ENGINEERING MATHEMATICS II

010.141

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MODULE 1: Linear Algebra, Matrices, Vectors, Determinants

NOTATIONS AND CONCEPTS

- In the text, matrices are denoted by capital boldface letters, such as **A**; entries at the intersection of the i-row and j-column are denoted by a_{ij} .
- A $m \times n$ matrix means m-rows and n-columns; a $n \times n$ matrix is called ***square***.
- In a square matrix the elements a_{11} down to a_{nn} are said to be on the ***main diagonal***.

MORE DEFINITIONS AND CONCEPTS

- Vector: a matrix with either one row or one column. Entries are called ***components***.
- Row or column vectors can be converted to column or row vectors by ***transposition***; transposition is indicated on matrices by a T superscript, such as $\mathbf{A}^{T}$; this means write the matrix with rows interchanged for columns.

MORE CONCEPTS

- Suppose the matrix **A** has m rows and n columns; then the transposed matrix $\mathbf{A}^\mathbf{T}$ has n rows and m columns, with the rows and columns of **A** interchanged.

$$\mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}^\mathbf{T} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

MATRIX EQUALITY

- Two matrices are equal, provided they are of the same size (same number of rows in each, and same number of columns in each) and provided that corresponding entries are identical.
- Thus $\mathbf{A} = \mathbf{B}$ iff $a_{ij} = b_{ij}$.
- Two matrices can be added or subtracted, provided they are of the same size; corresponding elements are thus added.

SCALAR MULTIPLICATION

- A matrix **A** may be multiplied by a scalar c by multiplying each element by c.
- Matrix addition is commutative and associative
- The zero matrix has the real number 0 at every element location; thus matrix addition has the zero property: **A** + 0 = **A**
- Matrix addition has the inverse property: **A** + [- **A**] = 0

ADDITION AND SCALAR MULTIPLICATION OF VECTORS

- **Problem 1**

$$7\mathrm{A}-5\mathrm{B}$$

where

$$\mathrm{A}=\begin{bmatrix} 3 & 0 & 4 \end{bmatrix}$$
$$\mathrm{B}=\begin{bmatrix} -1 & 8 & 2 \end{bmatrix}$$

$$7\mathrm{A}=\begin{bmatrix} 21 & 0 & 28 \end{bmatrix}$$
$$5\mathrm{B}=\begin{bmatrix} -5 & 40 & 10 \end{bmatrix}$$

$$7\mathrm{A}-5\mathrm{B}=\begin{bmatrix} 26 & -40 & 18 \end{bmatrix}$$
$$7\mathrm{A}^{\mathrm{T}}-5\mathrm{B}^{\mathrm{T}}=\begin{bmatrix} 26 \\ -40 \\ 18 \end{bmatrix}$$

ADDITION AND SCALAR MULTIPLICATION OF VECTORS

- **Problem 2**

$$5(C-2D)=?$$

$$C=\begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix} \qquad D=\begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}$$

$$C-2D=\begin{bmatrix} 5 \\ 9 \\ -5 \end{bmatrix} \qquad 5(C-2D)=\begin{bmatrix} 25 \\ 45 \\ -25 \end{bmatrix}$$

$$5(2D-C)=?$$

$$=\begin{bmatrix} 20 \\ -20 \\ 60 \end{bmatrix}-\begin{bmatrix} 45 \\ 25 \\ 35 \end{bmatrix}=\begin{bmatrix} 25 \\ -25 \\ -25 \end{bmatrix}$$

