Chap. 5 Euler-Bernoulli beam theory

- one of its dimensions much longer than the other two
  - civil engineering structure ... assembly on grid of beams with cross-sections having shapes such as T's or I's.
  - machine parts --- beam-like structures : lever arms, shafts, etc.
  - aeronautic structures --- wings, fuselages → can be treated as thin-walled beams
- "beam theory" --- important role, simple tool to analyze numerous structures
  valuable insight at a pre-design stage
- Euler-Bernoulli beam theory ... simplest, most useful
  - assumption --- ① cross-section of the beam is infinitely rigid in its own plane
    → in-plane displacement field → { 2 rigid body translations; 1 " rotation }

    ② the cross-section is assumed to remain plane

    ③ " normal to the deformed axis

5.1 The Euler-Bernoulli assumptions

- Fig. 5.1 --- "pure bending", beam deforms into a curve of constant curvature
  → a circle with center O, symmetric w.r.t. any plane perpendicular to its deformed axis
- Kinematic assumptions "Euler-Bernoulli" ... ① cross-section is infinitely rigid in its own plane

  ② " remains plane after deformation

  ③ " " normal to the deformed axis of the beam

  → valid for long, slender beams made of isotropic materials with solid cross-sections

5.2 Implications of the E-B assumptions

- $\left.\begin{array}{l} u_1(x_1, x_2, x_3) \\ u_2(\quad " \quad) \\ u_3(\quad " \quad) \end{array}\right\}$ displacement of an arbitrary point of the beam
  - E-B assumption ① → displacement field in the plane of X-S consists solely of 2 rigid body translations $\bar{u}_2(x_1)$, $\bar{u}_3(x_1)$

$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1), \quad u_3(x_1, x_2, x_3) = \bar{u}_3(x_1) \quad (5.1)$$

- E-B assumption ② → axial displacement field consists of { rigid body translation $\bar{u}_1(x_1)$; 2 " rotation $\Phi_2(x_1)$, $\Phi_3(x_1)$ } (Fig. 5.2)

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) + x_3 \Phi_2(x_1) - x_2 \Phi_3(x_1) \quad (5.2)$$

- E-B assumption ③ → equality of { the slope of the beam; the rotation of the section } (Fig. 5.4)

$$\Phi_3 = \frac{d\bar{u}_2}{dx_1}, \quad \Phi_2 = -\frac{d\bar{u}_3}{dx_1} \quad (5.3)$$

consequence of the sign convention

- to eliminate the sectional rotation from the axial displacement field

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) - x_3 \frac{d\bar{u}_3(x_1)}{dx_1} - x_2 \frac{d\bar{u}_2(x_1)}{dx_1} \quad (5.4a)$$

... important simplification of E-B : unknown displacements are functions of the span-wise coord., x_1, alone.

- Strain field

$$\epsilon_2 = 0, \quad \epsilon_3 = 0, \quad \gamma_{23} = 0 \quad (5.5a) \leftarrow \text{E-B①}$$

$$\gamma_{12} = 0, \quad \gamma_{13} = 0 \quad (\text{" } b) \leftarrow \text{" ③}$$

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1} = \underbrace{\frac{d\bar{u}_1(x_1)}{dx_1}}_{\bar{\epsilon}_1(x_1)} + x_3 \underbrace{\frac{d^2\bar{u}_3(x_1)}{dx_1^2}}_{\kappa_2(x_1)} - x_2 \underbrace{\frac{d^2\bar{u}_2(x_1)}{dx_1^2}}_{\kappa_3(x_1)} \quad (\text{" } c)$$

with

sectional axial strain — sectional curvature about $\bar{\imath}_2$, $\bar{\imath}_3$ axes

$$\Rightarrow \epsilon_1(x_1, x_2, x_3) = \bar{\epsilon}_1(x_1) + x_3 \kappa_2(x_1) - x_2 \kappa_3(x_1) \quad (5.7) \leftarrow \text{E-B ②}$$

- Assuming a strain field of the form Eqs. (5.5a), (5.5b), (5.7)

