

chap. 8 thin-walled beams

- typical aeronautical structures --- light-weight, thin-walled, beam-like structures \leftarrow complex loading environment
- combined axial, bending, shearing, torsional loads
- closed or open sections, or a combination of both
 - profound implications for the structural response (shearing and torsion)
- thin-walled beams --- specific geometric nature of the beam will be exploited to simplify the problem's formulation and solution process

Fig. 8.1 ~ 8.4 --- 8.1: closed section

8.2: open section

8.3: combination of both

8.4: multi-cellular section

8.1 Basic agns for thin-walled beams

C : geometry of the section, along the mid-thickness of the wall

s : length along the contour, orientation along C

$t(s)$: wall thickness

multi-cellular sections ... a number of different curves

8.1.1 the thin wall assumption

- wall thickness is assumed to be much smaller than the other representative dimensions
- In Fig. 8.1, $\frac{t(s)}{b} \ll 1$, $\frac{t(s)}{h} \ll 1$, or $\frac{t(s)}{\sqrt{b^2+h^2}} \ll 1$ (A.1)
- the thin-walled beam must also be long to enable the beam theory to be a reasonable approximation

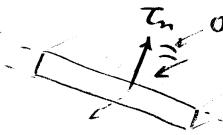
$$\frac{\sqrt{b^2+h^2}}{L} \ll 1$$

σ.1.2 stress flows

- As in Sec. 5.4.2, 5.5.2, the stress components acting in the plane of the x_2 - x_3 are assumed to be negligible as compared to the others.
- $\sigma_2 \ll \sigma_1, \sigma_3 \ll \sigma_1; T_{12} \ll T_{11}, T_{23} \ll T_{11}$
- only non-vanishing components ... $\left. \begin{array}{l} \text{axial stress } \sigma_1 \\ \text{transverse shear stress } T_{12}, T_{13} \end{array} \right\}$
- it is preferable to use the stress components parallel and normal to C
- T_s, T_n (Fig. A.5), rather than Cartesian components

$$T_n = \cos \alpha T_{12} + \sin \alpha T_{13} = T_{12} \frac{dx_3}{ds} - T_{13} \frac{dx_2}{ds} \quad (\text{A.2a})$$

$$T_s = -\sin \alpha T_{12} + \cos \alpha T_{13} = T_{12} \frac{dx_2}{ds} + T_{13} \frac{dx_3}{ds} \quad (\text{A.2b})$$

$$\cos \alpha = \frac{dx_3}{ds}, \sin \alpha = \underbrace{\frac{-dx_2}{ds}}_{\text{sign convention for } s} \quad (\text{Trig. A.5})$$


- principle of reciprocity of shear stress (Eq. (1.5)) → normal shear stress
- T_n must vanish at the two edges of the wall because the outer surface are stress free.
- no appreciable magnitude of this st. stress component can build up since the wall is very thin

⇒ $\{ T_n \text{ vanishes through the wall thickness}$

The only non-vanishing shear stress component: T_s , tangential shear stress

- Inverting Eqs. (A.2a), (A.2b), and $T_n \approx 0$,

$$T_{12} \approx T_s \frac{dx_3}{ds}, \quad T_{13} \approx T_s \frac{dx_2}{ds} \quad (\text{A.3})$$

- it seems reasonable to assume that T_s is uniformly distributed across the wall thickness (Fig. A.6) since the wall is very thin

- concept of "stress flows"

$$n(x_1, s) = \sigma_1(x_1, s) t(s) \quad \text{"axial stress flow," "axial flow"}$$

$$f(x_1, s) = T_s(x_1, s) t(s) \quad \text{"shearing," "shear flow"}$$

→ only necessary to integrate a stress flow along C , instead of over an area, to compute a force

$\delta^2.1.3$ Stress resultants

- integration over the beam's x - s area → integration along curve C

infinitesimal area of the x - s $dA = t \, ds$

- Axial force ... from Eq. (5.8)

$$N_1(x_1) = \int_A \sigma_1 \, dA = \int_C \sigma_1 t \, ds = \underbrace{\int_C n \, ds}_{\text{axial flow, Eq. (2.4a)}} \quad (\delta.5)$$

- Bending moments ... from Eq. (5.10)

$$M_2(x_1) = \int_C n x_3 \, ds, \quad M_3(x_1) = - \int_C n x_2 \, ds \quad (\delta.6)$$

- Shear forces ... from Eq. (5.9)

$$\sim V_2(x_1) = \int_C f \frac{dx_2}{ds} \, ds, \quad V_3(x_1) = \int_C f \frac{dx_3}{ds} \, ds \quad (\delta.7)$$

where Eq. (2.7), and shear flow, Eq. (2.4b), are used.

- Torque about 0

$$M_0(x_1) = \int_C \tau_p \times f \, ds$$

$\tau_p = x_2 \bar{i}_2 + x_3 \bar{i}_3$, position vector of point P (Fig. 2.7)

$ds = dx_2 \bar{i}_2 + dx_3 \bar{i}_3$, increment in curvilinear coord.

$$M_0(x_1) = \int_C (x_2 dx_3 - x_3 dx_2) f \bar{i}_1 = \int_C (x_2 \frac{dx_3}{ds} - x_3 \frac{dx_2}{ds}) f \bar{i}_1 \, ds$$

- from Fig. 2.7,

$$r_0 = x_2 \cos \alpha + x_3 \sin \alpha = x_2 \frac{dx_3}{ds} - x_3 \frac{dx_2}{ds} \quad (\delta.8)$$

- magnitude of the torque

$$M_{1.0}(x_1) = \int_C f r_0 \, ds \quad (\delta.9)$$

... torque = magnitude of the force \times perpendicular distance from the point to the line of action of the force

- Torque about an arbitrary point K of the x -s

$$M_{IK}(x_1) = \int_C f r_K ds \quad (\text{Fig. A.10})$$

r_K --- perpendicular distance from K to the line of action of the shear flow

A.1.4 Sign conventions

- the sign convention for the torque is independent of the choice of the curvilinear variable, s .

s : counterclockwise, s' : clockwise

$$\rightarrow f'(s') = -f(s), \quad r_0'(s') = -r_0(s) \quad (\text{Fig. A.9})$$

However, the resulting torque is unaffected by this choice

$$M_{IK} = \int_C f r_K ds = \int_C f' r_0' ds'$$

A.1.5 Local equilibrium eqn.

Fig. A.10 --- a differential element of the thin-walled beam

all the forces acting along axis $\bar{x}_1 \rightarrow$

$$-ndx + (n + \frac{\partial n}{\partial x_1} dx_1) ds - f dx_1 + (f + \frac{\partial f}{\partial s} ds) dx_1 = 0$$

$$\Rightarrow \frac{\partial n}{\partial x_1} + \frac{\partial f}{\partial s} = 0 \quad (\text{A.14})$$

--- any change in axial stress flow, n , along the beam axis must be equilibrated by a corresponding change in shear flow, f , along curve C that defines the x -s.

