3. Smith Chart and Impedance Matching

A. Construction and Applications of the Smith Chart

Smith Chart :

A graphical chart of normalized impedance (or admittance) in the Γ plane for analyzing and designing both lossless and lossy transmission-line circuits (P.H. Smith, 1939)

1) Parametric equations for the chart construction

For a lossless transmission line,

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = |\Gamma| e^{j\theta_\Gamma} = \Gamma_r + j\Gamma_i = \frac{z_L - 1}{z_L + 1} \qquad \text{(8-88, 97) (8-99)}$$

where the normalized impedance w.r.t. $Z_o = R_o = \sqrt{L/C}$ is

$$z_L \equiv \frac{Z_L}{Z_o} = \frac{R_L}{R_o} + j\frac{X_L}{R_o} = r + jx \qquad \text{(8-98)}$$

Inverse relation of (8-99) with (8-97, 98) :

$$z_L = \frac{1+\Gamma}{1-\Gamma} = \frac{1+|\Gamma| e^{j\theta_\Gamma}}{1-|\Gamma| e^{j\theta_\Gamma}} = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i} = r + jx \qquad \text{(8-100)(8-101)}$$

(8-101)$\times \dfrac{(1-\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) + j\Gamma_i}$:

$$r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} \qquad \text{(8-102)}$$

$$x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} \qquad \text{(8-103)}$$

Rearrangements of (8-102, 103) yield the parametric equations of constant-r and constant-x circles in the $\Gamma_r - \Gamma_i$ plane, respectively

$$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2 \qquad \text{(8-104)}$$

: Eqn. of const.-r circle of a radius $\frac{1}{1+r}$ centered at $\left(\frac{r}{1+r}, 0\right)$

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 \qquad \text{(8-105)}$$

: Eqn. of const.-x circle of a radius $\frac{1}{x}$ centered at $\left(1, \frac{1}{x}\right)$

