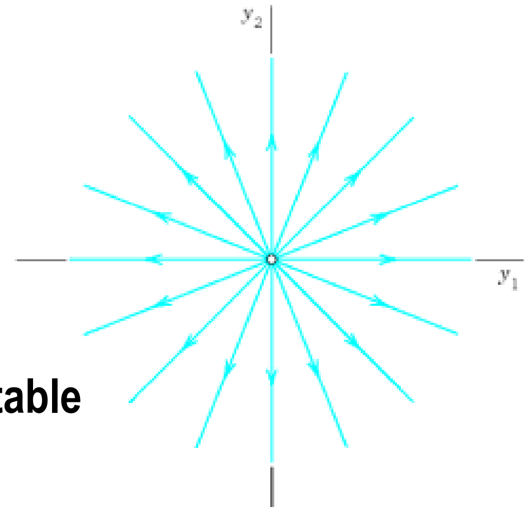
- **Unstable Critical point (불안정적 임계점):**
  - 안정적이 아닌 임계점
- **Stable and Attractive Critical point (안정적 흡인 임계점):**
  - 안정적 임계점이고, 임계점 근처 원판 내부의 한 점을 지나는 모든 궤적이 $t \to \infty$ 를 취할 때 임계점에 가까이 접근하는 경우



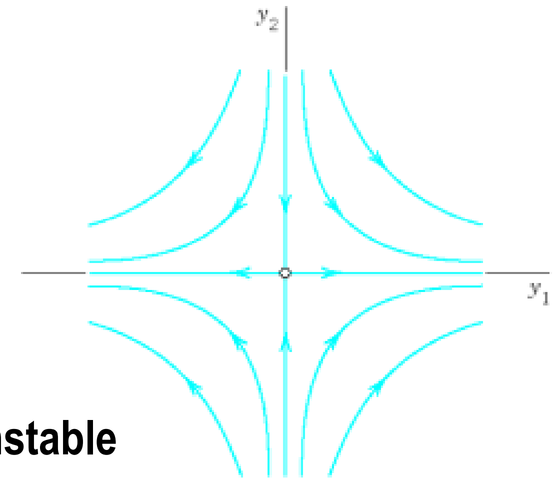
Criteria for Critical Points. Stability



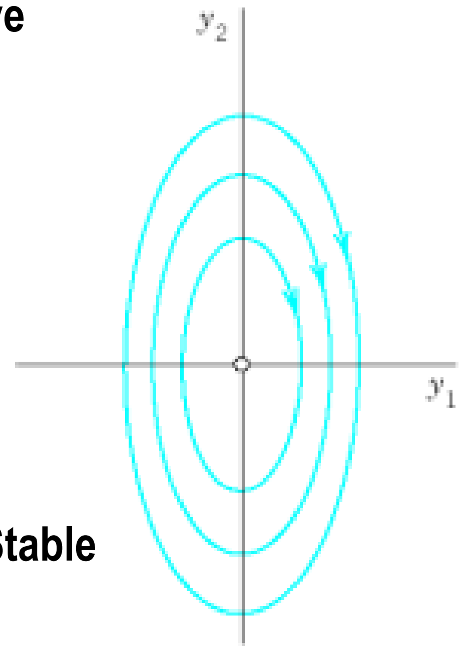
Stable & attractive



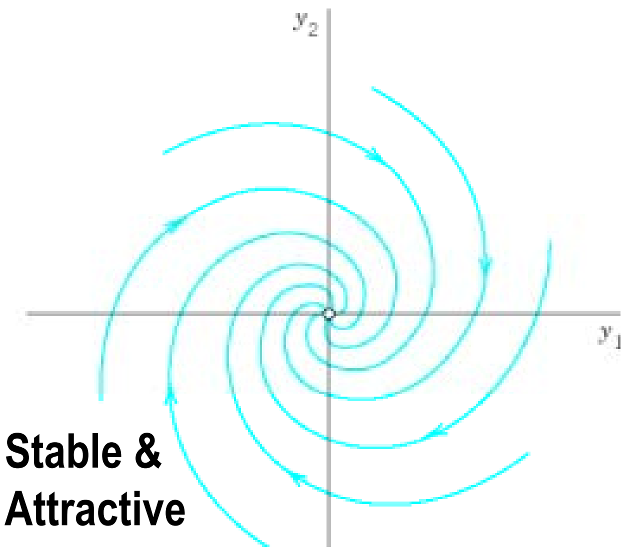
Unstable



Unstable



Stable



Stable & Attractive

Qualitative Methods for Nonlinear Systems

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- Qualitative Method
  - Method of obtaining qualitative information on solutions without actually solving a system.
  - particularly valuable for systems whose solution by analytic methods is difficult or impossible.