A.2 Bending of thin-walled beams

Fig. A.11 --- thin-walled beam subjected to axial forces and bending moments

E-B assumptions are applicable for either open or closed x -s

- Assuming a displacement field in the form of Eq. (6.1)

→ strain field given by Eq. (6.2a) ~ (6.2c)

... axial stresses distribution, from Eq. (6.15)

$$\sigma_1 = E \left[\frac{N_1}{S} - \frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta H} M_2 - \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta H} M_3 \right] \quad (\text{d.15})$$

$$A_H = H_{22}^c H_{33}^c - (H_{23}^c)^2$$

Using Eq. (d.4a), axial flow distribution

$$n(x_1, s) = E(s) t(s) \left[\frac{N_1(x_1)}{S} - \frac{x_2(s) H_{23}^c - x_3(s) H_{33}^c}{\Delta H} M_2(x_1) - \frac{x_2(s) H_{22}^c - x_3(s) H_{23}^c}{\Delta H} M_3(x_1) \right] \quad (\text{d.16})$$

d.3 Shearing of thin-walled beams

- bending moments in the thin-walled beams are accompanied by transverse shear

force → give rise to shear flow distributions

- evaluated by introducing the axial flow, given by Eq. (d.16) into the local equilibrium eqn., Eq. (d.14).

$$\frac{df}{ds} = -Et \left[\frac{1}{S} \frac{dN_1}{ds} - \frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta H} \frac{dM_2}{dx_1} - \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta H} \frac{dM_3}{dx_1} \right] \quad (\text{d.17})$$

- sectional equilibrium eqns., Eq. (6.16), (1.1d), (6.20) ↑
assuming that $p_1, q_2, q_3 = 0$,

$$\frac{df}{ds} = -E(s) t(s) \left[-\frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta H} V_3 + \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta H} V_2 \right] \quad (\text{d.18})$$

- Integration → shear flow distribution arising from V_2, V_3

$$f(s) = c - \int_0^s Et \left[-\frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta H} V_3 + \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta H} V_2 \right] ds \quad (\text{d.19})$$

c : integration constant corresponding to the value at $s=0$,
the procedure to determine this depends on whether $x-s$ is open or closed.

Since H_{22}^c , V_2 , V_3 are fn of π_1 alone,

$$f(s) = c + \frac{Q_3(s) H_{23}^c - Q_2(s) H_{33}^c}{A_H} V_3 - \frac{Q_3(s) H_{22}^c - Q_2(s) H_{23}^c}{A_H} V_2 \quad (\text{A.20})$$

where "stiffness static moments" or "stiffness first moment"

$$Q_2(s) = \int_0^s E x_3(t) t ds, \quad Q_3(s) = \int_0^s E x_2(t) t ds \quad (\text{A.21})$$

--- static moments for the portion of the x-s from $s=0$ to s

A.3.1 Shearing of open sections

• principle of reciprocity of shear stress, Eq. (15) \rightarrow shear flow vanishes at the end points of curve C

Fig. A.24 --- shear flow must vanish at points A and D since edges AF and DF are stress free

- If the origin of s is chosen to be located at such a stress free edge, the integration constant, c, in Eq. (A.20) must vanish.

• Procedure to determine the shear flow distribution over open x-s

① compute the location of the centroid of the x-s, and select a set of centroidal axes, \bar{x}_1 and \bar{x}_2 , and compute the sectional centroidal bending stiffnesses H_{22}^c , H_{33}^c and H_{23}^c . (principal centroidal axes $\rightarrow H_{23}^c = 0$)

② select suitable curvilinear coord. s to describe the geometry of x-s

③ evaluate the first stiffness moments using Eq. (A.21)

④ $f(s)$ is determined by Eq. (A.20)

A.3.2 Evaluation of stiffness static moments

• homogeneous, thin-walled rectangular strip oriented at an angle α (Fig. A.22)

$$Q_2(s) = \int_0^s E x_3 t ds = E \int_0^s (d_3 + s \sin \alpha) t ds = E st (d_3 + \frac{s}{2} \sin \alpha) \quad (\text{A.22})$$

-- Young's modulus \times the area st x coord. of the centroid of the local area
(i.e., area midpoint)

- similar result of the other stiffness static moment

$$Q_2(s) = E s t \left(d_2 + \frac{s}{2} \cos\theta \right) \quad (\text{Eq. 23})$$

Since the strip is made of a homogeneous material, E factors out of integral

$$\rightarrow Q_2(s) = E \underbrace{\int_c^s x_3 t ds}_C \quad \text{area static moment}$$

thin-walled homogeneous circular ar of radius R (Fig. 8.23)

$$ds = R d\theta,$$

$$Q_2(s) = \int_0^s E x_3 t ds = Et \int_0^\theta (d_2 + R \sin\theta) R d\theta = Et R^2 \left(\frac{d_2}{R} \theta + 1 - \cos\theta \right)$$

$$Q_2(s) = Et R^2 \left[\left(1 + \frac{d_2}{R} \right) \theta - \sin\theta \right]$$

stiffness static moment = E x area x distance to the area centroid

$$Q = EA x_{c,0}, \quad Q_2 = EA x_2$$

= "parallel axis theorem" (Sec. 6.2.1), but in this case, only the transport term remains since the static moment about the area centroid itself is zero, by definition.

8.3.3 shear flow distributions in open sections

Example 8.2 shear flow continuity conditions

- Two-wall joints --- equilibrium of forces along the beam's axis

→ $-f_1 + f_2 = 0$, or $f_1 = f_2$ --- the shear flow must be continuous at the junction J.

- Three-wall joint --- $-f_1 - f_2 - f_3 = 0$, or more generally

$$\sum f_i = 0 \quad (\text{Eq. 29})$$

--- the sum of the shear flows converging to a joint must vanish.

8.3.5 shear center for open sections

- problem is not precisely defined --- whereas the magnitudes of the transverse shear forces are given, their lines of action are not specified.

→ it is not possible to verify the torque equilibrium of the x-s.