... math. expression of the E-B assumptions

5.3 Stress resultants

- 3-D stress field → described in terms of sectional stresses called "stress resultants"

--- equipollent to specific components of the stress field

- 3 force resultants : { $N_1(x_1)$ axial force; $V_2(x_1)$, $V_3(x_1)$ transverse shearing forces }

$$N_1(x_1) = \int_A \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.8)$$

$$V_2(x_1) = \int_A \tau_{12}(x_1, x_2, x_3)\, dA, \quad V_3(x_1) = \int_A \tau_{13}(x_1, x_2, x_3)\, dA \qquad (5.9)$$

– 2 moment resultants : $M_2(x_1)$, $M_3(x_1)$ bending moments

$$M_2(x_1) = \int_A x_3\, \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.10a)$$

$$M_3(x_1) = -\int_A x_2\, \sigma_1(x_1, x_2, x_3)\, dA \qquad (\text{'' } b)$$

(+) equipollent bending moment about $\bar{\imath}_3$ (Fig. 5.5)

bending moments computed about point $P(x_{2p}, x_{3p})$

$$M_2^P(x_1) = \int_A (x_3 - x_{3p})\, \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.11a)$$

5.4 Beams subjected to axial loads

- distributed axial load $p_1(x_1)$ [N/m], concentrated axial load P_1 [N]
  - → axial displacement field $\bar{u}_1(x_1)$ ⇒ "bar" rather than "beam"

5.4.1 Kinematic description

- axial loads causes only axial displacement of the section

… Eq. (5.4) →

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) \qquad (5.12a)$$ – uniform over the c-s (Fig. 5.7)

$$u_2(\text{ '' }) = 0 \qquad (\text{'' } b)$$

$$u_3(\text{ '' }) = 0 \qquad (\text{'' } c)$$

axial strain field $\epsilon_1(\text{ '' }) = \bar{\epsilon}_1(x_1)$ (5.13)

5.4.2 Sectional constitutive law

- $\sigma_2 \ll \sigma_1$, $\sigma_3 \ll \sigma_1$ → transverse stress components $\simeq 0$, $\sigma_2 \simeq 0$, $\sigma_3 = 0$
  - generalized Hooke's law → $\sigma_1(x_1, x_2, x_3) = E\,\epsilon_1(x_1, x_2, x_3)$ (5.14)
    at the "infinitesimal" level
- inconsistency in E-B beam theory …

  Eq. (5.5a) → $\epsilon_2 = 0$, $\epsilon_3 = 0$

  Hooke's law → if $\sigma_2 = \sigma_3 = 0$, then $\epsilon_2 = -\nu\sigma_1/E$, $\epsilon_3 = -\nu\sigma_1/E$

  (Poisson's effect) … very small effect, and assumed to vanish

Eq. (5.13) → (5.14) : $\sigma_1(x_1, x_2, x_3) = E\,\bar{\epsilon}_1(x_1)$ (5.15)

– axial force

$$N_1(x_1) = \int_A \sigma_1(x_1, x_2, x_3)\, dA = \left[\int_A E\, dA\right] \bar{\epsilon}_1(x_1) = \underbrace{S}_{\text{axial stiffness}}\, \bar{\epsilon}(x_1) \qquad (5.16)$$

$S = EA$ for homogeneous material

… constitutive law for the axial behavior of the beam at the sectional level

5.4.3 Equilibrium eqn.s

- Fig. 5.8 ... infinitesimal slice of the beam of length dx_1

force equilibrium in axial dir. $\rightarrow$ $$\frac{dN_1}{dx_1} = -p_1 \qquad (5.18)$$

Eq. (1.4) ... equilibrium condition for a differential element of a 3-D solid

(5.18) ... " of a slice of the beam of differential length dx_1

5.4.4 Governing eqn.s

- Eq. (5.16) $\rightarrow$ Eq. (5.18), and using Eq. (5.6)

$$\frac{d}{dx_1}\left[S \frac{d\bar{u}_1}{dx_1}\right] = -p_1(x_1) \qquad (5.19)$$