Nonlinear systems

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}), \text{ thus } \begin{aligned} y_1' &= f_1(y_1, y_2) \\ y_2' &= f_2(y_1, y_2) \end{aligned}$$

linearization →

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h}(\mathbf{y}), \text{ thus } \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + h_1(y_1, y_2) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + h_2(y_1, y_2) \end{aligned}$$

Qualitative Methods for Nonlinear Systems

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- Theorem 1. Linearization
  - If f_1 and f_2 are continuous and have continuous partial derivatives in a neighborhood of the critical point (0,0), and if $\det \mathbf{A} \neq 0$, then the kind and stability of the critical point of nonlinear systems are the same as those of the linearized system

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \text{thus} \quad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned}$$

  - Exceptions occur if A has equal or pure imaginary eigenvalues; then the nonlinear system may have the same kind of critical points as linearized system or a spiral point.

Qualitative Methods for Nonlinear Systems

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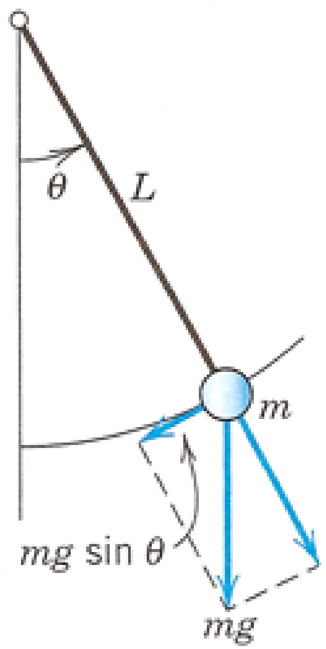
- Example 1. Free undamped pendulum
  - Determine the locations and types of critical points
  - Step 1: setting up the mathematical model
    - θ : the angular displacement, measured counterclockwise from the equilibrium position
    - The weight of the bob : *mg*
    - A restoring force tangent to the curve of motion of the bob : *mg* sin θ
    - By Newton's second law, at each instant this force is balanced by the force of acceleration *mL*θ``



$$\therefore\ mL\theta'' + mg\sin\theta = 0 \quad \rightarrow \quad \theta'' + k\sin\theta = 0 \quad \left(k = \frac{g}{L}\right)$$

Qualitative Methods for Nonlinear Systems

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- Step 2: Critical Points $(0,0),\ (\pm 2\pi,0),\ (\pm 4\pi,0),\ \cdots$ Linearization

$$\theta'' + k\sin\theta = 0 \xrightarrow{\text{set } y_1=\theta,\ y_2=\theta'} \begin{array}{l} y_1' = y_2 \\ y_2' = -k\sin y_1 \end{array}$$

$$y_2 = 0,\quad \sin y_1 = 0 \quad \rightarrow \quad \text{infinitely many critical points : } (n\pi\ ,\ 0),\ n=0,\ \pm 1,\ \pm 2,\ \cdots$$

$$\sin y_1 = y_1 - \frac{1}{6}y_1^3 + - \cdots \approx y_1 \longrightarrow \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}\mathbf{y},\quad \text{thus} \quad \begin{array}{l} y_1' = y_2 \\ y_2' = -ky_1 \end{array}$$