2) Construction of the Smith Chart

a) Smith Chart with $\Gamma_r - \Gamma_i$ rectangular coordinates

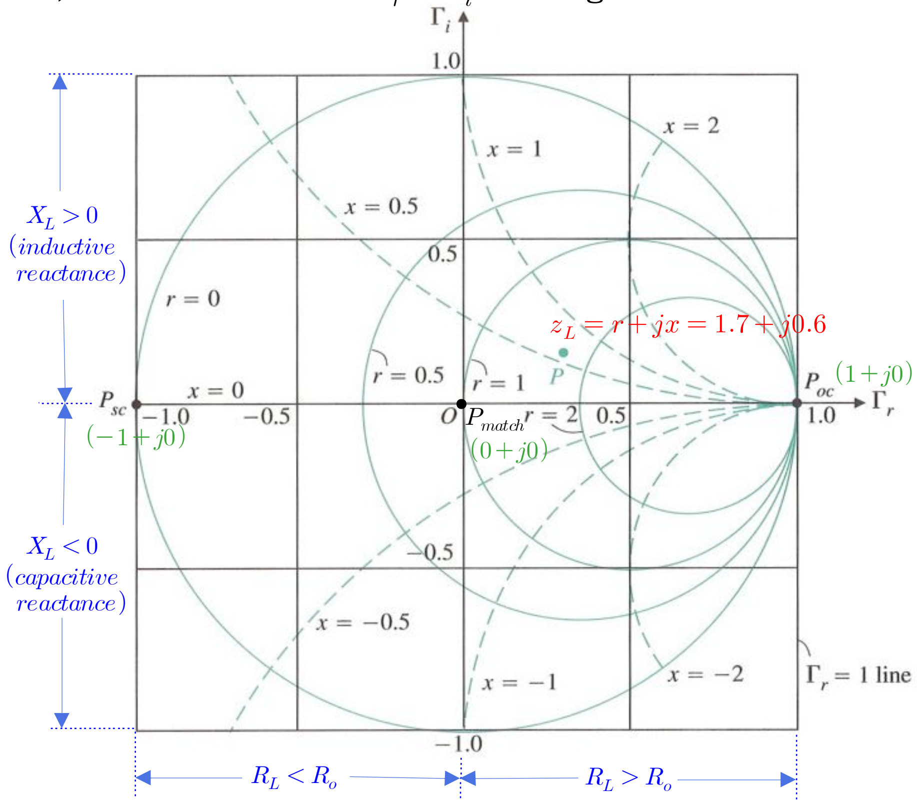


FIGURE 8−6

b) Smith Chart with $|\Gamma| - \theta_\Gamma$ polar coordinates

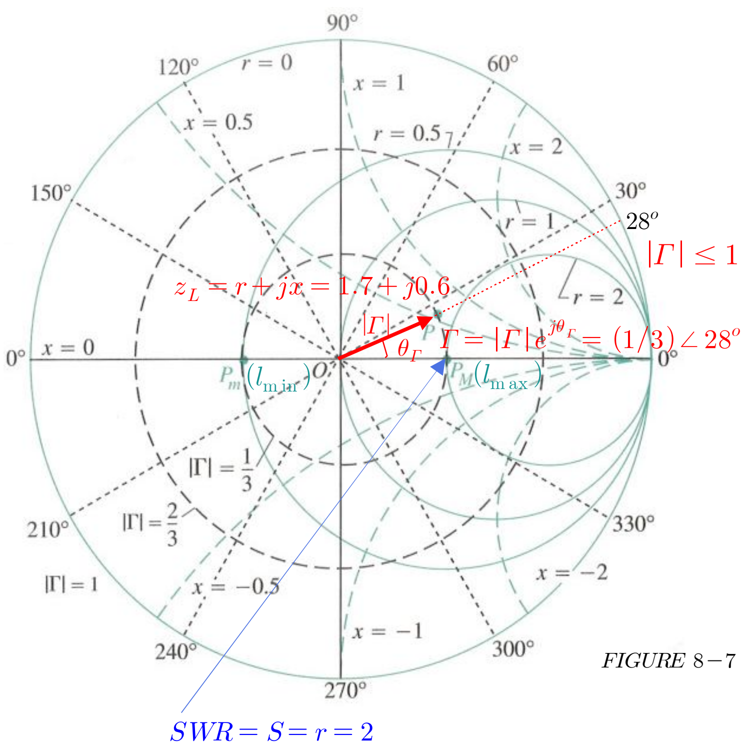


FIGURE 8−7

c) Commercially available Smith Chart

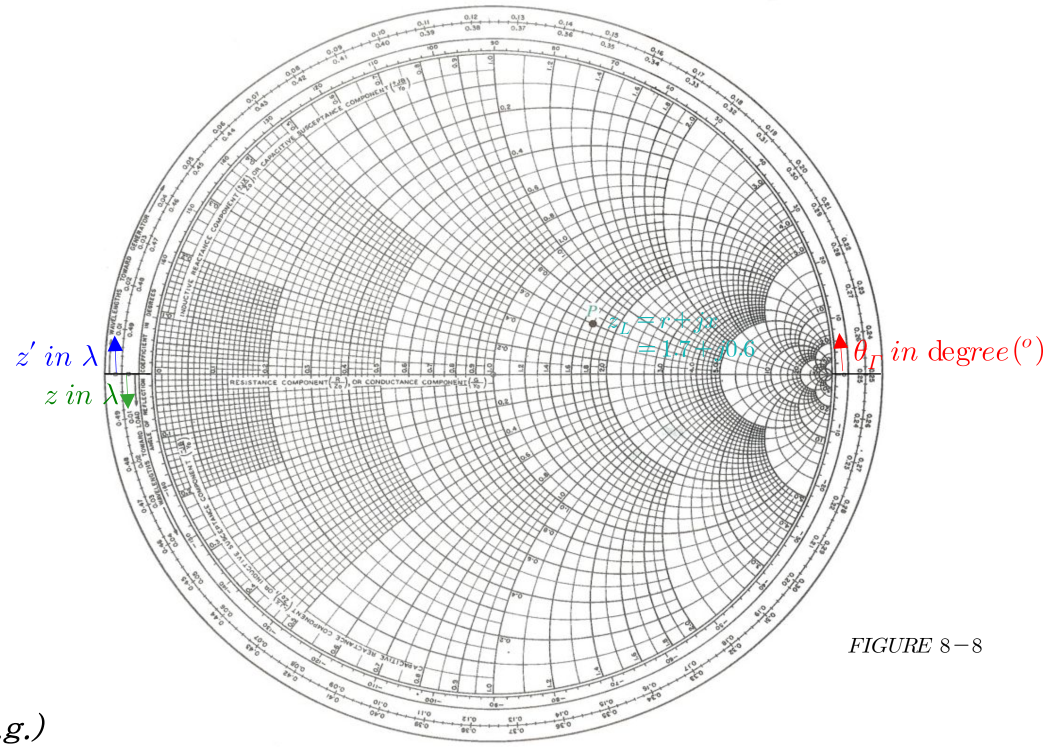
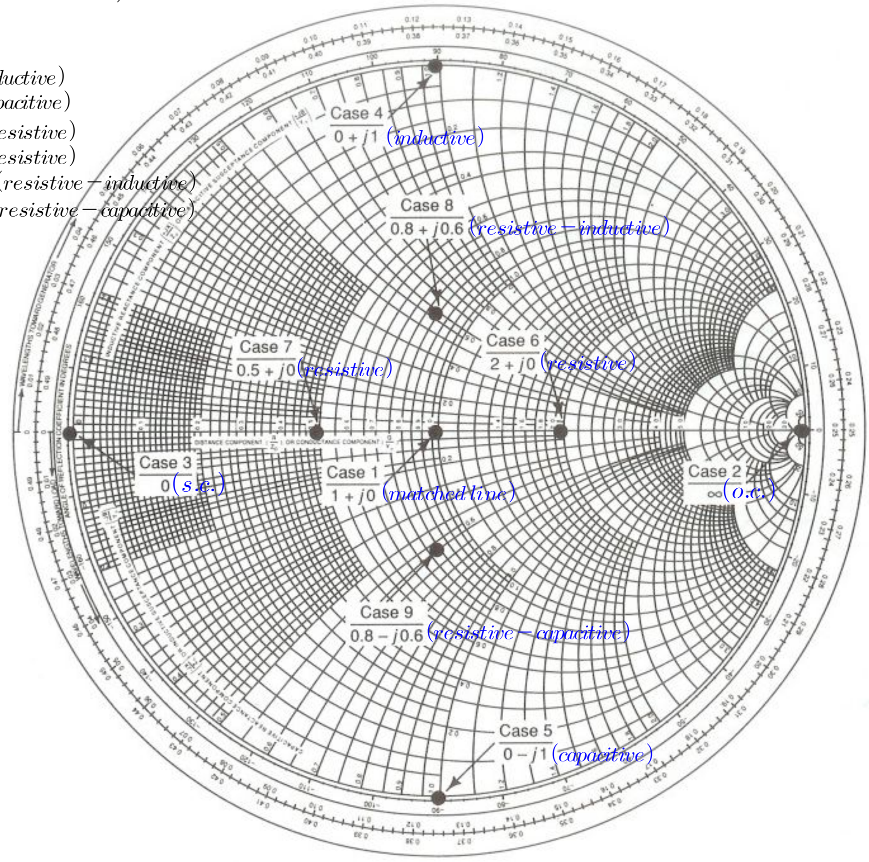