- Definition of the shear center

~ - Trig. A.30 -- subjected to horizontal and vertical shear forces V_2, V_3
with lines of action passing through K, (x_{2K}, x_{3K}) ,
no external torque applied, $M_{IK} = 0$

- 3 equilence conditions

① integration of the horizontal component of the shear flow over the x-s
must equal the applied horizontal shear force $\rightarrow \int_c f(\frac{dx_2}{ds}) ds = V_2$
--- will be satisfied since it simply corresponds to the def. of shear force,
Eq. (A.7a)

② " vertical " vertical $\rightarrow \int_c f(\frac{dx_3}{ds}) ds = V_3$,
identical to Eq. (A.7b)

③ torque generated by the distributed shear flow is equivalent to the
externally applied torque, about the same point

--- does require the line of action of the applied shear forces
about point K, the torque, $M_{IK} = \int_c f r_K ds$ (Eq. (A.10))

torque generated by the external forces w.r.t. point K = 0,
 $M_{IK} = 0 + 0 \cdot V_2 + 0 \cdot V_3 = 0$

$$\Rightarrow M_{IK} = \int_c f r_K ds = 0$$

--- point K cannot be an arbitrary point, its cords must
satisfy the torque equilience condition

$$M_{IK} = \int_c f r_K ds = 0 \quad (\text{Eq. 39})$$

"definition of the shear center location"

- Alternative definition

- perpendicular distance from an arbitrary point A to the line of
action

$$r_a = r_o - x_{2a} \frac{dx_3}{ds} + x_{3a} \frac{dx_2}{ds}$$

(x_{2a}, x_{3a}) : coord. of point A

- subtracting this eqn. from Eq. (A.11)

$$r_k = r_a - (x_{2k} - x_{2a}) \frac{dx_3}{ds} + (x_{3k} - x_{3a}) \frac{dx_2}{ds}$$

- substituting into the torque equivalence condition. Eq. (A.39)

$$\int_c f_r ds - (x_{2k} - x_{2a}) \left[\int_c f \frac{dx_3}{ds} ds \right] + (x_{3k} - x_{3a}) \left[\int_c f \frac{dx_2}{ds} ds \right]$$

$$= \int_c f_r ds - (x_{2k} - x_{2a}) V_3 + (x_{3k} - x_{3a}) V_2 = 0$$

- torque generated about point A by the shear flow distribution $M_{1a} = \int_c f_r ds$

$$M_{1a} = \int_c f_r ds = (x_{2k} - x_{2a}) V_3 - (x_{3k} - x_{3a}) V_2 \quad (\text{A.40})$$

- moment at A due to force and moment resultants at point K

$$M_{1a} = M_{1K} + (x_{2k} - x_{2a}) V_3 - (x_{3k} - x_{3a}) V_2$$

$$\therefore M_{1K} = 0 \text{ by Eq. (A.39)}$$

Eq. (A.39), (A.40) ... torque generated by the shear flow distribution associated with transverse shear force must vanish w.r.t. the shear center

- Summary

- "a beam bends without twisting if and only if the transverse shear loads are applied at the shear center."
- "if the transverse loads are not applied at the shear center, the beam will both bend and twist."
- If the x-s features a plane of symmetry, the shear center must lie in that plane of symmetry.

Example A.8 Shear center for an angle section

- lines of actions of two resultant of the shear flow distributions, R_1 and R_2 , will intersect at point IC \rightarrow produces no torque about this point \rightarrow must then be the shear center

A.3.7 Shearing of closed sections

- same governing eqn., Eq. (A.19), still applies, but no boundary condition is readily available to integrate this eqn.
- exception --- axis of symmetry, Fig. A.34
if V_3 acts in the plane of symm., $(\bar{i}_1, \bar{i}_3) \rightarrow$ mirror image of shear flow distribution
- point A --- joint equilibrium condition, Eq. (A.29) $\rightarrow f_1 + f_2 = 0 \} \rightarrow f_1 = f_2 = 0$
symmetry condition $\rightarrow f_1 = f_2$
shear flow vanishes at A and similarly B
- Fig. A.35 --- 1st step: beam is cut along its axis at an arbitrary point
 \rightarrow "auxiliary problem", shear flow distribution $f_0(s)$
- $f_0(s)$ creates a shear strain γ_s (Fig. A.36) \rightarrow infinitesimal axial displacement du_1 ,

$$du_1 = \gamma_s ds = \frac{\gamma_s}{G} ds = \frac{f_0(s)}{Gt} ds \quad (\text{A.43})$$

- total relative axial displacement at the cut, u_0

$$u_0 = \int_c \frac{f_0(s)}{Gt} ds$$

- last step: f_c is applied to eliminate the relative axial displacement, thereby returning the section to its original, closed state (f_c = "closing shear flow")

... total shear flow $f(s) = f_0(s) + f_c(s)$

$$u_t = \int_c \frac{f_0(s) + f_c}{Gt} ds = 0 \quad (\text{A.44})$$

... displacement compatibility eqn. for the closed section

$$f_c = - \frac{\int_c \frac{f_0(s)}{Gt} ds}{\int_c \frac{1}{Gt} ds} \quad (\text{A.45})$$

- summary of the procedure

- ① forces for an auxiliary problem
- ② forces by Eq. (A.41)
- ③ $f(s) = f_0(s) + f_c(s)$

2.3.3 Shearing of multi-cellular sections

- Fig. P. 39 ... a typical wing section with 2 closed cells
 - procedure similar to that used for a single closed section must be developed
- one cut per cell is required.
- shear flow distribution in the resulting open sections is evaluated using the procedure in Sec. P. 3.1 $\rightarrow f_0(s_1), f_0(s_2), f_0(s_3)$ along C_1, C_2, C_3
- closing shear flows are applied at each cut: f_{c1} and f_{c2}
- Then, shear flow distribution: $f_0(s_1) + f_{c1}, f_0(s_2) + f_{c2}, f_0(s_3) + (f_{c1} + f_{c2})$, along C_1, C_2, C_3

2 unknown f_{c1}, f_{c2} will be evaluated by enforcing the displacement compatibility condition for each cell.

- front cell (clockwise (+)) :

$$u_{t1} = \int_{C_1} \frac{f_0(s_1) + f_{c1}}{Gt} ds_1 + \int_{C_3} \frac{f_0(s_3) + (f_{c1} + f_{c2})}{Gt} ds_3 = 0$$

- aft cell (counterclockwise (-)) :

$$u_{t2} = \int_{C_2} \frac{f_0(s_2) + f_{c2}}{Gt} ds_2 + \int_{C_3} \frac{f_0(s_3) + (f_{c1} + f_{c2})}{Gt} ds_3 = 0$$

- can be recast as a set of 2 linear eqns. for f_{c1} and f_{c2}

$$\left\{ \begin{array}{l} \left[\int_{C_1+C_3} \frac{1}{Gt} ds \right] f_{c1} + \left[\int_{C_3} \frac{1}{Gt} ds \right] f_{c2} = - \int_{C_1+C_3} \frac{f_0(s)}{Gt} ds; \\ \left[\int_{C_3} \frac{1}{Gt} ds \right] f_{c1} + \left[\int_{C_2+C_3} \frac{1}{Gt} ds \right] f_{c2} = - \int_{C_2+C_3} \frac{f_0(s)}{Gt} ds \end{array} \right.$$

• Extension to multi-cellular section with N closed cells

- ① open section by N cuts, one per cell \rightarrow shear flow distribution in open section by the procedure in Sec. 8.2.1
- ② closing shear flows are applied at each cut and displacement compatibility conditions are imposed $\rightarrow N$ simultaneous eqns.
- ③ total shear flow distribution is found by adding the closing shear flow to that for the open section

8.4 The shear center

• Chap. 6 ... assumption that transverse loads are applied in "such a way that the beam will bend without twisting"

\rightarrow more precise statement: the lines of action of all transverse loads pass through the shear center

- If the shear forces are not applied at the shear center, the beam will undergo both bending and twisting.