MATRIX MULTIPLICATION

- Two matrices may be multiplied, and produce a third matrix, provided they are conformable to multiplication.
- If there exists $\mathbf{A}_{mn}$ and $\mathbf{B}_{np}$ then these two matrices are conformable and the resultant matrix is of size mp, that is, $\mathbf{C}_{mp}$
- The element in the ij position of **C** is found by

$$c_{ij} = a_{il}\, b_{lj} + \cdots + a_{in}\, b_{nj}$$

SOME MULTIPLICATION EXAMPLES

- **Example**

$$\begin{bmatrix} 4 & 3 \\ 7 & 2 \\ 9 & 0 \end{bmatrix}_{3,2} \cdot \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}_{2,2} = \begin{bmatrix} 8 + 3 & 20 + 18 \\ 14 + 2 & 35 + 12 \\ 18 & 45 \end{bmatrix}$$

- **Example**

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix}_{2,2} \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix}_{2,1} = \begin{bmatrix} 12 + 10 \\ 3 + 40 \end{bmatrix}_{2,1} = \begin{bmatrix} 22 \\ 43 \end{bmatrix}$$

SOME MULTIPLICATION RULES

- Multiplication is not commutative, in general.
- **AB** can be equal to 0, but this does not imply that either **A** or **B** is necessarily 0.
- Even if it is given that **AC** = **AD**, it is not necessarily true that **C** = **D**.
- Multiplication is distributive across addition, that is,

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}.$$

ROW AND COLUMN VECTOR MULTIPLICATION

- **Row × Column**

$$\begin{bmatrix}3 & 6 & 1\end{bmatrix}_{1,3} \cdot \begin{bmatrix}1\\2\\4\end{bmatrix}_{3,1} = \begin{bmatrix}3 + 12 + 4\end{bmatrix}_{1,1}$$

- **Column × Row**

$$\begin{bmatrix}1\\2\\4\end{bmatrix}_{3,1} \cdot \begin{bmatrix}3 & 6 & 1\end{bmatrix}_{1,3} = \begin{bmatrix}3 & 6 & 1\\6 & 12 & 2\\12 & 24 & 4\end{bmatrix}_{3,3}$$

SPECIAL MATRICES

- **Upper triangular**:

  Has all zero elements below the main diagonal; example:

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$$

- **Lower triangular** has all zero above the main diagonal.

  Diagonal matrix: has nonzero elements on the main diagonal only. Matrix is square

  If all elements of a diagonal are identical, this matrix is referred to as a **Scalar Matrix**.

IDENTITY AND TRANSPOSE OPERATIONS

- If a square diagonal matrix has the element 1 at each location on the diagonal, then it is known as the identity matrix, **I**.

  For any square matrix **A** there exists an **I** such that: $\mathbf{AI} = \mathbf{A}$

- The transpose of a product of two matrices is equal to the product of the transposed factors, in the reverse order, or: $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$

MORE MATRIX MANIPULATION

- If **a** is a row vector of order n, and **b** is a column vector, also of order n, then the inner product or dot product of **a** and **b** is the scalar which is the sum of the products of the respective elements

- **Example**

$$a = \begin{bmatrix} 4 & -1 & 5 \end{bmatrix}$$

$$b = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$a \cdot b = 8 - 5 + 40 = 43 \quad \text{(a scalar)}$$

- Thus, matrix multiplication amounts to combinations of dot products of row and column vectors

MULTIPLICATION OF MATRICES BY MATRICES AND BT VECTORS

- **Problem 3**

$$B = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$BB^T = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}_{3,2} \cdot \begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}_{2,3} = \begin{bmatrix} 4+9 & -6 & -3 \\ -6 & 4 & 2 \\ -3 & 2 & 1 \end{bmatrix}_{3,3}$$

LINEAR SYSTEM OF EQUATIONS, AND GAUSS ELIMINATION

- A linear system of m equations in n unknowns, x_i, i = 1, 2, …, n may be written as:

$$\begin{aligned} a_{11}\ x_1 + \cdots + a_{1n}\ x_n &= b_1 \\ &\vdots \\ a_{m1}\ x_1 + \cdots + a_{mn}\ x_n &= b_m \end{aligned}$$