FIGURE 8−8

(e.g.)

Case	$z_L = r + jx$
1	$1 + j0$ (*matched line*)
2	∞ (*o.c.*)
3	0 (*s.c.*)
4	$0 + j1$ (*inductive*)
5	$0 - j1$ (*capacitive*)
6	$2 + j0$ (*resistive*)
7	$0.5 + j0$ (*resistive*)
8	$0.8 + j0.6$ (*resistive − inductive*)
9	$0.8 - j0.6$ (*resistive − capacitive*)



3) Applications of the Smith Chart

a) SWR(S) and locations($l_{\min}$ & $l_{\max}$) of V_{min} and V_{max} on Smith Chart

Along the real Γ axis ($x=0$, $X_L=0$), $z_L = r\,(Z_L = R_L)$ in (8-99) :

$$\Gamma = \frac{R_L - R_o}{R_L + R_o} = \frac{r-1}{r+1} \overset{(8-91)}{=} \frac{S-1}{S+1} \qquad (8\text{-}106)$$

$\Rightarrow \quad S = r = R_L/R_o$ for $R_L > R_o$

$\Rightarrow \quad S$ is numerically equal to the value of r at P_M

[SWR circle (constant-$|\Gamma|$ circle) intersects the real Γ axis at P_M]

The points P_m and P_M also represent the first distances,

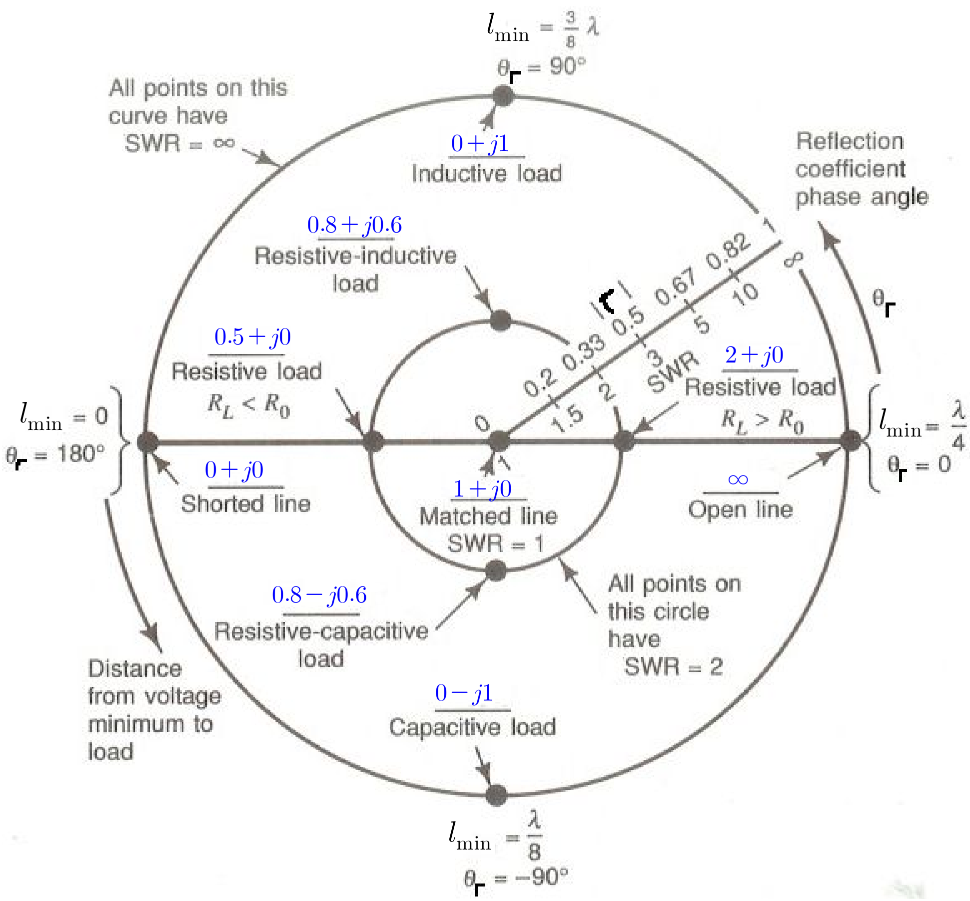
$l_{\min}$ and $l_{\max}$, from the load for $V_{\min}$ and $V_{\max}$, respectively.