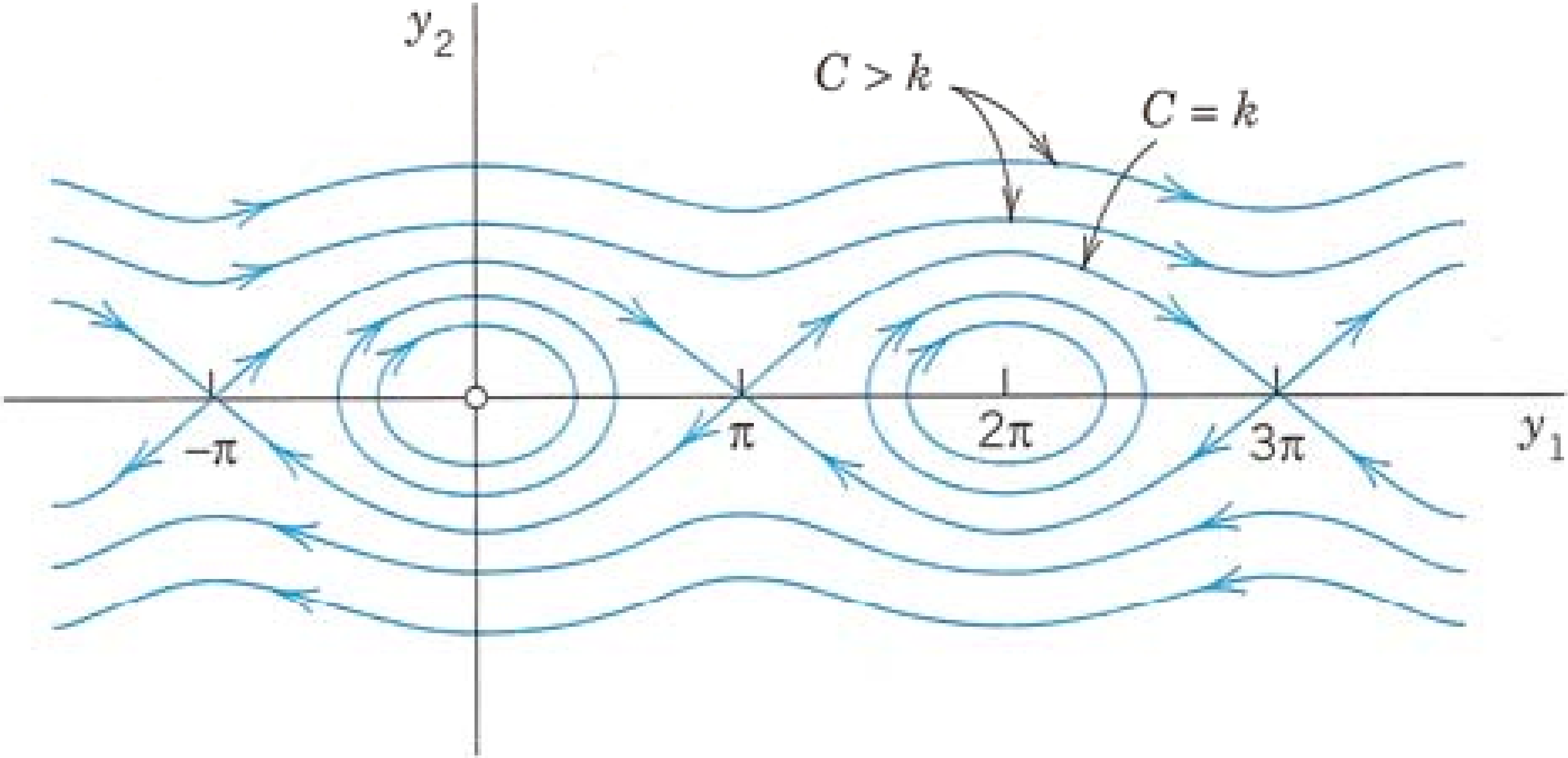
- Step 3: Critical Points $(\pm\pi,0),\ (\pm 3\pi,0),\ (\pm 5\pi,0),\ \cdots$ Linearization.

Consider the critical point (π , 0)

$$\theta'' + k\sin\theta = 0 \xrightarrow[\sin\theta = \sin(y_1+\pi) = -\sin y_1 = -y_1 + \frac{1}{6}y_1^3 - + \cdots \approx -y_1]{\text{set } y_1 = \theta - \pi,\ y_2 = (\theta-\pi)' = \theta'} \quad \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix}\mathbf{y}$$

critical points are all saddle points.