- 3 B.C. --- ① fixed (clamped) : $\bar{u}_1 = 0$

② free (unloaded) : $N_1 = 0 \rightarrow \frac{d\bar{u}_1}{dx_1} = 0$

③ subjected to a concentrated load P_1 : $N_1 = P_1 \rightarrow S\frac{d\bar{u}_1}{dx_1} = P_1$

5.4.5 The sectional axial stiffness

- homogeneous material ... $$S = EA \qquad (5.20)$$
- rectangular section of width b made of layered material of different moduli (Fig. 5.9)

$$S = \int_A E\, dA = \sum_{i=1}^{n} E^{[i]} \int_{A^{[i]}} dA^{[i]} = \sum_{i=1}^{n} E^{[i]} b \underbrace{(x_3^{[i+1]} - x_3^{[i]})}_{\text{weighting factor : thickness}}$$

"weighted average" of the Young's modulus

5.4.6 The axial stress distribution

- Eliminating the axial strain from Eq. (5.15) and (5.16)

$$\sigma_1(x_1, x_2, x_3) = \frac{E}{S} N_1(x_1) \qquad (5.21)$$

- homogeneous material

$$\sigma_1(x_1, x_2, x_3) = \frac{N_1(x_1)}{A} \qquad (5.22)$$

... uniformly distributed over the section

- sections made of layers presenting different moduli

$$\sigma_1^{[i]}(x_1, x_2, x_3) = E^{[i]} \frac{N_1(x_1)}{S} \qquad (5.23)$$

... stress in layer i is proportional to the modulus of the layer

Eq. (5.13) → axial strain distribution is uniform over the section,
i.e., each layer is equally strained (Fig. 5.10)

- strength criterion

$$\frac{E}{S}\,|N_{1\,max}^{tens}| \le \sigma_{allow}^{ten}, \quad \frac{E}{S}\,|N_{1\,max}^{comp}| \le \sigma_{allow}^{comp} \qquad (5.24)$$

in case compressive, buckling failure mode may occur → Chap. 14

5.5 Beams subjected to transverse loads

- Fig. 5.14 ... "transverse direction" distributed load, $p_2(x_1)$ [N/m]
  concentrated " P_2 [N]

  → bending moments, transverse shear forces, and $\left\{\begin{array}{l}\text{axial}\\ \text{transverse shearing}\end{array}\right\}$ stresses
  will be generated

5.5.1 Kinematic description

- Assumption ... transverse loads only cause $\left\{\begin{array}{l}\text{transverse displacement}\\ \text{curvature of the section}\end{array}\right.$

- General displacement field (Eq. (5.4)) →

$$u_1(x_1, x_2, x_3) = -x_2\frac{d\bar{u}_2(x_1)}{dx_1} \qquad (5.29a)$$

$$u_2(\;''\;) = \bar{u}_2(x_1) \qquad (\;''\; b)$$

$$u_3(\;''\;) = 0 \qquad (\;''\; c)$$

→ Fig. 5.15 ... linear distribution of the axial displacement component over the x-s

- only non-vanishing strain component

$$\epsilon_1(x_1, x_2, x_3) = -x_2\,\kappa_3(x_1) \qquad (5.30)$$

... linear distribution of the axial strain

5.5.2 Sectional constitutive law

- linearly elastic material, axial stress distribution

$$\sigma_1(x_1, x_2, x_3) = -E\,x_2\,\kappa_3(x_1) \qquad (5.31)$$

- Sectional axial force, by Eq. (5.8),

$$N_1(x_1) = \int_A \sigma_1(x_1, x_2, x_3)\,dA = -\left[\int_A E x_2\,dA\right]\kappa_3(x_1) \qquad (5.32)$$

- axial force = 0 since subjected to transverse loads only
  $\kappa_3 \neq 0$, then $[\cdots] = 0$

$$\Rightarrow \quad x_{2c} = \frac{1}{S} \int_A E x_2 \, dA = \frac{S_2}{S} = 0 \qquad (5.33)$$

location of the "modulus-weighted centroid" of the x-s

- If homogeneous material,

$$x_{2c} = \frac{E\int_A x_2 dA}{E\int_A dA} = \frac{1}{A}\int_A x_2 \, dA = 0 \qquad (5.34)$$

... x_2 is simply the area center of the section.