(7) and (8) on p.13 :

1st min. position: $l_{\min} = \left(\frac{\theta_\Gamma}{\pi} + 1\right)\frac{\lambda}{4}$ (7)

1st max. position: $l_{\max} = \frac{\theta_\Gamma \lambda}{4\pi}$ (for $\theta_\Gamma > 0$) or $\frac{\theta_\Gamma \lambda}{4\pi} + \frac{\lambda}{2}$ (for $\theta_\Gamma < 0$) (8)

(e.g.)



b) Normalized input impedance on Smith Chart

(8-77) $\Rightarrow$ $$Z_i(z') = R_o \frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}} \quad (8\text{-}107)$$

Normalized input impedance:

$$z_i(z') \equiv \frac{Z_i}{Z_o} = \frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}} = \frac{1+|\Gamma|e^{j\phi}}{1-|\Gamma|e^{j\phi'}} \quad (8\text{-}108)$$

where $\phi = \theta_\Gamma - 2\beta z'$ (8-109)

(cf) (8-108) for $z' = 0$ becomes $z_L = \dfrac{1+|\Gamma|e^{j\theta_\Gamma}}{1-|\Gamma|e^{j\theta_\Gamma}}$ (8-100)

Similarity in form of (8-108) and (8-100)

$\Rightarrow$ z_L can be transformed into z_i if Γ is transformed maintaining $|\Gamma|$ constant and decreasing the phase θ_Γ by $2\beta z' = 4\pi(z'/\lambda)$.

$\Rightarrow$ Wavelength Toward Generator (WTG) in cw direction $(z'\uparrow)$

= Outermost scale in units of λ around the perimeter

Wavelength Toward Load (WTL)in ccw direction $(z'\downarrow)$

= Inner scale in units of λ around the perimeter

Note) $\Delta z' = \lambda/2$ *in* ϕ $\Rightarrow$ $2\beta\Delta z' = 4\pi(\lambda/2/\lambda) = 2\pi$

: one complete rotation around the chart

(e.g. 8-7)

Given s.c. $(z_L = 0)$, lossless $R_o = 50$ (Ω), $z' = l = 0.1\,\lambda$.

Find $Z_i = ?$ using the Smith Chart.

(cf) Analytical solution:

From (8-82), $Z_{isc} = jR_o \tan\beta l = jR_o \tan(2\pi l/\lambda)$

$$= j50\tan\left(\frac{2\pi \times 0.1\lambda}{\lambda}\right) = j50\tan 36^o = j36.3\ (\Omega)$$

Graphical solution by using Smith Chart:

