Chap. 9 Virtual Work Principles

9.1 Introduction

- Mechanical work ··· scalar product of the force by the displacement through which it acts → scalar quantity → simpler to manipulate → very attractive

- Newton's equilibrium condition : the sum of all forces (regardless of externally applied loads, internal forces, and reaction forces) must vanish

- Analytical mechanics ··· powerful tools for complex problems
  - scalar quantities, simpler analysis procedure
  - reaction forces can often be eliminated if the work involved vanishes.
  - systematic development of procedure for approximate solutions (ex: finite element method)

- why still need Newton's formulation?
  -- to determine both magnitude and direction of all forces acting within a structure, to estimate failure condition

- Principle of tual Work (PVW) $\xrightarrow{\text{equivalent}}$ Newton's law

9.2 Equilibrium and work fundamentals

9.2.1 static equilibrium conditions

- Newton's 1st law ··· every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.
  → A particle at rest tends to remain at rest unless the sum of the externally applied force does not vanish.

→ A particle is at rest if and only if the sum of the externally applied forces vanishes.

→ A particle is in static equilibrium "

→ " $\Sigma \underline{F} = 0$ (9.1)

(1) the vector sum of all forces acting on a particle must be zero

(2) the vector polygon must be closed

(3) the component of the vector sum resolved in any coord. system must vanish. $\Sigma \underline{F} = F_1 \bar{i}_1 + F_2 \bar{i}_2 + F_3 \bar{i}_3 \rightarrow F_1 = F_2 = F_3 = 0$

- Newton's 3rd law : if particle A exerts a force on particle B, particle B simultaneously exerts on particle A a force of identical magnitude and opposite direction.

→ Two interacting particles exert on each other forces of equal magnitude, opposite direction, and sharing a common line of action.

- Euler's 1st law

- Fig. 9.1 … system consisting of N particles
  particle i subjected to an external force $\underline{F}_i$,
  $N-1$ interaction forces $\underline{f}_{ij}$, $j=1, 2, \ldots N$, $j \neq i$

Newton's 1st law →

$$\underline{F}_i + \sum_{j=1, j\neq i}^{N} \underline{f}_{ij} = 0 \qquad (9.2)$$

interaction forces … for rigid body, it will ensure the body shape remain unchanged

elastic body, stress resulting from deformation

planetary system, gravitational pull

- summation of N eqns. for N particles

$$\sum_{i=1}^{N} \underline{F}_i + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \underline{f}_{ij} = 0$$

By Newton's 3rd law,

$$\sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \underline{f}_{ij} = 0 \qquad (9.3)$$

Then,

$$\sum_{i=1}^{N} \underline{F}_i = 0 \qquad (9.4)$$

... Euler's 1st law for a system of particles

necessary condition for " to be in static equilibrium

but not a sufficient condition

- Euler's 2nd law

- taking a vector product of Eq. (9.2) by $\underline{r}_i$, then summing over all particles

$$\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i + \underbrace{\sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \underline{r}_i \times \underline{f}_{ij}}_{\shortparallel \; 0 \text{ due to Newton's 3rd law}} = 0 \qquad (9.5)$$

Then

$$\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i = \sum_{i=1}^{N} \underline{M}_i = 0 \qquad (9.6)$$

... Euler's 2nd law, } both necessary condition for the system of particles
" 1st " } to be in static equilibrium, but not a sufficient condition

9.2.2 Concepts of mechanical work

- Definition --- the work done by a force is the scalar product of the force by the displacement of its point of application

① force, displacement collinear : $\underline{F} = F\bar{u}$, $\underline{d} = d\bar{u}$, $W = Fd$

(+) if the same direction, (−) if the opposite direction

② not collinear : $W = F d \cos\theta$, θ angle between $\bar{u}$ and $\bar{v}$.

③ perpendicular : $\cos\theta = \cos\frac{\pi}{2} = 0, \quad W = 0$

- "incremental work" --- $dW = \underline{F} \cdot d\underline{r}$,

  total work --- $W = \int_{\underline{r}_i}^{\underline{r}_f} dW = \int_{\underline{r}_i}^{\underline{r}_f} \underline{F} \cdot d\underline{r}$ (9.7)

- $\underline{F} = F_1 \bar{e}_1 + F_2 \bar{e}_2 + F_3 \bar{e}_3$, $d\underline{r} = dr_1 \bar{e}_1 + dr_2 \bar{e}_2 + dr_3 \bar{e}_3$,

  $dW = \underline{F} \cdot d\underline{r} = F_1 dr_1 + F_2 dr_2 + F_3 dr_3$

- $d\underline{r} = dr\,\bar{u}$, $\underline{F} = F_{//}\,\bar{u} + F_{\perp}\,\bar{v} \rightarrow dW = (F_{//}\,\bar{u} + F_{\perp}\,\bar{v}) \cdot dr\,\bar{u}$

  $= F_{//}\, dr$

- superposition --- $\underline{F} = \underline{F}_1 + \underline{F}_2$, $dW = \underline{F} \cdot d\underline{r} = (\underline{F}_1 + \underline{F}_2) \cdot d\underline{r}$

  $= \underline{F}_1 \cdot d\underline{r} + \underline{F}_2 \cdot d\underline{r} = dW_1 + dW_2$

- why is work a quantity of interest for the static analysis?

  Ans. concept of "virtual work" that would be done by a force if it were to displace its point of application a fictitious amount.