8.4.1 Calculation of the shear center location

• involves two linearly independent loading cases

① $(\cdot)^{[2]}$, unit shear force $V_2^{[2]} = 1$, no shear force along \vec{i}_3 , $V_3^{[2]} = 0$
 \rightarrow shear flow $f^{[2]}(s)$

② $(\cdot)^{[3]}$, $V_3^{[3]} = 1$, $V_2^{[3]} = 0 \rightarrow f^{[3]}(s)$

- from Eq. (8.7), shear forces equivalent to $f^{[2]}(s)$

$$V_2^{[2]} = \int_C f^{[2]} \frac{dx_2}{ds} ds = 1, \quad V_3^{[2]} = \int_C f^{[2]} \frac{dx_3}{ds} ds = 0 \quad (8.51)$$

- shear center location K (x_{2K} , x_{3K}): Eq. (8.10) \rightarrow

$$M_{1K} = \int_C f^{[2]} r_K ds = \int_C f^{[2]} \left(r_0 - x_{3K} \frac{dx_3}{ds} + x_{3K} \frac{dx_2}{ds} \right) ds$$

r_K : distance from K to the tangent to contour C , Eq. (8.11)

- Rearranging

$$-x_{2k} \left[\int_c f^{(2)} \frac{dx_3}{ds} ds \right] + x_{3k} \left[\int_c f^{(3)} \frac{dx_2}{ds} ds \right] = - \int_c f^{(2)} r_0 ds$$

↑ ↑
0 1 by Eq. (A.51)

$$\rightarrow x_{3k} = - \int_c f^{(2)} r_0 ds \quad (A.52)$$

$$\text{Similarly, } x_{2k} = \int_c f^{(3)} r_0 ds \quad (A.53)$$

- alternate torque equilibrium condition, Eq. (A.40)

$$x_{3k} = x_{3a} - \int_c f^{(2)} r_a ds \quad (A.54)$$

$$x_{2k} = x_{2a} + \int_c f^{(3)} r_a ds \quad (A.55)$$

(x_{2a}, x_{3a}) : coord. of an arbitrary point A

- General procedure for determination of the shear center

① Compute the x-s centroid and select a set of centroidal axes
(sometimes convenient with principal centroidal axes)

② compute $f^{(2)}(s)$ corresponding to $V_2^{(2)} = 1, V_3^{(2)} = 0$ } according to

③ $f^{(3)}(s) \quad .. \quad V_2^{(3)} = 0, V_3^{(3)} = 1$ } Sec. A.3.1 or A.3.2

④ compute the coord. of shear center using Eqs (A.52) and (A.53)
or (A.54) and (A.55)

- if the x-s exhibits a plane of symmetry, simplified.

plane (\bar{i}_1, \bar{i}_2) is a plane of symm., the s.c. must be located in that plane

$\rightarrow x_{3k} = 0$, Eq. (A.52) can be bypassed

A.5 Torsion of thin-walled beams

• Chap. 7 ... Saint-Venant's theory of torsion for x-s of arbitrary shape
sol. of PDE is required to evaluate the warping or stress
function. However, approximate sol. can be obtained for
thin-walled beams

§ 5.1 Torsion of open sections

- Sec 7.4 ... torsional behavior of beams with thin rectangular x-s
- 7.5 ... thin-walled, open x-s of arbitrary shape, shear stresses are linearly distributed through the thickness, torsional stiffness $\sim (\text{wall thickness})^3$ (Eq. (7.61)), very limited torque carrying capability

§ 5.2 Torsion of closed section

- Fig. A.50 ... thin-walled, closed x-s of arbitrary shape subjected to an applied torque, assumed to be in a state of uniform torsion, axial strain and stress components vanish $\rightarrow \epsilon(s) = 0$
- local equilibrium eqn. for a differential element, Eq. (A.14) \rightarrow

$$\frac{df}{ds} = 0 \quad (\text{A.59})$$

\rightarrow shear flow must remain const. along curve C

$$f(s) = f = \text{const.} \quad (\text{A.60})$$

- const. shear flow distribution generates a torque M_1 ,

$$M_1 = \int_C f(s) r_o(s) ds = f \underbrace{\int_C r_o(s) ds}_{2A} \quad (\text{Eq. (A.56)})$$

$$M_1 = ZAf \quad (A : \text{enclosed area by } C) \quad (\text{A.61})$$

"Bredt - Batho formula"

- shear stress T_s resulting from torque M_1 ,

$$T_s(s) = \frac{M_1}{2At(s)} \quad (\text{A.62})$$

twist rate vs. applied torque --- simple energy argument

- strain energy stored in a differential slice of the beam of length dx ,

$$dA = \left[\frac{1}{2} \int_C r_s t \, ds \right] = \left[\frac{1}{2} \int_C \frac{\tau_s^2}{G} t \, ds \right] dx_1 \quad (A.63)$$

- introducing shear stress distribution, Eq. (A.62)

$$dA = \left[\frac{1}{2} \frac{M_1^2}{4A^2} \int_C \frac{ds}{G t(s)} \right] dx_1 \quad (A.64)$$

- now done by the applied torque

$$dW = \frac{1}{2} M_1 d\Phi_1 = \left[\frac{1}{2} M_1 \frac{d\Phi_1}{dx_1} \right] dx_1 = \left[\frac{1}{2} M_1 \kappa_1 \right] dx_1 \quad (A.65)$$

[twist rate = $\frac{d\Phi_1}{dx_1}$]

- 1st law of thermodynamics ... $dW = dA$

$$\kappa_1 = \frac{M_1}{4A^2} \int_C \frac{ds}{Gt} \quad (A.66)$$

→ proportionality between M_1 and κ_1 , torsional stiffness

$$H_{11} = \frac{4A^2}{\int_C \frac{ds}{Gt}} \quad (A.67)$$

- arbitrary shaped closed x -s of const. wall thickness, homogeneous material

$$H_{11} = \frac{4Gt A^2}{l} ; \quad (l: \text{perimeter of } C) \quad (A.68)$$

... maximum $H_{11} \rightarrow$ thin-walled circular tube (maximize the numerator)

- sign convention

- A : area enclosed by curve C that defines the section's configuration

$$2A = \int_C r_0(s) ds$$

$r_0(s)$: perpendicular distance from the origin, O , to the tangent to C , its sign depends on the direction of the curvilinear variable, s

A is (+) when s describes C while leaving A to the left
(-) in the opposite

- Fig. A.50 ... $f > 0, A > 0 \rightarrow M_1 = ZAf > 0$

s' : clockwise direction, $f' = -f, A' = -A$,

$$M_1 = ZA'f' = ZAf > 0$$

8.5.3 Comparison of open and closed sections

- closed section --- shear stress is uniformly distributed through the thickness
- open " " " linear distribution "
- torsional stiffness $\propto (\text{enclosed area})^2$ for closed section, Eq. (8.67)
- " $(\text{thickness})^3$ " open " , Eq. (7.64)