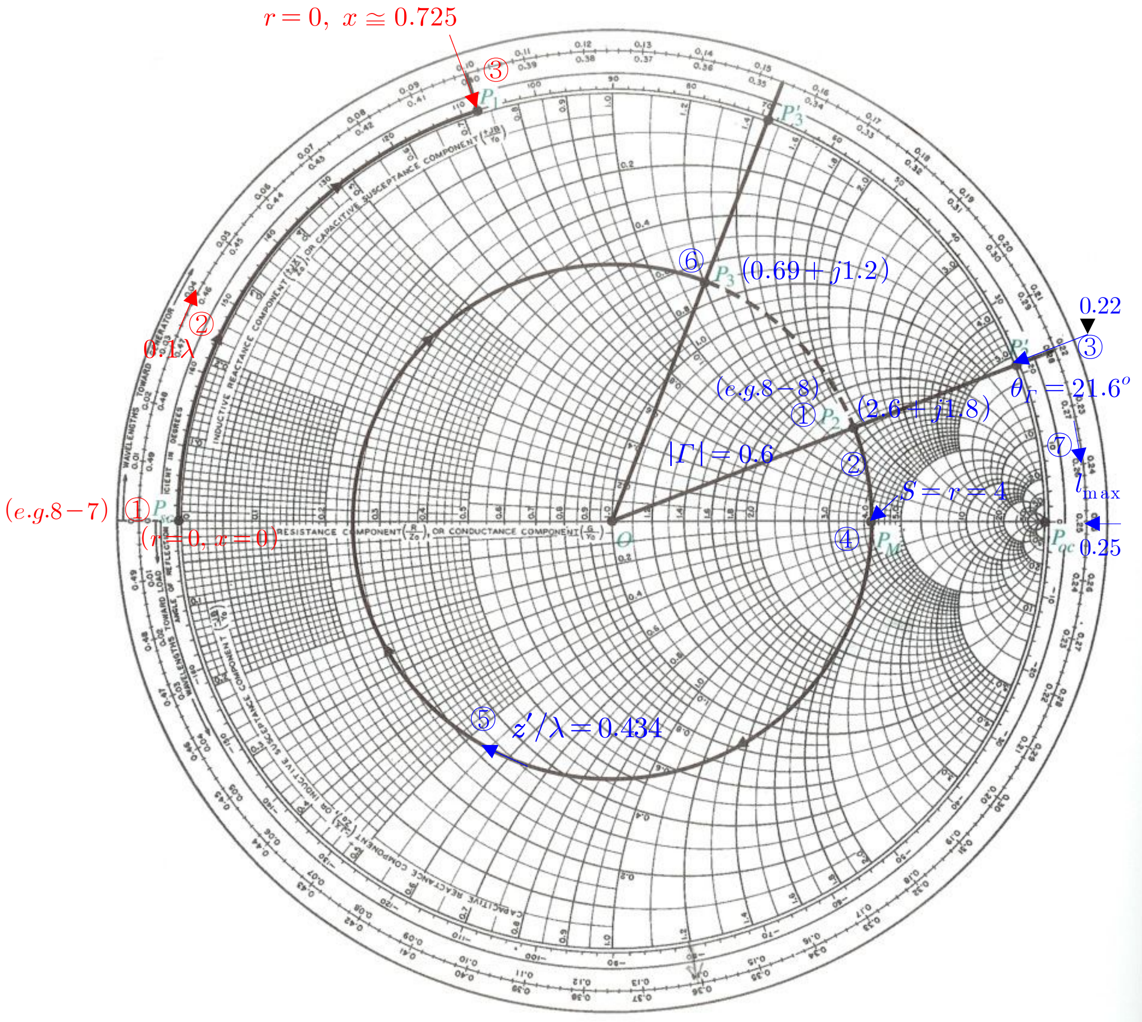
① Start at $P_{sc}(r=0,\ x=0)$

→ ② Move along WTG by 0.1 to P_1

→ ③ At P_1, read $r=0,\ x \cong 0.725$

$\Rightarrow$ $\therefore$ $z_i = r + jx = 0 + j0.725$

$\Rightarrow$ $Z_i = R_o z_i = 50(j0.725) = j36.3\ (\Omega)$



(e.g. 8-8) FIGURE 8-9 Smith-chart calculations for Examples 8-7 and 8-8.

Given lossless $Z_o = R_o = 100\,(\Omega)$, $z' = l = 0.434\,\lambda$, $Z_L = 260 + j180\,(\Omega)$.

Find (a) Γ, (b) S, (c) Z_i, (d) $l_{\max}$ using the Smith Chart.

(cf) Analytical solutions:

(a) (8-88) $\Rightarrow \Gamma = \dfrac{Z_L - Z_o}{Z_L + Z_o} = |\Gamma|\,e^{j\theta_\Gamma}$

$$\Rightarrow \quad \Gamma = \frac{(260 + j180) - 100}{(260 + j180) + 100} = \frac{16 + j18}{36 + j18} = \frac{(16 + j18)(36 - j18)}{36^2 + 18^2}$$

$$= \frac{900 + j360}{1620} = 0.5556 + j0.2222 \cong \underline{0.6\angle 21.6^o}$$

(b) (8-90) $\Rightarrow \quad S = \dfrac{1 + |\Gamma|}{1 - |\Gamma|} = \dfrac{1 + 0.6}{1 - 0.6} = \underline{4}$

(c) (8-79) $\Rightarrow \quad Z_i = R_o \dfrac{Z_L + jR_o \tan\beta l}{R_o + jZ_L \tan\beta l} = 100 \dfrac{(260 + j180) + j100\tan(2\pi \times 0.434\lambda/\lambda)}{100 + j(260 + j180)\tan(2\pi \times 0.434\lambda/\lambda)}$

$$= 100 \frac{260 + j[180 + 100\tan(0.864\pi)]}{[100 - 180\tan(0.868\pi)] + j260\tan(0.868\pi)} = \underline{69 + j120\ (\Omega)}$$

(d) (8) on p.13 $\Rightarrow \quad l_{\max} = \dfrac{\theta_\Gamma \lambda}{4\pi} = \dfrac{21.6^o \lambda}{4 \times 180^o} = \underline{0.03\,\lambda}$

Graphical solution by using Smith Chart:

(a) ① Start at $P_2(z_L = Z_L/R_o = 2.6 + j1.8)$

→ ② Draw a circle of radius $\overline{OP_2}$: $\frac{\overline{OP_2}}{\overline{OP_2'}} = |\Gamma| = 0.6$