9.3 Principle of virtual work

- PVW --- "arbitrary virtual displacement." "arbitrary test " ".
  " " fictitious " "

  "arbitrary" --- displacement can be chosen arbitrarily without any restrictions imposed on their magnitudes or orientations

  "virtual", "test", "fictitious" --- do not affect the forces acting on the particle

9.3.1 PVW for a single particle

- Fig. 9.2 -- particle in static equilibrium under a set of externally applied loads, fictitious displacement of $\underline{\delta}$

- virtual work done

$$W = [\Sigma \underline{F}] \cdot \underline{s} = 0 \qquad (9.8)$$

$\Sigma \underline{F} = 0$ due to Newton's 1st law

- assume that one of the externally applied forces, $\underline{F}_1$, is an elastic spring force. If for a real, arbitrary displacement, $\underline{d}$, the spring force will change to become $\underline{F}_1'$, the sum of externally applied forces, $\Sigma \underline{F}_i'$

  $\rightarrow \Sigma \underline{F}' \neq 0$

- for a virtual or fictitious displacement, do not affect the loads applied to the particle, it remains in static equilibrium, Eq (9.8) holds
- if Eq. (9.8) is satisfied for all arbitrary virtual displacements, then $\Sigma \underline{F}_i = 0$, and the particle is in static equilibrium.

∘ Principle 3 (PVW for a particle) --- A particle is in static equilibrium if and only if the virtual work done by the externally applied forces vanishes for all arbitrary virtual displacements.

9.3.2 Kinematically admissible virtual displacements

∘ "arbitrary virtu. displacements" --- including those that violate the kinematic constraints of the problem

- "kinematically inddmissible direction", "infeasible direction" --- $\underline{s}$ in the track example

  $\rightarrow \underline{s} = s_2 \bar{\imath}_2$ kinematically admissible

- reaction forces acts along the kinematically inadmissible direction

∘ Modified version of PVW --- "a particle is in static equilibrium if and only if the virtual work done by the externally applied forces vanishes for all arbitrary kinematically admissible virtual displacements"

  $\rightarrow$ constraint (reaction) forces are automatically eliminated

  $\rightarrow$ fewer number of eqns

9.3.3 Use of infinitesimal displacements as virtual displacements

- special notation commonly used to denote virtual displacements

$$\underline{s} = \delta \underline{u} \qquad (9.13)$$

- virtual work done by a force undergoing virtual displacement → δW

- convenient to use virtual displacements of infinitesimal magnitude

→ often simplifies algebraic developments

① Displacement dependent forces

- automatically remain unaltered

② Rigid bodies

- 2 points P, Q of a rigid body → must satisfy the rigid body dynamics

$$\underline{v}_P = \underline{v}_Q + \underline{\omega} \times \underline{r}_{QP}$$

$$\rightarrow \frac{d\underline{u}_P}{dt} = \frac{d\underline{u}_Q}{dt} + \frac{d\underline{\psi}}{dt} \times \underline{r}_{QP}$$

$$\rightarrow d\underline{u}_P = d\underline{u}_Q + d\underline{\psi} \times \underline{r}_{QP}$$

- it is possible to write

$$\delta \underline{u}_P = \delta \underline{u}_Q + \delta \underline{\psi} \times \underline{r}_{QP} \qquad (9.14)$$

... field of kinematically admissible virtual displacements for a rigid body

" δ " : virtual, fictitious displacement, leave the forces unchanged, allowed to violate the kinematic constraints

" d " : real, infinitesimal displacement, no requirement for forces, cannot violate the kinematic constraints

- $\delta \underline{\psi}$: vector quantity, but finite rotations are scalar quantity
- virtual displacements of infinitesimal magnitude greatly simplifies the treatment

9.3.4 PVW for a system of particles

- for a particle i,

$$\delta W_i = \left(\underline{F}_i + \sum_{j=1, j\neq i}^{N} \underline{f}_{ij}\right) \cdot \delta \underline{u}_i \qquad (9.15)$$

- sum of the virtual work - all particles must also vanish

--- a system of particles is in static equilibrium iff

$$\delta W = \sum_{i=1}^{N} \left\{ \left[\underline{F}_i + \sum_{j=1, j\neq i}^{N} \underline{f}_{ij} \right] \cdot \delta \underline{u}_i \right\} \qquad (9.16)$$

$= 0$ for all virtual displacements, $\delta \underline{u}_i$, $i = 1, 2, \cdots, N$

$\rightarrow$ 3N scalar eqn.s for a system of N particles $\rightarrow$ 3N D.O.F.'s

- Internal and external VW

- "internal forces" -- act and reacted within the system

"external " " --- act on the system but reacted outside the system

$$\delta W_E = \sum_{i=1}^{N} \underline{F}_i \cdot \delta \underline{u}_i$$

$$\delta W_I = \sum_{i=1}^{N} \left[\sum_{j=1, j\neq i}^{N} \underline{f}_{ij} \right] \cdot \delta \underline{u}_{ij} \qquad (9.17)$$

Eq. (9.16) $\rightarrow$ $\delta W = \delta W_E + \delta W_I = 0 \qquad (9.18)$

Principle 4 (PVW) --- A system of particles is in static equilibrium iff the sum of the virtual work done by the internal and external forces vanishes for all arbitrary virtual displacements.

- actual displacements : $W = W_E + W_I = 0 \qquad (9.19)$

- Euler's law

- virtual displacement of a particle i

$$\delta \underline{u}_i = \delta \underline{u}_0 + \delta \underline{\psi} \times \underline{r}_i \qquad (9.20)$$

$\uparrow$ virtual translation of a rigid body; $\uparrow$ virtual rotation $\rightarrow$ 6 independent virtual quantities, far few than 3N

$$\delta W = \sum_{i=1}^{N} \left\{ \left[\underline{F}_i + \sum_{j=1, j\neq i}^{N} \underline{f}_{ij} \right] \cdot (\delta \underline{u}_0 + \delta \underline{\psi} \times \underline{r}_i) \right\}$$

$$= \left(\sum_i \underline{F}_i\right) \cdot \delta \underline{u}_0 + \left(\sum_i \sum_j \underline{f}_{ij}\right) \cdot \delta \underline{u}_0$$

$$+ \sum_i \underline{F}_i \cdot (\delta \underline{\psi} \times \underline{r}_i) + \sum_i \sum_j \underline{f}_{ij} \cdot (\delta \underline{\psi} \times \underline{r}_i)$$