- Fig. A.51 --- circular shape, thin-walled tube, of mean radius R_m

$$H_{II, \text{open}} = 2\pi G R_m t^3 / 3, \text{ Eq. (7.64)}$$

$$H_{II, \text{closed}} = 2\pi G R_m^3 t, \text{ Eq. (7.19)}$$

$$\frac{H_{II, \text{closed}}}{H_{II, \text{open}}} = 3 \left(\frac{R_m}{t} \right)^2 \quad (8.69)$$

- Maximum shear stress τ_{\max} subjected to the same torque, M_I ,

$$\tau_{\max}^{\text{open}} = G \ K_{I, \text{open}} t = G \frac{M_I t}{H_{II, \text{open}}} = \frac{3M_I}{2\pi R_m t^2}$$

$$\tau_{\max}^{\text{closed}} = R_m G K_{I, \text{closed}} = G \frac{M_I R_m}{H_{II, \text{closed}}} = \frac{M_I}{2\pi R_m^2 t}$$

$$\frac{\tau_{\max}^{\text{open}}}{\tau_{\max}^{\text{closed}}} = 3 \left(\frac{R_m}{t} \right) \quad (8.70)$$

- Example --- $R_m = 20t$

① H_{II} --- that of closed section will be 1,200 times larger than that of the open section

② τ_{\max} --- " open " 60 "

" closed " \rightarrow closed section can carry a 60 times larger torque

8.5.4 Torsion of combined open and closed sections

- presenting a combination of open and closed curves (Fig. 8.52)

- twist rate is identical for { the trapezoidal box
rectangular strips

- torques they carry - - - $\begin{cases} M_1^{\text{box}} = H_{11}^{\text{box}} \kappa_1 \\ M_1^{\text{strip}} = H_{11}^{\text{strip}} \kappa_1 \end{cases}$

- torsional stiffnesses - - - $\begin{cases} H_{11}^{\text{box}} = 4St A^2/l, \text{ Eq. (2.62)} \\ H_{11}^{\text{strip}} = Gwt^3/3, \text{ Eq. (2.64)} \end{cases}$

- total torque $M_1 = M_1^{\text{box}} + 2M_1^{\text{strip}}$

$$M_1 = H_{11}^{\text{box}} \left(1 + 2 \frac{H_{11}^{\text{strip}}}{H_{11}^{\text{box}}} \right) \kappa_1 = H_{11}^{\text{box}} \left[1 + \frac{2}{3} \frac{wl}{(b_1+b_2)^2} \left(\frac{t}{h}\right)^2\right] \kappa_1$$

- - - for thin-walled section, $\frac{t}{h} \ll 1$, $H_{11} \approx H_{11}^{\text{box}}$

→ torsional stiffness of the section is nearly equal to that of the closed trapezoidal box alone.

$$M_1^{\text{box}} = H_{11}^{\text{box}} \kappa_1 \approx H_{11}^{\text{box}} \frac{M_1}{H_{11}^{\text{box}}} = M_1$$

$$M_1^{\text{strip}} = H_{11}^{\text{strip}} \kappa_1 \approx \frac{H_{11}^{\text{strip}}}{H_{11}^{\text{box}}} M_1$$

- Max. shear stress - - - from Eqs. (2.62), (2.65)

$$\tau_{\max}^{\text{box}} = \frac{M_1^{\text{box}}}{2At} = \frac{l}{2At} M_1$$

$$\tau_{\max}^{\text{strip}} = \frac{3M_1^{\text{strip}}}{20t^2} = \frac{3}{wt^2} \frac{H_{11}^{\text{strip}}}{H_{11}^{\text{box}}} M_1$$

ratio

$$\frac{\tau_{\max}^{\text{strip}}}{\tau_{\max}^{\text{box}}} = \frac{l}{b_1+b_2} \left(\frac{t}{h}\right)$$

- - - the max. shear stress in the strip is far smaller than that in the trapezoidal box

2.5.5. Torsion of multi-cellular sections

- 4-cell, thin-walled x-s subjected to a torque M_1 (Fig. 2.53)

- only uniform torsion exists, and hence the axial stress flow vanishes

\rightarrow Eq. (A.14) reduces to $\frac{\partial f}{\partial s} = 0 \rightarrow$ shear flow is constant

• Free-body diagrams of the portion of the section

- Fig. 8.54-(1) ... axial stress flow = 0, $f_A = f_B$

- " (2) ... $f_C = f_D$

- " (3) ... $f_C + f_F + f_S - f_B = 0, \sum f_i = 0 \quad (A.71)$

--- "the sum of the shear flows going into a joint must vanish."

• const. shear flows are assumed to act in each cell of the section
(Fig. 8.55)

- Eq (A.71) $\rightarrow f^{[4]} + (f^{[3]} - f^{[4]}) + (f^{[2]} - f^{[3]}) + (-f^{[2]}) = 0$

• determination of the const. shear flow in each cell

① total torque = sum of the torques carried by each individual cell

"Bredt-Batho formula"

$$M_1 = \sum_{i=1}^N M_1^{[i]} = 2 \sum_{i=1}^N A^{[i]} f^{[i]} \quad (A.72)$$

② compatibility condition ... twist rates of the various cells are identical.

$$\chi_1^{[1]} = \chi_1^{[2]} = \dots = \chi_1^{[i]} = \dots = \chi_1^{[N]} \quad (A.73)$$

- Eq. (A.66) \rightarrow

$$\begin{aligned} \chi_1^{[i]} &= \int_{C^{[i]}} \frac{M_1^{[i]}}{4(A^{[i]})^2} \frac{ds}{Gt} = \int_{C^{[i]}} \frac{2A^{[i]} f^{[i]}}{4(A^{[i]})^2} \frac{ds}{Gt} \\ &= \frac{1}{2A^{[i]}} \int_{C^{[i]}} \frac{f^{[i]}}{Gt} ds \end{aligned} \quad (A.74)$$

Eqs. (A.72), (A.73) ... N_{cells} eqns. for N_{cells} shear flows

8.6 Coupled bending-torsion problems

• Chap. 6 ... arbitrary x-s subjected to complex loading conditions

2 important restrictions

① no torques

② transverse shear forces are assumed to be applied in such a way that the beam will bend without twisting