→ ③ At P_2', read $\theta_\Gamma = 21.6^o$

(or $z'/\lambda = 0.220 \Rightarrow (0.25 - 0.22) \times 4\pi = 0.12\pi = 21.6^o$)

$\Rightarrow \therefore \quad \Gamma = |\Gamma| e^{j\theta_\Gamma} = \underline{0.6\angle 21.6^o}$

(b) ④ Read r at P_M where the $|\Gamma| = 0.6$ circle intersects with the positive-real axis

$\Rightarrow \therefore \quad S = r = \underline{4}$

(c) ⑤ Move P_2 or P_2' along WTG by $z'/\lambda = 0.434$ up to P_3 or P_3'

→ ⑥ At P_3, read $r = 0.69$, $x = 1.2$

$\Rightarrow \quad Z_i = R_o z_i = 100(0.69 + j1.2) = \underline{69 + j120 \ (\Omega)}$

(d) ⑦ At P_M where the voltage is a maximum,

distance from the load: $l_{\max} = (0.25 - 0.22)\lambda = \underline{0.03\lambda}$

c) Normalized admittance on Smith Chart

Admittance corresponding to Z :

$$Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} = G + jB \quad (S) \qquad (16)$$

where $G = R/(R^2 + X^2)$ (S) : conductance (17)

$B = -X/(R^2 + X^2)$ (S) : susceptance (18)

Normalized load admittance:

$$y_L \equiv \frac{Y_L}{Y_o} = \frac{1}{z_L} = \frac{Z_o}{Z_L} = Z_o Y_L = g + jb \qquad (8\text{-}112,\ 113)$$

(8-100) $z_L = \frac{1+\Gamma}{1-\Gamma}$ in (8-112) : $y_L = \frac{1}{z_L} = \frac{1-\Gamma}{1+\Gamma}$ (19)

Normalized input impedance of a quarter-wave lossless line:

(8-78) $Z_i = Z_o \frac{1 + \Gamma e^{-2\gamma l}}{1 - \Gamma e^{-2\gamma l}}$

$$\Rightarrow z_i(l = \lambda/4) = \frac{1 + \Gamma e^{-j2\beta l}}{1 - \Gamma e^{-j2\beta l}} = \frac{1 + \Gamma e^{-j\pi}}{1 - \Gamma e^{-j\pi}} = \frac{1-\Gamma}{1+\Gamma} \overset{(19)}{=} y_L \qquad (20)$$

Note) (8-111) $Z_i = \frac{Z_o^2}{Z_L} \Rightarrow \frac{Z_i}{Z_o} = \frac{Z_o}{Z_L} \Rightarrow z_i = y_L$ (20)

Therefore, z_L can be transformed into y_L by rotating $\lambda/4\,(180^o)$ on the Smith Chart.

$\Rightarrow$ z_L and y_L are diametrically opposite to each other on the $|\Gamma|$-circle

($r \rightarrow g,\ \ x \rightarrow b,\ \ o.c. \leftrightarrow s.c.\ \Rightarrow$ *admittance chart*)

(e.g. 8-9)

Given $Z = 95 + j20\ (\Omega)$, find Y.

Assume $Z_o = R_o = 50\ (\Omega)$,

then $z = (95 + j20)/50 = 1.9 + j0.4$

$$\Rightarrow\ Y = y/R_o = (0.5 - j0.1)/50 = 0.01 - j0.002\ \ (S)$$

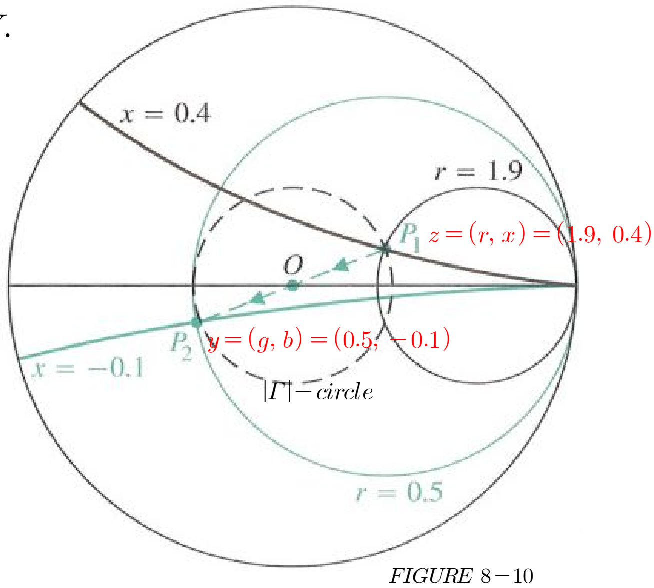


FIGURE 8−10

(e.g. 8-10)

Given lossless o.c. $Z_o = R_o = 300\ (\Omega)$, $z' = l = 0.04\lambda$, find Y_i .