$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{b} \cdot (\underline{c} \times \underline{a})$

$$= \delta \underline{u}_0 \cdot \left(\sum_i \underline{F}_i\right) + \delta \underline{u}_0 \cdot \cancelto{0}{\left(\sum_i \sum_j \underline{f}_{ij}\right)}$$

due to Eqs. (9.3), (9.5)

$$+ \delta \underline{\psi} \cdot \left(\sum_i \underline{r}_i \times \underline{F}_i\right) + \delta \underline{\psi} \cdot \cancelto{0}{\left(\sum_i \sum_j \underline{r}_i \times \underline{f}_{ij}\right)}$$

$$= \delta \underline{u}_0 \cdot \underbrace{\left[\sum_i \underline{F}_i\right]}_{= 0} + \delta \underline{\psi} \cdot \underbrace{\left[\sum_i \underline{r}_i \times \underline{F}_i\right]}_{= 0}$$

← Euler's 1st, 2nd law Eqs. (9.4), (9.6)

... necessary but not sufficient cond. for static equilibrium

9.4 Principle of virtual work applied to mechanical systems

- Rigid body ... Eq. (9.20) kinematically admissible virtual displacement field (3-dimensional)

  → 2 vector eqn.s Eqs. (9.4) and (9.6), or 6 scalar eqn.s

- 2-dimensional or planar mechanism, Eq. (9.20) →

$$\delta \underline{u}_i = \delta \underline{u}_0 + \delta \phi \, \bar{i}_3 \times \underline{r}_i \qquad (9.21)$$

$\delta \underline{\psi} \to \delta \phi \, \bar{i}_3$

9.4.1 Generalized coord. and forces

- Not convenient to work with Cartesian coord. in many cases

  → will be represented in terms of N "generalized coord."

$$\underline{u} = \underline{u}(q_1, q_2, \cdots, q_N)$$

→ virtual displacement

$$\delta \underline{u} = \frac{\partial \underline{u}}{\partial q_1} \delta q_1 + \frac{\partial \underline{u}}{\partial q_2} \delta q_2 + \cdots + \frac{\partial \underline{u}}{\partial q_N} \delta q_N$$

- virtual work done by a force $\underline{F}$

$$\delta W = \underline{F} \cdot \delta \underline{u} = \underbrace{\left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_1}\right)} \delta q_1 + \left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_2}\right) \delta q_2 + \cdots + \left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_N}\right) \delta q_N$$

↳ generalized force, $Q_i = \underline{F} \cdot \dfrac{\partial \underline{u}}{\partial q_i}$ (9.22)

- Then,

$$\delta W = Q_1 \delta q_1 + Q_2 \delta q_2 + \cdots + Q_N \delta q_N = \sum_{i=1}^{N} Q_i \delta q_i \quad (9.23)$$

virtual work = generalized forces × generalized virtual displacements

- externally applied load or internal force

$$\delta W_I = \sum_{i=1}^{N} Q_i^I \delta q_i, \quad \delta W_E = \sum_i Q_i^E \delta q_i \quad (9.24)$$

- PVW, Eq. (9.18)

$$\delta W_I + \delta W_E = \sum_i Q_i^I \delta q_i + \sum_i Q_i^E \delta q_i = \sum_i [Q_i^I + Q_i^E] \delta q_i = 0$$

$$\rightarrow \quad Q_i^I + Q_i^E = 0, \quad i = 1, 2, \cdots, N \quad (9.25)$$

- if arbitrary virtual displacements, reaction forces must be included in Q_i^E

" kinematically admissible " , " are eliminated.

- pendulum with a torsional spring (Fig. 9.13)
- virtual work by the gravity load $\delta W_E = -mg\, \bar{i}_2 \cdot \delta \underline{u}_T$

$$\underline{u}_T = R(\cos\phi\, \bar{i}_1 + \sin\phi\, \bar{i}_2), \quad \delta \underline{u}_T = R(-\sin\phi\, \bar{i}_1 + \cos\phi\, \bar{i}_2)\delta\phi$$

$$\delta W_E = \underbrace{-mgR\cos\phi}_{Q_\phi^E}\, \delta\phi = Q_\phi^E\, \delta\phi$$

- virtual work done by the internal force (restoring moment of the spring)

$$\delta W_I = -k\phi\, \delta\phi = Q_\phi^I\, \delta\phi$$

- PVW ... $Q_\phi^I + Q_\phi^E = -mgR\cos\phi - k\phi = 0$

$$\cos\phi \simeq 1 \rightarrow \phi = -mgR/k$$

9.5 P.V.W applied to truss structures

- truss ... like a simple rectilinear spring of stiffness const. $k = EA/L$

bar slenderness → 100

9.5.1. Truss structures

- Elongation - displacement equations

— Fig. 9.29, displacement $\underline{\Delta} = \Delta_1 \bar{i}_1 + \Delta_2 \bar{i}_2$

e : elongation $\quad (L+e)^2 = (L_1+\Delta_1)^2 + (L_2+\Delta_2)^2$

— Δ_1, and Δ_2 small compared to the bar's length → can be linearized

$$e = \Delta_1 \frac{L_1}{L} + \Delta_2 \frac{L_2}{L} = \Delta_1 \cos\theta_1 + \Delta_2 \sin\theta \qquad (9.$$

… elongation is the projection of the relative displacement along the bar's direction

- Internal virtual work for a bar

— Fig. 9.30 … general planar tuss member.

virtual work done by the root and tip forces

$$\delta W = \underline{F}^r \cdot \delta \underline{u}^r + \underline{F}^t \cdot \delta \underline{u}^t = F\bar{b} \cdot (\delta \underline{u}^t - \delta \underline{u}^r)$$

virtual work by the internal forces

$$\delta W_I = -\underline{F}^r \cdot \delta \underline{u}^r - \underline{F}^t \cdot \delta \underline{u}^t = -F\bar{b} \cdot (\delta \underline{u}^t - \delta \underline{u}^r) \qquad ($$

virtual elongation $\delta e = \bar{b} \cdot (\delta \underline{u}^t - \delta \underline{u}^r)$