$\Rightarrow$ the axis system is located at the { modulus-weight centroid / area center if homogeneous material

※ center of mass $x_{2m} = \dfrac{\rho\int_A x_2 dA}{\rho\int_A dA} = \dfrac{\int_A x_2 dA}{\int_A dA} = x_{2c}$, center of mass --> 3 coincide

- Bending moment, by Eq. (5.31)

$$M_3(x_1) = \left[\int_A E x_2^2 \, dA\right] \kappa_3(x_1) = H_{33}^c \, \kappa_3(x_1) \qquad (5.35)$$

"centroidal bending stiffness" about axis $\bar{i}_3$

... constitutive law for the bending behavior of the beam

bending moment & the curvature

bending stiffness (or "flexural rigidity")

$$\rightarrow \quad M_3(x_1) = H_{33}^c \, \kappa_3(x_1) \qquad (5.37)$$

: "moment-curvature" relationship

5.5.3 Equilibrium eqn.s

- Fig. 5.16 ... infinitesimal slice of the beam of length dx_1

$M_3(x_1)$, $V_2(x_1)$ acting at a face at location x_1,

@ $x_1 + dx_1$, evaluated using a Taylor series expansion, and h.o. terms ignored

$\rightarrow$ 2 equilibrium eqn.s

vertical force $\rightarrow$ $\dfrac{dV_2}{dx_1} = -p_2(x_1)$ (5.38a)

moment about O $\rightarrow$ $\dfrac{dM_3}{dx_1} + V_2 = 0$ (5.38b)

$$\rightarrow \quad \frac{d^2 M_3}{dx_1^2} = p_2(x_1) \qquad (5.39)$$

5.5.4 Governing eqn.s

- Eq. (5.37) $\rightarrow$ Eq. (5.39), and recalling Eq. (5.6) $\rightarrow$

$$\frac{d^2}{dx_1^2}\left[H_{33}^c \frac{d^2 \bar{u}_2}{dx_1^2}\right] = p_2(x_1) \qquad (5.40)$$

4^{th} order DE

- 4 B.C. ... ① clamped end : $\bar{u}_2 = 0$, $\dfrac{d\bar{u}_2}{dx_1} = 0$

② simply supported (pinned) : $\bar{u}_2 = 0, \quad \dfrac{d^2\bar{u}_2}{dx_1^2} = 0$

③ free (or unloaded) end : $\underbrace{\dfrac{d^2\bar{u}_2}{dx_1^2} = 0}_{\text{bending moment}}, \quad \underbrace{-\dfrac{d}{dx_1}\left[H_{33}^c \dfrac{d^2\bar{u}_2}{dx_1^2}\right] = 0}_{\text{shear force}}$

④ end subjected to a concentrated transverse load P_2 : $P_2 = V_2 = -\dfrac{dM_3}{dx_1}$

$$\frac{d^2\bar{u}_2}{dx_1^2} = 0, \quad -\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right] = P_2$$

⑤ rectilinear spring (Fig. 5.17) : $\underbrace{-V_2(L) = k \cdot \bar{u}_2(L)}_{\text{sign convention}}$

$$\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right]_{x_1=L} - k\bar{u}_2(L) = 0, \quad \frac{d^2\bar{u}_2}{dx_1^2} = 0$$

(+) when the spring is located at the left end

⑥ rotational spring (Fig. 5.18) : $-M_3(L) = k\,\Phi_3(L)$

$$H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\Big|_{x_1=L} + k\frac{d\bar{u}_2}{dx_1} = 0, \quad -\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right] = 0$$