→ Now can be removed

- Fig. 8.65 ... concentrated transverse load P_2 acting at the tip and at point of application, A , with coord. (x_{2a}, x_{3a}) , $p_1(x_1)$, $p_2(x_1)$, $p_3(x_1)$... distributed loads

solution procedure

- ① Compute location of the centroid, $C (x_{2c}, x_{3c})$
- ② " orientation of the principal axes of bending $\bar{i}_1^*, \bar{i}_2^*, \bar{i}_3^*$ and the principal bending stiffness (Sec. 6.6)
- ③ " location of the shear center, $K (x_{2k}, x_{3k})$ (Sec. 8.4)
- ④ " torsional stiffness (Chap. 7, or Sec. 8.5.2)
- ⑤ Solve the extensional problem, Eq. (6.20) with appropriate BC's.
- ⑥ " two decoupled bending problems Eqs. (6.31), (6.32) in principal centroidal axes of bending planes
- ⑦ " torsional problem

$$\frac{d}{dx_1^*} \left(H_{11}^* \frac{d\Phi_1^*}{dx_1^*} \right) = - [q_1^*(x_1^*) + (x_{2a}^* - x_{2k}^*) P_3^*(x_1^*) - (x_{3a}^* - x_{3k}^*) P_2^*(x_1^*)], \quad (8.76)$$

B.C. : $\dot{\Phi}_1^* = 0$ @ root

$$H_{11}^* \frac{d\Phi_1^*}{dx_1^*} = Q_1^* + (x_{2a}^* - x_{2k}^*) P_3^* - (x_{3a}^* - x_{3k}^*) P_2^* \quad @ \text{tip}$$

\rightarrow axis system defined by the principal centroidal axes of bending

\rightarrow more convenient to recast the governing eqn. in a coord. system for which axis \bar{i}_1 is aligned with the axis of a beam

- knowledge of centroid and shear center \rightarrow complete decoupling of a problem \rightarrow 4 independent problems
 - axial problem
 - Σ bending problems
 - torsional "

If no torque and all transverse loads are applied at the s.c.
 \rightarrow R.H.S. of Eq. (8.17) = 0 $\rightarrow \bar{\Phi}_2(x_1) = 0$, the beam does not twist
 If not, the beam twists, rigid body rotation $\bar{\Phi}_2(x_1)$ about the s.c.

8.7 Warping of thin-walled beams under torsion

- thin-walled beam subjected to an applied torque
 \rightarrow shear stress generated \rightarrow out-of-plane deformations, "warping"
 in x_3 ... magnitude is typically small, but dramatic effect on the torsional behavior
- particularly pronounced for non-uniform torsion of open sections
 - twist rate varies along the span
 - \leftrightarrow contrasts with Saint-Venant theory (Sec. 7.3.2)
 - uniform torsion, const. twist rate

8.7.1 Kinematic description

- Fig. 8.70 --- thin-walled beam subjected to a tip concentrated torque Q_1
- displacement field --- similar to that for Saint-Venant sol.
 --- each x -s is assumed to rotate like a rigid body about R
 ("center of twist", (x_{2r}, x_{3r})) \leftarrow unknown yet

$$u_1(x_1, s) = \underbrace{\Psi(s)}_{\text{unknown warping fn.}} \underbrace{x_1}_{\text{twist rate}} \quad (8.80a)$$

$$u_2(x_1, s) = -(x_3 - x_{3r}) \bar{\Phi}_2(x_1) \quad (8.80b)$$

$$u_3(x_1, s) = (x_2 - x_{2r}) \bar{\Phi}_2(x_1) \quad (8.80c)$$

- strain field

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1} = \Psi(s) \frac{dx_1}{ds},$$

$$\epsilon_2 = \frac{\partial u_2}{\partial x_2} = 0, \quad \epsilon_3 = \frac{\partial u_3}{\partial x_3} = 0, \quad \gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \left[\frac{\partial \Psi}{\partial x_2} - (x_3 - x_{3r}) \right] K_1, \quad (\text{A. A1})$$

$$\gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = \left[\frac{\partial \Psi}{\partial x_3} + (x_2 - x_{2r}) \right] K_1$$

- non-uniform torsion is assumed $\rightarrow \frac{dx_i}{ds} \neq 0$

\Rightarrow axial strain $\neq 0$,

in-plane strain components = 0 since rigid body rotation assumed
shear strain components ~ partial derivatives of warping fn. and
twist rate

A.7.2 Stress-strain relations

- non-vanishing components of the stress

$$\sigma_1 = E \epsilon_1 = E \Psi(s) \frac{dx_1}{ds}$$

$$T_{12} = G \gamma_{12} = \left[\frac{d\Psi}{dx_2} - (x_3 - x_{3r}) \right] G K_1 \quad (\text{A. A2})$$

$$T_{13} = G \gamma_{13} = \left[\frac{d\Psi}{dx_3} + (x_2 - x_{2r}) \right] G K_1$$

- only non-vanishing shear stress component for thin-walled beams $\rightarrow T_s$

$$T_s = T_{12} \frac{dx_2}{ds} + T_{13} \frac{dx_3}{ds}$$

$$= \underbrace{\left[\frac{\partial \Psi}{\partial x_2} \frac{dx_2}{ds} + \frac{\partial \Psi}{\partial x_3} \frac{dx_3}{ds} \right]}_{\text{total derivative of } \Psi \text{ w.r.t. } s} + \underbrace{(x_3 - x_{3r}) \frac{dx_3}{ds} - (x_2 - x_{2r}) \frac{dx_2}{ds}}_{\text{distance from the twist center to the tangent to C, Eq. (A.11)}}$$

total derivative of
 Ψ w.r.t. s

distance from the twist center
to the tangent to C, Eq. (A.11)

$$T_s = \left(\frac{d\Psi}{ds} + r_r \right) G K_1 \quad (\text{A. A3})$$

--- for open and closed sections

A.7.3 Warping of open sections

o shear stress distribution in open-section --- linearly distributed across the wall thickness and 0 along the wall mid-line

- $\tau_s = 0$ along curve C, Eq. (d.83)

$$\tau_s = \left(\frac{d\Psi}{ds} + r_r \right) \sigma_{x_1} = 0 \quad (\text{d.84})$$

- warping function relation

$$\frac{d\Psi}{ds} = -r_r = -\left(r_o - x_{2r} \frac{dx_3}{ds} + x_{3r} \frac{dx_2}{ds} \right) \quad (\text{d.85})$$

- purely geometric function, $\Gamma(s)$

$$\frac{d\Gamma}{ds} = -r_o \quad (\text{d.86})$$

- warping function

$$\Psi(s) = \Gamma s + x_{2r} x_3 - x_{3r} x_2 + c_1 \quad (\text{d.87})$$

o uniform torsion, $\frac{dK_1}{dx_1} = 0 \rightarrow$ axial strain/stress = 0

→ c_1 and (x_{2r}, x_{3r}) cannot be determined, simply represents a rigid body displacement field, does not affect the state of stress/strain

- non-uniform torsion { varying applied torque

constrained warping displacement at a boundary or

→ non-vanishing axial strain/stress although at some point acted upon by a torque alone

but, still $N_1, M_2, M_3 = 0$

- axial force $N_1 = 0$ --- Eq. (d.82a), (d.87)

$$\int_c \sigma_1 t ds = 0$$

$$\int_c E\Gamma t ds + x_{2r} \int_c E x_3 t ds - x_{3r} \int_c E x_2 t ds + c_1 \int_c Et ds = 0$$