$$Y_i = y_i/Z_o = (0 + j0.26)/300 \cong j0.00087\ \ (S) = j0.87\ \ \ (mS)$$

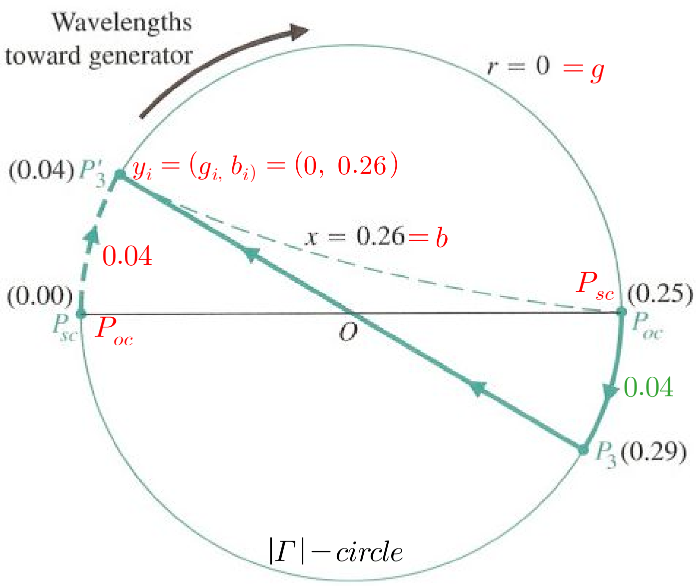


FIGURE 8−11

B. Transmission-Line Impedance Matching

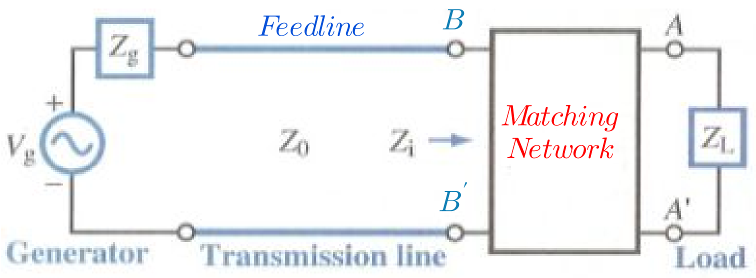
1) Impedance-Matching Network

$Z_L = Z_o \;\Rightarrow\;$ Matched line

(No reflection, no distortion, no power loss at the load)

In general, $Z_L \neq Z_o$

$\Rightarrow$ needs an impedance-matching network having $Z_i = Z_o$
[No reflection at the line terminal BB', usually consists of inductors(L) and capacitors(C) to avoid ohmic losses]



For a lossless line $(Z_o = R_o + j0)$, $Z_L = R_L + jX_L = R_o$ for matching

i.e., $X_L = 0$ for matching.

2) Single-stub method for impedance matching

= short circuit (or open circuit) connected at a lossless transmission line in parallel with another line connecting the load.

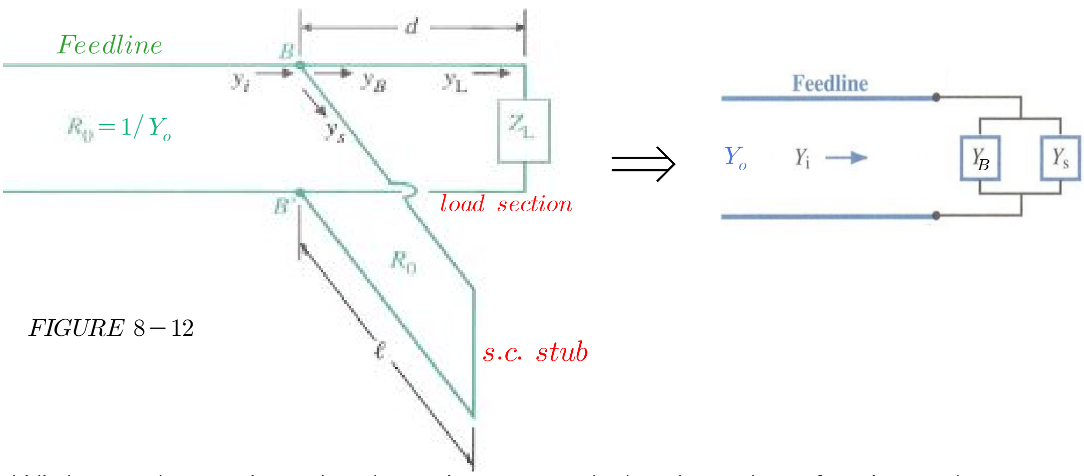


FIGURE 8-12

Wish to determine the location d and the length l for impedance-(or admittance) matching such that

$$Y_i = Y_o = Y_B + Y_s \quad (8\text{-}114) \qquad \Rightarrow \qquad 1 = y_B + y_s \quad (8\text{-}115)$$

where $y_B = 1 + jb_B$ (find d) and $y_s = -jb_B$ (find l) (8-116, 117)

Procedure for single-stub impedance-matching:

① Start at $z_L \rightarrow y_L$

$\rightarrow$ ② Find intersections of the $|\Gamma|$-circle and the $g=1$ circle.