Qualitative Methods for Nonlinear Systems



Nonhomogeneous Linear Systems of ODEs

- Nonhomogeneous of Linear Systems: $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}, \quad \mathbf{g} \neq \mathbf{0}$
  - Assume **g**(*t*) and the entries of the n x n matrix **A**(*t*) to be continuous on some interval *J* of the *t*-axis.
  - General solution : $\mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)}$
    - $\mathbf{y}^{(h)}$: A general solution of the homogeneous system $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$ on *J*
    - $\mathbf{y}^{(p)}$: A particular solution(containing no arbitrary constants) of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ on *J*
- Methods for obtaining particular solutions
  - Method of Undetermined Coefficients
  - Method of the Variation of Parameter

Nonhomogeneous Linear Systems of ODEs

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- Method of undetermined coefficients:
  - Components of **g** :
  - (1) constants
  - (2) positive integer powers of *t*
  - (3) exponential functions
  - (4) cosines and sines.

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

Nonhomogeneous Linear Systems of ODEs

- Example 1. Method of undetermined coefficients. Modification rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

  - A general solution of the homogeneous system : $\mathbf{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$
  - Apply the Modification Rule by setting $\mathbf{y}^{(p)} = \mathbf{u} t e^{-2t} + \mathbf{v} e^{-2t}$
  - Equating the te^{-2t}-terms on both sides: $-2\mathbf{u} = \mathbf{A}\mathbf{u} \quad \Rightarrow \quad \mathbf{u} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (\text{with any } a \neq 0)$
  - Equating the other terms:

$$\mathbf{u} - 2\mathbf{v} = \mathbf{A}\mathbf{v} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} \quad \Rightarrow \quad a = -2, \quad \mathbf{v} = \begin{bmatrix} k \\ k+4 \end{bmatrix} \quad (\text{choose } k = 0)$$

  - General solution :

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}$$

Nonhomogeneous Linear Systems of ODEs

- Example 1. solution by the method of variation of parameters

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

  - General solution of the homogeneous system :

$$\mathbf{y}^{(h)} = c_1 \mathbf{y}^{(1)} + \cdots + c_n \mathbf{y}^{(n)} = \mathbf{Y}(t)\mathbf{c} \qquad (\therefore \quad \mathbf{Y}' = \mathbf{A}\mathbf{Y})$$

  - Particular solution : $\mathbf{y}^{(p)} = \mathbf{Y}(t)\mathbf{u}(t)$

$$\left(\mathbf{y}^{(p)}\right)' = \mathbf{Y}'\mathbf{u} + \mathbf{Y}\mathbf{u}' \quad \Rightarrow \quad \mathbf{Y}'\mathbf{u} + \mathbf{Y}\mathbf{u}' = \mathbf{A}\mathbf{Y}\mathbf{u} + \mathbf{g}$$

$$\Rightarrow \quad \mathbf{Y}\mathbf{u}' = \mathbf{g} \quad \Rightarrow \quad \mathbf{u}' = \mathbf{Y}^{-1}\mathbf{g} \quad \Rightarrow \quad \mathbf{u} = \int_{t_0}^{t} \mathbf{Y}^{-1}(\tilde{t})\mathbf{g}(\tilde{t})\,d\tilde{t}$$

Nonhomogeneous Linear Systems of ODEs

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- Example 1. solution by the method of variation of parameters (cont.)

$$\mathbf{Y}^{-1} = \frac{1}{-2e^{-6t}}\begin{bmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix}$$

$$\mathbf{u}' = \mathbf{Y}^{-1}\mathbf{g} = \frac{1}{2}\begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix}\begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -4 \\ -8e^{2t} \end{bmatrix} = \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix}$$

$$\mathbf{u} = \int_0^t \begin{bmatrix} -2 \\ -4e^{2\tilde{t}} \end{bmatrix} d\tilde{t} = \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$

$$\mathbf{y}^{(p)} = \mathbf{Yu} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix}\begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix} = \begin{bmatrix} -2te^{-2t} - 2e^{-2t} + 2e^{-4t} \\ -2te^{-2t} + 2e^{-2t} - 2e^{-4t} \end{bmatrix} = \begin{bmatrix} -2t - 2 \\ -2t + 2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-4t}$$