$\underbrace{\qquad}_{\text{origin of the axis is selected}} \qquad \underbrace{\qquad}_{\text{to be at the centroid}}$

$\underbrace{\qquad}_{\text{axial stiffness}}$

$$c_1 = -\frac{1}{S} \int_c E\Gamma t ds \quad (\text{d.88})$$

8.8 Equivalence of the shear and twist centers

• S.C. ... defined by torque equipollence condition, Eq. (8.37)

center of twist ... introduced for the analysis of thin-walled beams under torsion

$$\text{Eq. (8.53)} \rightarrow \text{Eq. (8.16)}$$

$$x_{2K} = \int_c f^{(3)} r_0 ds = - \int_c f^{(3)} \frac{d\Gamma}{ds}$$

- Integrating by parts

$$x_{2K} = \int_c \Gamma \frac{df^{(3)}}{ds} ds - [f^{(3)} \Gamma]_{\text{boundary}}$$

$\stackrel{0}{\text{since}} f^{(3)} = 0 \text{ at boundary}$

by Eq. (8.58)

$$x_{2K} = - \int_c \frac{Et}{I_{b2c}} x_3 \Gamma ds = - \frac{1}{I_{b2c}} \int_c EP x_3 ds = x_{2r}$$

by Eq. (8.89)

Similarly, $x_{3K} = x_{3r}$

→ Equivalence of the shear and twist center for open sections

.. also holds for closed sections

direct consequence of Betti's reciprocity theorem, Eq. (10.11)

- bonding moment $M_2 = \int_c \sigma_1 x_3 t ds = 0$

$$\int_c E \Gamma x_3 t ds + x_{2r} \underbrace{\int_c E x_1^2 t ds}_{H_{22}^c} - x_{3r} \underbrace{\int_c E x_2 x_3 t ds}_{H_{23}^c} + c_1 \underbrace{\int_c E x_3 t ds}_{(1)} = 0$$

$$H_{23}^c = 0$$

(principal centroidal
axes of bending)

$$x_{2r} = - \frac{1}{H_{22}^c} \int_c E \Gamma x_3 t ds \quad (\text{Eq. A9})$$

- $M_3 = 0$

$$x_{3r} = \frac{1}{H_{33}^c} \int_c E \Gamma x_2 t ds \quad (\text{Eq. A90})$$

2.7.5 Warping of closed sections

• shear stress distribution ... const. through the wall thickness in closed rec.

$$T_s = \frac{M_1}{2At} = H_{11} \frac{x_1}{2At}, \quad A = \text{area enclosed by } C$$

Eq. (A.67)

- Eq. (2.83) \rightarrow

$$\frac{d\Psi}{ds} = \frac{T_s}{Gx_1} - r_r = \frac{H_{11}}{2AGt} - r_r \quad (\text{Eq. A94})$$

--- governing eqn. for $\Psi(s)$ in closed sections

- process of integration of Eq. (A.94) ... close to that for open sections

① purely geometric function $\Gamma(s)$

$$\frac{d\Gamma}{ds} = \frac{H_{11}}{2AGt} - r_o \quad (\text{Eq. A95})$$

arbitrary B.C. is used to integrate Eq. (A.95)

② c_1 and (x_{2r}, x_{3r}) can be determined by the vanishing of F_1, M_2, M_3

2.7.6. Warping of multi-cellular sections

• Sec. 2.5.5 ... sh flow distribution, $f(s)$, due to applied torque

$$f(s) = G(s) x_1, \quad T_s = G(s) \frac{x_1}{t}$$

- governing eqn. for the warping function

$$\frac{d\Psi}{ds} = \frac{G(s)}{Gt} - r_r \quad (\text{Eq. A97})$$

- determination of the warping function --- exactly mirrors that for open and closed sections, except the following

$$\frac{d\Gamma}{ds} = \frac{Gt}{Gt - r_0} \quad (\text{A.78})$$

§. 9 Non-uniform torsion

- non-uniform torsion --- both shear and axial stresses generated by differential warping \rightarrow markedly different behavior from that under uniform torsion

- axial stress distribution --- uniform across the wall thickness

axial flow $\tau_w = t\delta_1$

- Although the axial stress does not vanish, the resulting axial force and bending moment do vanish. \rightarrow local equilibrium eqn. Eq. (A.14) is not necessarily satisfied.

- For this local equilibrium to hold, a shear flow, f_w , "warping shear flow" is generated to satisfy the local equilibrium

$$\frac{\partial \tau_w}{\partial x_1} + \frac{\partial f_w}{\partial s} = 0$$

- Introducing Eq. (A.12a) for the case of open sections

$$\frac{\partial f_w}{\partial s} = -Et \bar{\Psi} \frac{d^2 K_i}{dx_1^2} \quad (\text{A.99})$$

--- can be integrated by the procedure in Sec. §. 3.

\rightarrow simple C-channel (Fig. A.75)

Question of overall equilibrium --- does the warping shear flow generate resultant transverse shear force? (No), Eq. (A.7) \rightarrow

$$V_{2W} = \int_C f_w \frac{dx_2}{ds} ds = - \underbrace{\int_C \tau_2 \frac{\partial f_w}{\partial s} ds}_{\text{integration by parts}} + [\tau_2 f_w]_{\text{boundary}}$$

$\stackrel{!!}{}$ since $f_w = 0$ at the edge of the contour

$$\text{Eq. (A.97)}, \quad V_{2W} = \frac{d^2 K_i}{dx_1^2} \int_C E \bar{\Psi} \tau_2 t ds = 0 \quad \leftarrow \text{Eq. (A.91)}$$

Similarly, $V_{3w} = 0$

- torque resultant about the shear center generated by the warping shear flow
- Eq. (A.10) \rightarrow

$$M_{1wK} = \int_c f_w \tau_K ds = - \int_c f_w \frac{d\Psi}{ds} ds \quad (\text{A.100})$$

Integrating by parts

$$M_{1wK} = \int_c \Psi \frac{df_w}{ds} ds - [f_w \Psi]_{\substack{\text{boundary} \\ 0}} \quad (\text{A.101})$$

Introducing Eq. (A.99)

$$M_{1wK} = -H_w \frac{d^2 \kappa_1}{dx_1^2}, \quad H_w = \int_c E \Psi^2 t ds \quad (\text{A.102})$$

\uparrow "warping stiffeners"

- total torque = that by the twist rate + " due to warping

$$M_{1K} = H_{11} \kappa_1 - H_w \frac{d^3 \kappa_1}{dx_1^3} \quad (\text{A.104})$$

\uparrow additional contribution
generated by from the warping shear flow, $= 0$ for uniform torsion
shear stress distribution