Then, Eq. (9.28) → $\quad \delta W_I = -F \delta e \qquad ($

$$\delta e = (\sin\theta \, \bar{i}_1 + \cos\theta \, \bar{i}_2) \cdot (\delta u_1^t \bar{i}_1 + \delta u_2^t \bar{i}_2 - \delta u_1^r \bar{i}_1 - \delta u_2^r \bar{i}_2)$$

$$= (\delta u_1^t - \delta u_1^r) \sin\theta + (\delta u_2^t - \delta u_2^r) \cos\theta$$

9.5.2 Solution using Newton's law

- Fig. 9.31 --- five-bar planar truss

Newton's law → equilibrium conditions at 4 joints A, B, C, D

→ total 8 scalar eqn.s ("method of joints")

$$P_A - F_{AD} = 0\,, \qquad H_A + F_{AB} = 0 \qquad (9.31)$$

$$\vdots \qquad\qquad \vdots$$

– isostatic problem, 5 bar axial force + 3 reaction forces

9.5.3 Solution using kinematically admissible virtual displacements

- Eq. (9.31) → 5 corresponding to equilibrium in an unconstrained direction, multiplied by virtual displacements (kinematically admissible)

$$[P_A - F_{AD}]\,\delta u_1^A + [P_B - F_{BC} - F_{BD}\sin\theta]\,\delta u_1^B \qquad (9.32)$$
$$+[-F_{AB} - F_{BD}\cos\theta]\,\delta u_2^B + [F_{BC}]\,\delta u_1^C + [P_C - F_{CD}]\,\delta u_2^C = 0$$

- Regrouping

$$\overbrace{P_A\,\delta u_1^A + P_B\,\delta u_1^B + P_C\,\delta u_2^C}^{\delta W_E} \underbrace{- F_{AB}\,\delta u_2^B - F_{AD}\,\delta u_1^A - F_{BC}(\delta u_1^B - \delta u_1^C) - F_{BD}(\delta u_1^B\sin\theta + \delta u_2^B\cos\theta) - F_{CD}\,\delta u_2^C}_{} = 0 \qquad (9.33)$$

$$\delta W_I = - F_{AB}\,\delta e_{AB} - F_{AD}\,\delta e_{AD} - F_{BC}\,\delta e_{BC} - F_{BD}\,\delta e_{BD} - F_{CD}\,\delta e_{CD} \qquad (9.35)$$

$$\rightarrow \quad \delta W = \delta W_E + \delta W_I = 0 \qquad (9.36)$$

Principle 5 (PVW) A structure is in static equilibrium iff the sum of the internal and external virtual work vanishes for all kinematically admissible virtual displacements.

9.5.4 Solution using arbitrary virtual displacements

- Eq. (9.31) → 8 equilibrium multiplied by a virtual displacement

$$[P_A - F_{AD}]\,\delta u_1^A + [H_A + F_{AB}]\,\delta u_2^A + [P_B - F_{BC} - F_{BD}\sin\theta]\,\delta u_1^B \qquad (9.37)$$
$$+ \cdots$$

- Regrouping

$$\overbrace{P_A\,\delta u_1^A + P_B\,\delta u_1^B + P_C\,\delta u_2^C + H_A\,\delta u_2^A + V_D\,\delta u_1^D + H_D\,\delta u_2^D}^{\delta W_E} \qquad (9.38)$$
$$\underbrace{- \cdots}_{\delta W_I}$$

Principle 6 (PVW) A structure is in static equilibrium iff the sum of the internal and external work vanishes for all virtual displacements.

9.6 Principle of complementary virtual work

∘ Fig. 9.33 --- basic eqn.s of linear elasticity (Chap. 1)

→ 3 groups { equilibrium eqns, strain-displacement relationships, constitutive laws }

- strain compatibility eqns --- do not form an independent set of eqns and are not required to solve elasticity problems
- However, it is a over-determined problem since 6 strain components are expressed in terms of 3 displacement components only

∴ Solution of any elasticity problem requires 3 groups of basic eqn.s (Fig. 9.33)

→ PVW alone does not provide enough information to solve the problems

⇒ PCVW will augment equilibrium eqns and constitutive laws to derive complete solutions, entirely equivalent to the compatibility eqns

9.6.1 Compatibility eqn.s for a planar truss

∘ Compatibility conditions

- Fig. 9.34 ··· 2-bar truss, arbitrary elongations e_A, e_C

  configuration of the truss compatible with these elongations is easily found

  → intersection of 2 circles (of radii $L_A + e_A$, $L_C + e_C$) → O′

- Fig 9.35 ··· 3-bar truss, again arbitrary elongations e_A, e_C

  but configuration of bar B is now uniquely defined, since it must join B and O′

$e_B = L_B' - L_B$, 3 elongations are no longer independent

- same conclusion can be reached by the elongation - displacement relationship instead of the geometric reasoning

elongation ... projection of displacement vector along bar's direction, Eq (9.27)

$$e_A = u_1 \cos\theta + u_2 \sin\theta, \qquad e_C = u_1 \cos\theta - u_2 \sin\theta \qquad (9.43)$$

... for a 2-bar truss, final configuration is uniquely determined if the 2 displacement components, u_1 and u_2, are given.