- Example of two equations in three unknowns:

$$\begin{aligned} a_{11}\ x_1 + a_{12}\ x_2 + a_{13}\ x_3 &= b_1 \\ a_{21}\ x_1 + a_{12}\ x_2 + a_{23}\ x_3 &= b_2 \end{aligned}$$

MATRIX FORM OF A LINEAR SYSTEM

- a_{jk} are called **coefficients of the system**.
- If the b_i's are all 0, the system is called **homogeneous**.
- If one $b_i \neq 0$, the system is called **non-homogeneous**.
- Solution vector $\mathbf{x}$ is a set of x_i whose components are a solution
- If $\mathbf{A}$ is the coefficient matrix and $\mathbf{x}$ is the solution vector and $\mathbf{b}$ is the column vector of the right-hand side of the set of equations, then: $\mathbf{Ax} = \mathbf{b}$ is the matrix form.

AUGMENTED MATRIX

- If the column vector **b** is written along side the coefficient matrix **A**, the result is the augmented matrix:
- The symbol "~" is used to indicate the augmented matrix.
- The augmented matrix can be useful in solving the set of equations.

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

- The Gauss method of elimination is frequently used.

SYSTEMS OF EQUATIONS

- In a system of equations there may be a unique solution, infinitely many solutions, or no solution at all.
- Gauss elimination techniques use the following principles:
  - Rule 1- rows may interchanged;
  - Rule 2- one row may be multiplied by a non-zero constant
  - Rule 3- a multiple of one row may be added to another row
  - Rule 4- if done, the result is a row-equivalent system, with the same solution (or lack of)

SOME MORE DEFINITIONS

- A linear system is over-determined if it has more equations than unknowns; under-determined if fewer equations than unknowns; determined if m = n, that is, # equations = # unknowns
- System is consistent if it has at least one solution, and inconsistent if it has no solutions
- Thus, infinite solutions; one solution, no solution

GAUSS ELIMINATION

- Gauss elimination is a standard method for solving linear systems
- Consider the system:

Step 1: Eliminate x_1

Pivot ⌛ $x_1 - x_2 + x_3 = 0$

Eliminate ⌛

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 0 \\ 20x_1 + 10x_2 &= 80 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 0 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

To obtain a

Triangular Form

GAUSS ELIMINATION (cont)

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 0 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 30x_2 - 20x_3 &= 80 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right]$$

Step 2: Eliminate x_2

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ \boxed{30x_2} - 20x_3 &= 80 \\ 0 &= 0 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

GAUSS ELIMINATION (cont)

$$
\begin{aligned}
x_1 - x_2 + x_3 &= 0 \\
10x_2 + 25x_3 &= 90 \\
-95x_3 &= -190 \\
0 &= 0
\end{aligned}
\qquad
\left[\begin{array}{ccc|c}
1 & -1 & 1 & 0 \\
0 & 10 & 25 & 90 \\
0 & 0 & -95 & -190 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

Step 3: Back-Substitution

$$
\begin{aligned}
-95x_3 &= -190 \\
10x_2 + 25x_3 &= 90 \\
x_1 - x_2 + x_3 &= 0
\end{aligned}
$$

Determination of x_3, x_2, x_1.

$$
\begin{aligned}
x_3 &= 2 \\
x_2 &= \frac{1}{10}\left(90 - 25x_3\right) = 4 \\
x_1 &= x_2 - x_3 = 2
\end{aligned}
$$

GAUSS ELIMINATION

➢ **Problem 4**

$$\begin{array}{ccccc} 6x & + & 4y & = & 2 \\ 3x & - & 5y & = & -34 \end{array}$$