(−) when at the left end

5.5.5 The sectional bending stiffness

- Homogeneous material,

$$H_{33}^c = E I_{33}^c \qquad (5.41)$$

$$I_{33}^c = \int_A x_2^2\, dA \qquad (5.42)$$

: purely geometric quantity, the area second moment of the section computed about the area center

- rectangular section of width b made of layered materials (Fig. 5.9)

$$H_{33}^c = \int_A E x_2^2\, dA = \sum_{i=1}^{n} E^{[i]} \int_{A^{[i]}} x_2^2\, dA^{[i]} = \frac{b}{3}\sum_{i=1}^{n} E^{[i]}\left[(x_2^{[i+1]})^3 - (x_2^{[i]})^3\right] \qquad (5.43)$$

"weighted average" of the Young's moduli

5.5.6 The axial stress distribution

- local axial stress ··· eliminating the curvature from Eq. (5.37), (5.37)

$$\sigma_1(x_1, x_2, x_3) = -E x_2 \frac{M_3(x_1)}{H_{33}^c} \qquad (5.44)$$

- homogeneous material : $\sigma_1(x_1, x_2, x_3) = -x_2 \dfrac{M_3(x_1)}{I_{33}} \qquad (5.45)$

··· linearly distributed over the section, independent of Young's modulus

- various layers of materials

$$\sigma_1^{[i]}(x_1, x_2, x_3) = -E^{[i]} x_2 \frac{M_3(x_1)}{H_{33}^c} \qquad (5.46)$$

... axial STRAIN distribution is linear over the section ← Eq (5.30),

" stress " ... piecewise linear (Fig. 5.20)

· strength criterion

$$\frac{|x_2^{max}|}{H_{33}^c} E |M_3^{max}| \le \sigma_{allow}^{comp}, \quad \frac{|x_2^{min}|}{H_{33}^c} E |M_3^{max}| \le \sigma_{allow}^{ten}$$

↑ max. (+) bending moment in the beam

- layers of various material

... must be computed at the $\left\{\begin{matrix} \text{top} \\ \text{bottom} \end{matrix}\right\}$ locations of each ply

5.5.7 Rational design of beams under bending

° "Neutral axis" ... along axis $\bar{\imath}_3$, which passes through the section's centroid

- material located near the N. A. carries almost no stress

- " contributes little to the bending stiffness

⇒ Rational design ... removal of the material located at and near the N.A. and relocation away from that axis

° Fig. 5.21 ... $\left\{\begin{matrix} \text{rectangular} \\ \text{ideal} \end{matrix}\right\}$ section, same mass $m = bh\rho$

↑ a thin web would be used to keep the 2 flanges.

- ratio of bending stiffness

$$\frac{H_{ideal}}{H_{rect}} = \frac{E \cdot 2\left[\frac{b(h/2)^3}{12} + \frac{bh}{2} d^2\right]}{E \frac{bh^3}{12}} = \frac{1}{4} + 12\left(\frac{d}{h}\right)^2$$

For $d/h = 10$, $\boxed{} \simeq 1200$

- ratio of max. axial stress

$$\frac{\sigma_{rect}^{max}}{\sigma_{ideal}^{max}} = \frac{E \frac{h}{2} M_3 I_{ideal}}{I_{rect} E\left(d + \frac{h}{4}\right) M_3} = \frac{\frac{1}{4} + 12\left(\frac{d}{h}\right)^2}{\frac{1}{2} + 2\left(\frac{d}{h}\right)}$$

For $d/h = 0$, $\boxed{} \simeq 6(d/h) = 60$

→ ideal section can carry a 60 times larger bending moment

- ideal section ⇒ "I beam", but prone to instabilities of web and flange buckling

5.6 Beams subjected to combined axial and transverse loads

Sec. 5.4, 5.5 ··· convenient to locate the origin of the axes system at the centroid of the beam's x-s.