- 3-bar truss (Fig. 9.35)

$$e_A = u_1 \cos\theta + u_2 \sin\theta, \quad e_B = u_1, \quad e_C = u_1 \cos\theta - u_2 \sin\theta \qquad (9.44)$$

→ it is not possible to express the 2 displacement components in terms of 3 elongations. Because 3 elongations form an over-determined set for the 2 unknown displacement components

- however, it is possible to eliminate 2 displacement components to obtain the compatibility eqn.

$$e_A - 2 e_B \cos\theta + e_C = 0 \qquad (9.45)$$

... 3 elongations in terms of 2 displacement components → 1 compatibility condition

- 2-bar truss ... isostatic, order of redundancy, # of compatibility eqn. = 0

3 " ... hyperstatic, # of compatibility eqn. = order of redundancy of the hyperstatic problem

- 3-bar truss ... 3 force components, 2 equilibrium eqn.s → hyperstatic of degree 1

||

3 elongations, 2 displacement components → 1 compatibility eqn.s

9.6.2 PCVW for trusses

- 3-bar truss under applied load

  - Fig. 9.36 ... assumed to undergo compatible deformations so that the 3 bar elongations satisfy the elongation-displacement relationship, Eq. (9.44)

$$\delta W' = -[e_A - u_1\cos\theta - u_2\sin\theta]\,\delta F_A - [e_B - u_1]\,\delta F_B - [e_C - u_1\cos\theta + u_2\sin\theta]\,\delta F_C = 0 \qquad (9.46)$$

"complementary VW"   "virtual forces"

$$\delta W' = -e_A\,\delta F_A - e_B\,\delta F_B - e_C\,\delta F_C + u_1(\delta F_A\cos\theta + \delta F_B + \delta F_C\cos\theta) + u_2\sin\theta(\delta F_A - \delta F_C) = 0 \qquad (9.47)$$

  - free body diagram → equilibrium eqns

$$F_A\cos\theta + F_B + F_C\cos\theta = P, \quad F_A - F_C = 0$$

  ... A set of forces that satisfies these equilibrium eqns is said to be "statically admissible"

  - "statically admissible virtual forces"

$$\begin{cases} \delta F_A\cos\theta + \delta F_B + \delta F_C\cos\theta = 0 \\ \delta F_A - \delta F_C = 0 \end{cases} \qquad (9.48)$$

  ... do not include the externally applied loads since $\delta P = 0$, geometry of the system is given → $\delta\theta = 0$

  - Eq. (9.47) becomes much simpler due to Eq. (9.48)

$$\delta W' = -e_A\,\delta F_A - e_B\,\delta F_B - e_C\,\delta F_C = 0 \qquad (9.49)$$

for all statically admissible virtual forces

  - Eq. (9.49) ... "internal complementary VW"

$$\delta W_I' = -e_A\,\delta F_A - e_B\,\delta F_B - e_C\,\delta F_C = -\sum_{i=1}^{N_b} e_i\,\delta F_i \qquad (9.50)$$

Eq. (9.49) $\rightarrow$ $\delta W' = \delta W_I' = 0$ (9.51)

for all statically admissible virtual forces

- 3-bar truss under prescribed displacement

- Fig. 9.37 --- instead of a concentrated load, downward vertical displacement is prescribed of magnitude Δ.

- "driving force" D required to obtain the specified displacement, as yet unknown.

Eq. (9.46) $\rightarrow$
$$\delta W' = -[e_A - u_1\cos\theta - u_2\sin\theta]\,\delta F_A - [e_B - u_1 + \Delta]\,\delta F_B - [e_C - u_1\cos\theta + u_2\sin\theta]\,\delta F_C = 0 \quad (9.52)$$
$$= -e_A\,\delta F_A - e_B\,\delta F_B - e_C\,\delta F_C - \Delta\,\delta F_B + u_1(\cos\theta\,\delta F_A + \delta F_B + \cos\theta\,\delta F_C) + u_2\sin\theta(\delta F_A - \delta F_C) = 0 \quad (9.53)$$

- set of statically admissible virtual forces that satisfy the following equilibrium eqns

$$\underbrace{\delta F_A\cos\theta + \delta F_B + \delta F_C\cos\theta = 0, \quad \delta F_A - \delta F_C = 0}_{\text{equilibrium at joint O}}, \quad \underbrace{\delta F_B + \delta D = 0}_{\text{" B}} \quad (9.54)$$

- If the virtual forces are required to be statically admissible, Eq. (9.53) will be simpler

$$\delta W' = \underbrace{\Delta\,\delta D}_{\text{external complementary VW}} \underbrace{- e_A\,\delta F_A - e_B\,\delta F_B - e_C\,\delta F_C}_{\delta W_I'} = 0 \quad (9.55)$$

$\hookrightarrow$ $\delta W_E' = \Delta\,\delta D$ (true displacement × virtual (9.56)

Eq. (9.55) $\rightarrow$ $\delta W' = \delta W_E' + \delta W_I' = 0$ (9.57)

for all statically admissible virtual forces

Principle 7 (PCVW) A truss undergoes compatible deformations iff the sum of the internal and external complementary VW vanishes for all statically admissible virtual forces.

- if CVW is required to vanish for all arbitrary virtual forces, i.e., for all independently chosen arbitrary δF_A, δF_B, δF_C, δD

  → Eq. (9.55) → $e_A = e_B = e_C = \Delta = 0$ → truss cannot deform

  → NOT correct

- For a statically admissible virtual force, must satisfy Eq. (9.54),

  3 eqns for 4 statically admissible virtual forces

  → possible to express 3 of the virtual forces in terms of the 4th:

  $\delta F_B = -2\delta F_A \cos\theta$, $\delta F_C = \delta F_A$, $\delta D = 2\delta F_A \cos\theta$

  → PCVW : $\delta W' = \Delta(2\delta F_A \cos\theta) - e_A \delta F_A - e_B(-2\delta F_A \cos\theta) - e_C \delta F_A$

  $$= [2\Delta\cos\theta - e_A + 2e_B\cos\theta - e_C]\,\delta F_A = 0$$

  $\underbrace{}_{}$ 0 ... compatibility eqn.

9.6.3 CVW

CVW : work done by virtual forces acting through real displacements
VW : " real " " virtual "

↑ real quantities remain fixed

- Fig. 9.37 ... not necessarily linear elastic material

$$W = \int_0^u ku\,du = \frac{1}{2}ku^2 = \frac{1}{2}Fu$$

$\longrightarrow$ linearly elastic material

$$W' = \int_0^F \frac{F}{k}\,dF = \frac{1}{2k}F^2 = \frac{1}{2}Fu = W$$

only when linearly elastic material

Fig. 9.38 ... shaded areas for "VW" and "CVW"

9.6.4 Applications to trusses

- planar truss with a number of bars connected at N nodes
  - PVW --> 2N equilibrium eqns

  PCVW --> n eqns produced for a hyperstatic truss of order n

for an isostatic truss, no compatibility eqns.