$y_{B1} = 1 + jb_{B1}$ and $y_{B2} = 1 + jb_{B2}$

$\rightarrow$ ③ Determine d_1, d_2 from the angles between y_L and y_{B1}, y_{B2}

$\rightarrow$ ④ Determine l_1, l_2 from the angles between P_{sc}(extreme right) and $y_{s1} = -jb_{B1}$, $y_{s2} = -jb_{B2}$

(e.g. 8-11)

Given lossless $Z_o = R_o = 50\,(\Omega)$, $Z_L = 35 - j47.5\,(\Omega)$, find d and l.

① $z_L = Z_L/R_o = 0.70 - j0.95$

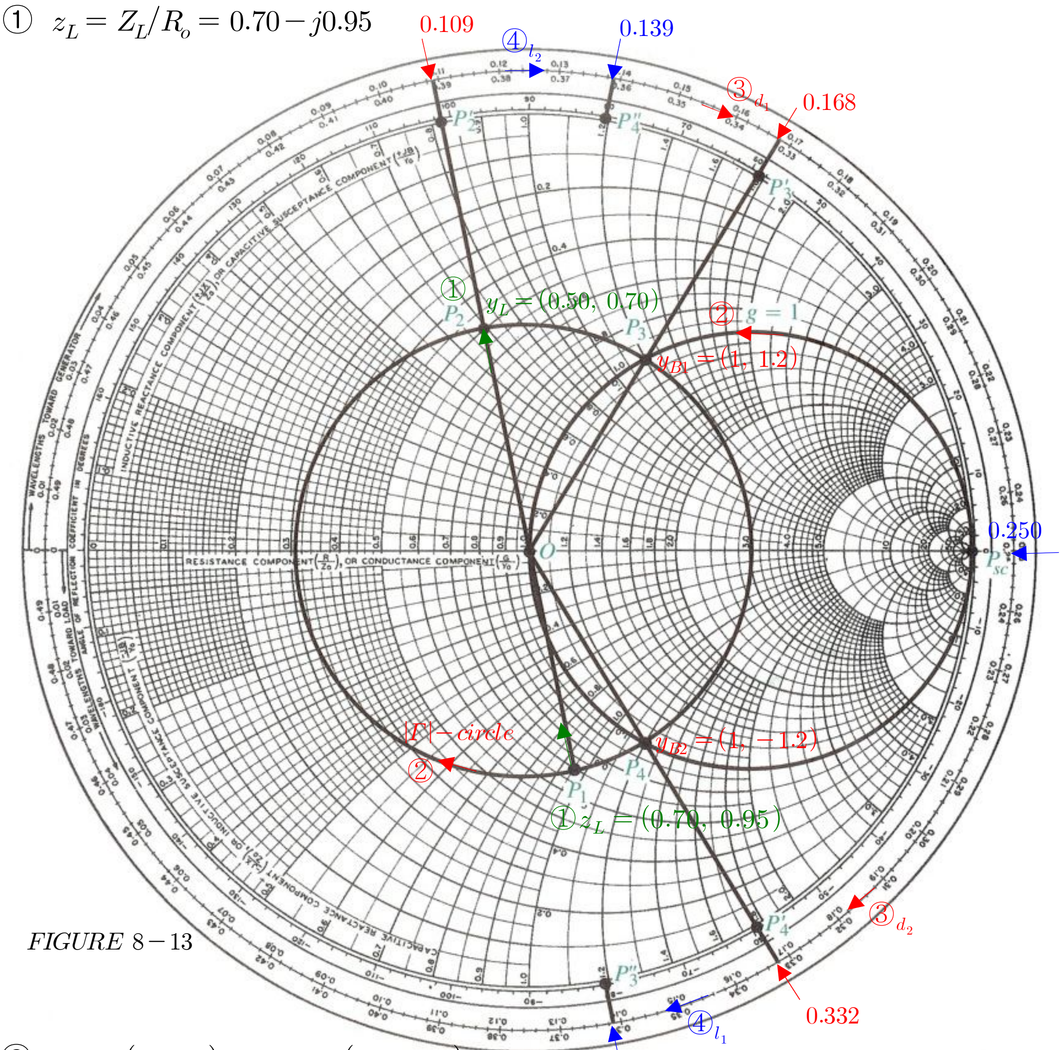


FIGURE 8−13

0.361

② $y_{B1} = (1,\ 1.2)$, $y_{B2} = (1,\ -1.2)$

③ $d_1 = (0.168 - 0.109)\lambda = \underline{0.059\lambda}$ and $d_2 = (0.332 - 0.109)\lambda = \underline{0.223\lambda}$

④ $l_1 = (0.361 - 0.250)\lambda = \underline{0.111\lambda}$ and $l_2 = (0.139 + 0.250)\lambda = \underline{0.389\lambda}$

Homework Set 5

1) P.8-18 2) P.8-20 3) P.8-21
4) P.8-24 5) P.8-27