Augmented matrix is:

$$\begin{array}{ccccc} 6 & + & 4 & = & 2 \\ 3 & - & 5 & = & -34 \end{array} \rightarrow \begin{array}{ccccc} 1 & + & \frac{2}{3} & = & \frac{1}{3} \\ 1 & - & \frac{5}{3} & = & -\frac{34}{3} \end{array} \rightarrow \begin{array}{ccccc} 1 & + & \frac{2}{3} & = & \frac{1}{3} \\ 0 & - & \frac{7}{3} & = & -\frac{35}{3} \end{array}$$

$$\begin{array}{ccccc} 1 & + & \frac{2}{3} & = & \frac{1}{3} \\ 0 & + & 1 & = & 5 \end{array} \rightarrow \begin{array}{ccccc} 1 & + & 0 & = & \frac{1}{3}-\frac{10}{3} \\ 0 & + & 1 & = & 5 \end{array} \rightarrow \begin{array}{ccccc} 1 & + & 0 & = & -3 \\ 0 & + & 1 & = & 5 \end{array}$$

Therefore

$$\begin{array}{ccc} x & = & -3 \\ y & = & 5 \end{array}$$

GAUSS ELIMINATION (cont)

➢ Problem 5

$$\begin{array}{rcrcl} 3\,x & - & 0.5y & = & 0.6 \\ 1.5x & + & 4.5y & = & 6 \\ \hline \end{array}$$

$$\begin{array}{ccccc} 3 & - & 0.5 & = & 0.6 \\ 1.5 & + & 4.5 & = & 6 \\ \hline \end{array} \rightarrow \begin{array}{ccccc} 1 & - & \frac{1}{6} & = & 0.2 \\ 1 & + & 3 & = & 4 \\ \hline \end{array} \rightarrow \begin{array}{ccccc} 1 & - & \frac{1}{6} & = & 0.2 \\ 0 & + & \frac{19}{6} & = & 3.8 \\ \hline \end{array}$$

$$\begin{array}{ccccc} 1 & - & \frac{1}{6} & = & 0.2 \\ 0 & + & 1 & = & \frac{6}{19}\cdot\frac{19}{5} \\ \hline \end{array} \rightarrow \begin{array}{ccccc} 1 & - & \frac{1}{6} & = & 0.2 \\ 0 & + & 1 & = & 1.2 \\ \hline \end{array} \rightarrow \begin{array}{ccccc} 1 & + & 0 & = & 0.4 \\ 0 & + & 1 & = & 1.2 \\ \hline \end{array}$$

Therefore

$$x = 0.4$$
$$y = 1.2$$

GAUSS ELIMINATION (cont)

- **Problem 6**

$$\begin{array}{rrrcr} 0 & 7 & 3 & = & -12 \\ 2 & 8 & 1 & = & 0 \\ -5 & 2 & -9 & = & 26 \end{array} \rightarrow \begin{array}{rrrcr} 1 & 4 & \frac{1}{2} & = & 0 \\ 0 & 7 & 3 & = & -12 \\ -5 & 2 & -9 & = & 26 \end{array} \rightarrow \begin{array}{rrrcr} 1 & 4 & \frac{1}{2} & = & 0 \\ 0 & 1 & \frac{3}{7} & = & -\frac{12}{7} \\ 0 & 22 & -\frac{13}{2} & = & 26 \end{array} \rightarrow$$

$$\begin{array}{rrccc} 1 & 4 & \frac{1}{2} & = & 0 \\ 0 & 1 & \frac{3}{7} & = & -\frac{12}{7} \\ 0 & 0 & \left(-\frac{66}{7}-\frac{13}{2}\right) & = & \left(26+\frac{264}{7}\right) \end{array} \rightarrow \begin{array}{rrrcr} 1 & 4 & \frac{1}{2} & = & 0 \\ 0 & 1 & \frac{3}{7} & = & -\frac{12}{7} \\ 0 & 0 & 1 & = & -4 \end{array} \rightarrow \begin{array}{rrrcr} 1 & 4 & \frac{1}{2} & = & 0 \\ 0 & 1 & 0 & = & 0 \\ 0 & 0 & 1 & = & -4 \end{array} \rightarrow$$

$$\begin{array}{rrrcr} 1 & 4 & 0 & = & 2 \\ 0 & 1 & 0 & = & 0 \\ 0 & 0 & 1 & = & -4 \end{array} \rightarrow \begin{array}{rrrcr} 1 & 0 & 0 & = & 2 \\ 0 & 1 & 0 & = & 0 \\ 0 & 0 & 1 & = & 0 \end{array}$$