- PCVW ... enables the development of the force method,
  - in general, $n \ll N$, only a few eqns generated → simpler solution procedure
  - But major drawback ... must be statically admissible virtual forces, self-equilibrating, requires much more extensive work for generation of the eqn.s → PVW is used much more widely used.

9.6.6 Unit load method for trusses

- PCVW → "unit load method" ... determine deflections at specific points of structures
- Fig. 9.40 ... 2-bar truss
  PCVW, imagine the displacement Δ prescribed at O,
  external complementary work $\delta W_E' = \Delta \delta D$, δD : virtual driving force
- PCVW → $\delta W_E' + \delta W_I' = 0$, $\Delta \delta D = -\delta W_I'$ (9.62)
  for all statically admissible virtual forces
- Internal CVW : $\delta W_I' = -e_A \delta F_A - e_C \delta F_C$
  Then, Eq. (9.62) → $\Delta \delta D = e_A \delta F_A + e_C \delta F_C$
  For a more general truss consisting of N_b bars,
  $$\Delta \delta D = \sum_{i=1}^{N_b} e_i \delta F_i \qquad (9.63)$$
  for all s " a " v " f.,
  ... δD, δF_A, δF_C ... a set of s.a.v.f., free body diagrams
  → $\delta F_A - \delta F_C = 0$, $\delta D - (\delta F_A + \delta F_C) \cos\theta = 0$
  ... 2 equilibrium eqns of the system linking 3 virtual forces
- Unit load method ... the virtual driving force is selected to be a unit load, $\delta D = 1$ → $\delta F_A = \delta F_C = \Delta \delta D / (2\cos\theta) = 1/(2\cos\theta)$
- Simplified notation ... when $\delta D = 1$ → $\delta F_A = \hat{F}_A$, $\delta F_C = \hat{F}_C$
  Eq. (9.63) → $\Delta = \sum_{i=1}^{N_b} \hat{F}_i e_i$ (9.64)

$\bar{F}_i$: actual forces that develop due to the externally applied load must satisfy all equilibrium conditions, and the associated elongations must be compatible.

$\hat{\cdot}_i$ and the unit driving forces ⋯ a set of statically admissible forces. must satisfy the equilibrium eqns, but the associated elongations are NOT required to be compatible.

- For a linearly elastic material, $e_i = \frac{F_i L_i}{E_i A_i}$,

$$\text{Eq. (9.64)} \rightarrow \quad \Delta = \sum_i \frac{\hat{F}_i F_i L_i}{E_i A_i} \qquad (9.65)$$

- To determine rotation of the structure, → "unit moment method"

$$\Phi\, \delta M = -\delta W_I' \qquad (9.66)$$

9.7 Internal virtual work in beams and solids

9.7.1 Beam bending

- plane $(\bar{i}_1, \bar{i}_2)$ ⋯ plane of symmetry

  $M_3(x_1)$ bending moment, $\Phi_3(x_1)$ rotation, $\bar{u}_2(x_1)$ transverse displacement

- infinitesimal slice of a beam (Fig. 9.48) → curvature of the differential element $\kappa_3 = \Phi_3' = \bar{u}_2''$
- work done by the moment acting on the left-hand side : $-M_3\Phi_3$

  ($(-)$ since moment and rotation are counted $(+)$ about opposite axes)

  " other side : $M_3(\Phi_3 + d\Phi_3)$

net work done by the 2 moments : $dW = M_3 d\Phi_3 = M_3 \left(\frac{d\Phi_3}{dx_1}\right) dx_1$

total internal work done by the moment distribution acting along the beam :

$$W_I = -\int_0^L M_3 \frac{d\Phi_3}{dx_1} dx_1 = -\int_0^L M_3 \kappa_3 \, dx_1 \qquad (9.69)$$

$(-)$ due to internal moment, which is opposite to externally applied moment

· internal VW

$$\delta W_I = -\int_0^L M_3 \, \delta \kappa_3 \, dx_1 \qquad (9.70)$$

internal CVW

$$\delta W_I' = -\int_0^L \kappa_3 \, \delta M_3 \, dx_1$$

9.7.2 Beam twisting

· Fig. 9.49 ... differential rotation of 2 x-s's → twist rate of the differential element, $\kappa_1 = \Phi_1'$

- work done by the torque acting on the left-hand side : $-M_1 \Phi_1$

  ((−) due to that the torque and rotation are (+) about opposite axes)

  " other side : $M_1 (\Phi_1 + d\Phi_1)$

  net work by 2 torques : $dW = M_1 d\Phi_1 = M_1 (d\Phi_1 / dx_1) \, dx_1$

  total internal work done by the torque distribution

$$W_I = -\int_0^L M_1 \frac{d\Phi_1}{dx_1} dx_1 = -\int_0^L M_1 \kappa_1 \, dx_1 \qquad (9.71)$$

  ((−) due to the internal torque)

· internal VW

$$\delta W_I = -\int_0^L M_1 \, \delta \kappa_1 \, dx_1 \qquad (9.72)$$

internal CVW

$$\delta W_I' = -\int_0^L \kappa_1 \, \delta M_1 \, dx_1$$

based on the kinematics of Saint-Venant's theory of uniform torsion

9.7.3 Three-dimensional solid

- work done by each 6 stress components are computed separately and then are summed up.