5.6.1 Kinematic description

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) - (x_2 - \underset{\substack{\uparrow \\ \text{location of centroid}}}{x_{2c}}) \frac{d\bar{u}_2(x_1)}{dx_1} \quad (5.73a)$$

$$u_2(\quad " \quad) = \bar{u}_2(x_1) \quad (\ " \ b)$$

$$u_3(\quad " \quad) = 0 \quad (\ " \ c)$$

· strain field

$$\epsilon_1(x_1, x_2, x_3) = \bar{\epsilon}_1(x_1) - (x_2 - x_c)\kappa_3(x_1) \quad (5.74)$$

5.6.2 Sectional constitutive law

∘ axial stress distribution

$$\sigma_1(x_1, x_2, x_3) = E\bar{\epsilon}_1(x_1) - E(x_2 - x_{2c})\kappa_3(x_1) \quad (5.75)$$

- axial force

$$N_1 = \int_A [E\bar{\epsilon}_1(x_1) - E(x_2 - x_{2c})\kappa_3(x_1)]\, dA \qquad \Rightarrow N_1 = S\bar{\epsilon}_1$$

$$= \underbrace{\left[\int_A E\, dA\right]}_{S \text{ (axial stiffness)}} \bar{\epsilon}_1(x_1) - \underbrace{\left[\int_A E(x_2 - x_{2c})\, dA\right]}_{= \int_A E x_2\, dA - x_{2c}\int_A E\, dA = S_3 - S x_{2c} = 0} \kappa_3(x_1)$$

- bending moment

$$M_3^c = -\int_A (x_2 - x_{2c})[E\bar{\epsilon}_1(x_1) - E(x_2 - x_{2c})\kappa_3(x_1)]\, dA \qquad \Rightarrow M_3^c = H_{33}^c \kappa_3$$

$$= -\underbrace{\left[\int_A E(x_2 - x_{2c})\, dA\right]}_{= 0} \bar{\epsilon}_1(x_1) + \underbrace{\left[\int_A E(x_2 - x_{2c})^2\, dA\right]}_{H_{33}^c \text{ (bending stiffness)}} \kappa_3(x_1)$$

⇒ "decoupled sectional constitutive law"

2 crucial steps { ① displacement field must be in the form of Eq. (5.73)
② bending moment must be evaluated w.r.t the centroid

··· Thus, centroid plays a crucial role.

5.6.3 Equilibrium eqn.s

∘ Fig. 5.47 --- infinitesimal slice of the beam of length dx_1

- force equilibrium in horizontal dir.

$$\frac{dN_1}{dx_1} = -p_1 \quad \Rightarrow \quad (5.68)$$

- vertical equilibrium

$$\frac{dV_2}{dx_1} = -p_2 \qquad \Rightarrow \quad (5.76a)$$

- equilibrium of moments about the centroid

$$\frac{dM_3}{dx_1} + V_2 = \underbrace{(x_{2a} - x_{2c})}_{\text{moment arm of the axial load w.r.t the centroid}} p_1 \qquad (5.77)$$

5.6.4 Governing eqns

$$\begin{cases} \dfrac{d}{dx_1}\left[S\,\dfrac{d\bar{u}_1}{dx_1}\right] = -p_1(x_1) & (5.78a) \Rightarrow (5.19) \\[2ex] \dfrac{d^2}{dx_1^2}\left[H_{33}^c\,\dfrac{d^2\bar{u}_2}{dx_1^2}\right] = p_2(x_1) + \dfrac{d}{dx_1}\left[(x_{2a}-x_{2c})\,p_1(x_1)\right] & (\text{〃}\ b) \end{cases}$$

$\Rightarrow$ almost similar to (5.40) except $\dfrac{d}{dx_1}\left[(x_{2a}-x_{2c})\,p_1(x_1)\right]$

... "decoupled" eqns.
$$\left.\begin{cases} (5.78a) \rightarrow \bar{u}_1(x_1) \\ (\text{〃}\ b) \rightarrow \bar{u}_2(x_1) \end{cases}\right\}$$ can be independently solved

{ If axial loads are applied @ centroid, extension and bending are "decoupled"
  〃 not 〃 〃 , 〃 "coupled"