- equilibrium eqn. for a differential element of the beam under torsional load ... Eq. (7.15)

$$\frac{d}{dx_1} \left(H_{11} \frac{d\bar{\kappa}_1}{dx_1} - H_w \frac{d^3 \bar{\kappa}_1}{dx_1^3} \right) = -q, \quad (\text{A.105})$$

A.10 structural idealization

- Actual thin-walled beam structures ... "stringers" added to increase the bending stiffness

- can be idealized by separating the axial and shear stress carrying components into distinct entities called { stringers } sheets

{ axial stress ... assumed to be carried only in the stringers
bending sheets

- Fig 8.79 (a) --- "box beam", "L" shaped longitudinal members located away from the centroid \rightarrow much larger contribution to the bending stiffness
- (b) --- sheet-stringer idealization \rightarrow considerably simplified analysis procedure for stress distribution

8.10.1 Sheet-stringer approximation of a thin-walled beam

- Fig. A.20 --- no discrete stringers or with far smaller $x-s$ area
 \rightarrow still possible to construct a sheet-stringer model
 - Idealized structures
 - ① axial stresses are carried solely by the stringers
 - ② shear .. the sheets
 - Approach to estimate the areas of the stringers
 - ① triangular equivalence method (Sec. 6.2.) \rightarrow guarantee the same bending stiffness and centroid location
 - ② linear distribution of axial stress, $\sigma_1 = \sigma_1^{[1]} + (\sigma_1^{[2]} - \sigma_1^{[1]}) s/b$,
 - $\sigma_1^{[1]}$: stresses at point A
 - $\sigma_1^{[2]}$: .. B
 - s : local position along the contour of width b
 - \rightarrow the areas, $A_1^{[1]}$ and $A_1^{[2]}$, of the stringers need to be determined
 - 2 constraints : - axial stresses at A and B are the same as the actual
 - force and moment equivalences are maintained
 - Force equivalence
- $$F_1 = \int_0^b [\sigma_1^{[1]} + (\sigma_1^{[2]} - \sigma_1^{[1]}) s/b] t ds = \frac{1}{2} (\sigma_1^{[1]} + \sigma_1^{[2]}) bt$$
- $$= \sigma_1^{[1]} A_1^{[1]} + \sigma_1^{[2]} A_1^{[2]}$$
- bending moment equivalence
- $$M_A = \int_0^b [..] s t ds = \frac{bt^2}{6} (\sigma_1^{[1]} + 2\sigma_1^{[2]}) = b\sigma_1^{[2]} A_1^{[2]}$$

- Solution

$$A^{[1]} = \frac{bt}{6} \left(z + \frac{\sigma_1^{[1]}}{\sigma_1^{[2]}} \right), \quad A^{[2]} = \frac{bt}{6} \left(z + \frac{\sigma_1^{[1]}}{\sigma_1^{[2]}} \right) \quad (\text{d.110})$$

~ 2 special cases

① Uniform axial stress $\sigma_1^{[1]} = \sigma^{[2]} \rightarrow A^{[1]} = A^{[2]} = bt/2 \quad (\text{d.111})$

② Pure bending $\sigma_1^{[1]} = -\sigma_1^{[2]} \rightarrow " = bt/6 \quad (\text{d.112})$

- different stress distributions are considered, equivalent idealized areas need to be recomputed.

d.10.2 Axial stress in the stringers

- the same approach as developed in chap. 6,

axial stress $\sigma_1^{[r]}$ acting in the r-th stringer

$$\sigma_1^{[r]} = E^{[r]} \left[\frac{N_1}{S} + x_3^{[r]} \frac{H_{22}^C M_2 + H_{23}^C M_3}{AH} - x_2^{[r]} \frac{H_{23}^C M_2 + H_{22}^C M_3}{AH} \right] \quad (\text{d.113})$$

uniform stress is assumed in a small "lumped" area,

$$\rightarrow \text{net axial force} = A^{[r]} \sigma_1^{[r]}$$

d.10.3 Shear flow in the sheet components

- local equilibrium condition, Eq. (A.14) $\rightarrow \frac{\partial f}{\partial s} = 0$, since no axial stress $\rightarrow f = \text{const.}$ $\quad (\text{d.114})$

Stringer equilibrium

- Fig. d.81, axial equilibrium for the r-th stringer

$$(\sigma_1 + \frac{\partial \sigma_1}{\partial x_1} dx_1, -\sigma_1) A^{[r]} + f_2 dx_1 - f_1 dx_1 = 0$$

$$\Delta f^{[r]} = f_2 - f_1 = -A^{[r]} \frac{\partial \sigma_1}{\partial x_1} \quad (\text{d.115})$$

- Eq. (d.113) \rightarrow (d.115),

$$\Delta f^{[r]} = -E^{[r]} A^{[r]} \left[\frac{H_{22}^C V_2 - H_{23}^C V_3}{\Delta H} x_2^{[r]} - \frac{H_{23}^C V_2 - H_{22}^C V_3}{\Delta H} x_3^{[r]} \right] \quad (\text{d.116})$$

$$\Delta H = H_{22}^C H_{23}^C - (H_{23}^C)^2$$

- general thin-walled x-s -- shear flow distribution is governed by a differential eqn., Eq. (8.20),

sheet-stringer idealization -- .. difference eqn., Eq. (8.116)

- integration const. needs to be determined { open section -- 0 at stress-free edge
closed .. sec. 8.3.7

• shear flow resultants

- Fig. 8.12 -- curved sheet carrying a const. shear flow, f_{12} , and connecting 2 stringers, shear force resultant

$$V_3 = \int_1^2 i_3 \cdot f_{12} ds = f_{12} \int_1^2 dx_3 = f_{12} (x_3^{[2]} - x_3^{[1]})$$

similarly, $V_2 = f_{12} (x_2^{[2]} - x_2^{[1]})$

$$V_r = \sqrt{V_2^2 + V_3^2} = f_{12} \sqrt{(x_2^{[2]} - x_2^{[1]})^2 + (x_3^{[2]} - x_3^{[1]})^2} = f_{12} L_{12} \quad (8.118)$$

direction parallel to the line connecting the 2 stringers

- moment resulting from the shear flow distribution w.r.t. point O

$$M_o = \int_1^2 f_{12} s_0 ds = f_{12} \int_1^2 s_0 ds = f_{12} \int_A z dA = f_{12} Z \hat{A}$$

\hat{A} : area of the sector defined by the 2 stringers (Fig. 8.82)

- distance e of line of action from O

$$e = Z \hat{A} : \frac{f_{12}}{\sqrt{L_{12}}} = \frac{Z \hat{A}}{L_{12}} \quad (8.119)$$

8.10.4 Torsion of sheet-stringer sections

- open section --- linear shear stress distribution through thickness, inefficient at carrying torsional loads

$$H_{11} = G \frac{bt^3}{3}$$

If different thickness for individual sheets

$$H_{11} = \sum_{\text{sheets}} H_{11i} = \sum \frac{\sigma_i b_i t_i^3}{3} \quad (8.120)$$