· Axial stresses

- Fig. 9.50 ... infinitesimal differential element of a solid

  work done by the force, $\sigma_1 dx_2 dx_3$, acting on the left-hand side :

  $-(\sigma_1 dx_2 dx_3) u_1$

  ((−) due to that force and displacement are counted (+) in opposite directions) work done by the force acting on the other side : $(\sigma_1 dx_2 dx_3)(u_1 + du_1)$

net work by the 2 forces : $dW = (\sigma_1 dx_2 dx_3) du_1$

$= (\sigma_1 dx_2 dx_3)\left(\frac{\partial u_1}{\partial x_1}\right) dx_1$

- total internal work done by the axial stress distribution

$$W_I = -\int_V \sigma_1 \frac{du_1}{dx_1} dx_1 dx_2 dx_3 = -\int_V \sigma_1 \epsilon_1 dV \qquad (9.73)$$

((−) due to the internal axial stresses)

· Shear stresses

- Due to the principle of reciprocity, shear stress components will act on right, left edges, also on the top, bottom edges (Fig. 9.50 right)
- work done by the force, $\tau_{12} dx_1 dx_3$, acting on the bottom edges :

$-(\tau_{12} dx_1 dx_3) u_1$

((−) due to that force and displacement are opposite)

" top edge : $(\tau_{12} dx_1 dx_3)(u_1 + du_1)$

net work done by these 2 forces : $dW = (\tau_{12} dx_1 dx_3) du_1$

$= (\tau_{12} dx_1 dx_3)\left(\frac{\partial u_1}{\partial x_2}\right) dx_2$

- work done by $\tau_{12} dx_2 dx_3$, acting on the left edge : $-(\tau_{12} dx_2 dx_3) u_2$

((−) due to that the force and displacement are counted (+) in opposite direction.)

" right edge : $(\tau_{12} dx_2 dx_3)(u_2 + du_2)$

- net work done by these 2 forces : $dW = (\tau_{12} dx_2 dx_3) du_2$

$= (\tau_{12} dx_2 dx_3)\left(\frac{\partial u_2}{\partial x_1}\right) dx_1$

- total internal work by the shear stress distribution

$$W_I = -\int_V \tau_{12}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) dx_1 dx_2 dx_3 = -\int_V \tau_{12} \gamma_{12} dV \qquad (9.74)$$

((−) due to the internal shear stresses)

- total work done by all 6 stress components

$$W_I = -\int_V (\sigma_1\epsilon_1 + \sigma_2\epsilon_2 + \sigma_3\epsilon_3 + \tau_{23}\gamma_{23} + \tau_{13}\gamma_{13} + \tau_{12}\gamma_{12})\, dV \qquad (9.75)$$

$$= -\int_V \underline{\sigma}^T \underline{\epsilon}\, dV \qquad (9.76)$$

- internal virtual work $\quad \delta W_I = -\int_V \underline{\sigma}^T \delta\underline{\epsilon}\, dV$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (9.77)

" $\quad$ CVW $\quad \delta W_I' = -\int_V \underline{\epsilon}^T \delta\underline{\sigma}\, dV$

9.7.4 E-B beam

- viewed as a 3-dim. solid ... in E-B beam, all strain components vanish, except for the axial strain. Eq (9.75) →

$$W_I = -\int_V \sigma_1 \epsilon_1\, dV = -\int_0^L \int_A \sigma_1 (\bar{\epsilon}_1 + x_3\kappa_2 - x_2\kappa_3)\, dA\, dx_1$$

$$= -\int_0^L \Big\{ \underbrace{\Big[\int_A \sigma_1 dA\Big]}_{N_1,\ \text{by Eq. (5.8)}} \bar{\epsilon}_1 + \underbrace{\Big[\int_A \sigma_1 x_3 dA\Big]}_{M_2} \kappa_2 + \underbrace{\Big[-\int_A \sigma_1 x_2 dA\Big]}_{M_3,\ \text{by Eq. (5.10)}} \kappa_3 \Big\} dx_1$$

$$W_I = -\int_0^L (N_1\bar{\epsilon}_1 + M_2\kappa_2 + M_3\kappa_3)\, dx_1 \qquad (9.78)$$

- Interval VW $\quad \delta W_I = -\int_0^L (N_1\delta\bar{\epsilon}_1 + M_2\delta\kappa_2 + M_3\delta\kappa_3)\, dx_1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (9.79)

" $\quad$ CVW $\quad \delta W_I' = -\int_0^L (\bar{\epsilon}_1\delta N_1 + \kappa_2\delta M_2 + \kappa_3\delta M_3)\, dx_1$

9.7.6 Unit load method for beams

- if Δ is prescribed at a point of the beam, PCVW, Eq. (9.57) →

$$\Delta\, \delta D + \delta W_I' = 0$$

for statically admissible virtual forces, (δD: virtual driving force)

- by Eq. (9.79b),

$$\Delta\, \delta D = \int_0^L (\bar{\epsilon}_1\delta N_1 + \kappa_2\delta M_2 + \kappa_3\delta M_3)\, dx_1 \qquad (9.80)$$

- $\delta D = 1$ and $\delta N_1 = \hat{N}_1$, $\delta M_2 = \hat{M}_2$, $\delta M_3 = \hat{M}_3$: resulting statically admissible axial forces and bending moments

$$\Delta = \int_0^L (\hat{N}_1 \bar{\epsilon} + \hat{M}_2 \kappa_2 + \hat{M}_3 \kappa_3)\, dx_1 \qquad (9.81)$$

- if { linearly elastic material
  the origin of the axis system is at the centroid of the c-s

  → sectional constitutive law, Eq. (6.13) can be applicable

$$\Delta = \int_0^L \left[\frac{\hat{N}_1 N_1}{S} + \frac{\hat{M}_2 (H_{33}^c M_2 + H_{23}^c M_3)}{\Delta_H} + \frac{\hat{M}_3 (H_{23}^c M_2 + H_{22}^c M_3)}{\Delta_H} \right] dx_1 \qquad (9.82)$$

- if the principal axes of bending

$$\Delta = \int_0^L \left[\frac{\hat{N}_1 N_1}{S} + \frac{\hat{M}_2 M_2}{H_{22}^c} + \frac{\hat{M}_3 M_3}{H_{33}^c} \right] dx_1 \qquad (9.83)$$

9.8 Application of the unit load method to hyperstatic problems

∘ unit load method ··· determination of 2 sets of statically admissible forces corresponding to 2 distinct loading cases

{ ① associated with the externally applied loads
  ② 〃 the unit load

→ applies equally to iso- and hyperstatic systems

∘ hyperstatic systems { displacement or stiffness method
  force or flexibility method

↳ ... focuses on the determination of internal forces/moments and reactions

key step : development of the compatibility eqns

PCVW ··· equivalent to the compatibility eqns

→ logical to combine the force method with PCVW

- force method ... intuitively described as "method of cuts"

  for each cut, the order of the hyperstatic system is decreased by 1.

  statically admissible forces are then solely obtained from the equilibrium eqns.

- 2 crucial steps

  ① determine the relative displacements at the cuts under the externally applied loads alone

  ② evaluate the internal forces applied at the cuts that are required to eliminate the relative displacements at the cuts

  ← PCVW is a powerful tool to solve both problems

- Fig. 9.65 ... single bar of truss
  - R : set of self-equilibrating forces applied at the cut
  - C external VW $\delta W_E' = d_1\,\delta R - d_2\,\delta R = (d_1 - d_2)\,\delta R$

    relative displacement at the cut : $\Delta = d_1 - d_2$
  - PCVW, Eq. (9.57) ... $\delta W_E' + \delta W_I' = 0$

$$\rightarrow \quad \Delta\,\delta R = -\delta W_I' \qquad (9.84)$$

... very similar to Eq. (9.62), but Δ : relative displacement at the cut, δR : set of self-equilibrating virtual forces applied at the cut

- Right of Fig. 9.65 ... a cantilevered beam
  - C external VW : $\delta W_E' = \theta_1\,\delta M - \theta_2\,\delta M = (\theta_1 - \theta_2)\,\delta M$
  - PCVW →

$$\Phi\,\delta M = -\delta W_I' \qquad (9.85)$$

9.8.1 Force method for trusses

Fig. 9.66 ... 3-bar hyperstatic truss, hyperstatic system of order 1, a single cut is applied at the middle bar

Then, the actual system is viewed as a superposition of 2 problems.

① isostatic system subjected to the externally applied loads

- unit load method is directly applicable

$$\Delta_c = \sum_{i=1}^{N_b} \frac{\hat{F}_i F_i L_i}{(EA)_i} \qquad (9.86)$$

where F_i : bar forces subjected to the externally applied loads

$\hat{F}_i$: statically admissible virtual forces corresponding to the self-equilibrating unit load system applied at the cut

$$F_A = F_C = P/(2\cos\theta), \quad F_B = 0$$

$$\hat{F}_A = \hat{F}_C = -1/(2\cos\theta), \quad \hat{F}_B = 1$$

$$\rightarrow \quad \Delta_c = -\left(\frac{1}{(EA)_A} + \frac{1}{(EA)_C}\right)\frac{PL}{4\cos^3\theta}$$

② internal force system ... relative displacement at the cut, Δ_1, due to a unit internal force in bar B

- Eq. (9.84) $\rightarrow$

$$\Delta_1 = \sum_{i=1}^{N_b} \frac{\hat{F}_i^2 L_i}{(EA)_i} \qquad (9.87)$$

$$\Delta_1 = \frac{L}{(EA)_B\, 4\cos^3\theta} \; \frac{\bar{k}_A + \bar{k}_C + 4\bar{k}_A\bar{k}_C\cos^3\theta}{\bar{k}_A\bar{k}_C}$$

③ superposition of 2 loading cases

- compatibility condition at the cut

$$\Delta_c + R\Delta_1 = 0 \qquad (9.88)$$

$$\rightarrow \quad R = -\frac{\Delta_c}{\Delta_1} = \frac{\bar{k}_A + \bar{k}_C}{\bar{k}_A + \bar{k}_C + 4\bar{k}_A\bar{k}_C\cos^3\theta} P \qquad (9.89)$$

- bar forces

$$F_i + R\hat{F}_i, \quad i = 1, 2, \ldots, N_b \qquad (9.90)$$

9.8.2 Force method for beams

- beam structures become hyperstatic due to the presence of multiple supports

- Fig. 9.70 ... cantilevered beam w/ additional mid-span support

→ additional reaction R

- eliminating or cutting the appropriate No. of supports to render the beam isostatic

i) Δ_c is computed by unit load method, Eq. (9.83)

$$\Delta_c = \int_0^L \frac{M_3 \hat{M}_3}{H_{33}^c} dx_1 \qquad (9.91)$$

$M_3(x_1)$: bending moment distribution in the isostatic beam subjected to the externally applied loads

$\hat{M}_3(x_1)$: statically admissible bending moment "

a set of self-equilibrating unit forces applied at the support

ii) Δ_1 : relative deflection at the support due to a set of self-equilibrating, unit loads. Eq. (9.84)

$$\Delta_1 = \int_0^L \frac{\hat{M}_3^2}{H_{33}^c} dx_1 \qquad (9.92)$$

iii) displacement compatibility eqn. at the support

$$\Delta_c + R\Delta_1 = 0 \qquad (9.93)$$

$$R = -\frac{\Delta_c}{\Delta_1} \qquad (9.94)$$

→ reaction forces $F_A + R\hat{F}_A$ @ the root

bending moments $M_A + R\hat{M}_A$

" distribution $M_3(x_1) + R\hat{M}_3(x_1)$

- Alternative way to eliminate the support (or "releasing one constraint")

--- replacement of the root clamp by a simple support (Fig. 9.71)

i) Φ_c : relative root rotation in the isostatic structure, Eq. (9.85)

ii) Φ_1 :

iii) root rotation compatibility eqn. : $\Phi_c + M_A \Phi_1 = 0$, $M_A = -\Phi_c/\Phi_1$