Therefore,

$$x = 2, \quad y = 0, \quad z = -4$$

GAUSS ELIMINATION (cont)

- **Problem 7**

$$\begin{array}{rrrrr} x & +y & -z & = & 9 \\ & 8y & +6z & = & -6 \\ -2x & +4y & -6z & = & 40 \end{array}$$

$$\begin{array}{rrrcr} 1 & 1 & -1 & = & 9 \\ 0 & 8 & 6 & = & -6 \\ -2 & 4 & -6 & = & 40 \end{array} \rightarrow \begin{array}{rrrcr} 1 & 1 & -1 & = & 9 \\ 0 & 1 & \frac{3}{4} & = & -\frac{3}{4} \\ 0 & 6 & -8 & = & 58 \end{array} \rightarrow \begin{array}{rrrcr} 1 & 1 & -1 & = & 9 \\ 0 & 1 & \frac{3}{4} & = & -\frac{3}{4} \\ 0 & 0 & -\frac{25}{2} & = & \frac{125}{2} \end{array} \rightarrow$$

$$\begin{array}{rrrcr} 1 & 1 & -1 & = & 9 \\ 0 & 1 & \frac{3}{4} & = & -\frac{3}{4} \\ 0 & 0 & 1 & = & -5 \end{array} \rightarrow \begin{array}{rrrcr} 1 & 1 & -1 & = & 9 \\ 0 & 1 & 0 & = & 3 \\ 0 & 0 & 1 & = & -5 \end{array} \rightarrow \begin{array}{rrrcr} 1 & 1 & 0 & = & 4 \\ 0 & 1 & 0 & = & 3 \\ 0 & 0 & 1 & = & -5 \end{array} \rightarrow$$

$$\begin{array}{rrrcr} 1 & 0 & 0 & = & 1 \\ 0 & 1 & 0 & = & 3 \\ 0 & 0 & 1 & = & -5 \end{array}$$

Therefore,

$$x = 1, \quad y = 3, \quad z = -5$$

GAUSS ELIMINATION (cont)

The form of the system and of the matrix in the last step of Gauss elimination is called the **echelon form**

The **Reduced system**

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
c_{22}x_2 + \cdots + c_{2n}x_n &= b_1^* \\
&\vdots \\
k_{rr}x_r + \cdots + k_{rn}x_n &= \tilde{b}_r \\
0 &= \tilde{b}_{r+1} \\
&\vdots \\
0 &= \tilde{b}_m
\end{aligned}$$

with $r \le m$

$$a_{11},\ c_{22},\ \cdots,\ k_{rr} \neq 0$$

GAUSS ELIMINATION (cont)

There are thus three possible cases:

a) No solution: if $r < m$ and one of $\tilde{b}_{r+1}, \cdots, \tilde{b}_m \neq 0$

b) Precisely one solution: If $r = n$ and $\tilde{b}_{r+1}, \cdots, \tilde{b}_m = 0$

c) Infinitely many solutions: If $r < n$ and $\tilde{b}_{r+1}, \cdots, \tilde{b}_m = 0$

RANK, INDEPENDENCE, AND VECTOR SPACE

Objective: Existence and uniqueness of solutions

Given a set of m vectors, each with m components, a linear combination of these vectors is of the form:

$$c_1\,\mathbf{a}_{(1)} + \bullet \quad \bullet \quad \bullet \quad + c_m\,\mathbf{a}_{(m)} = \mathbf{0}$$

The vectors are said to be linearly independent if the only set of scalars c_i which satisfy the equation is a set of zero elements.

RANK

- For a given matrix, the maximum number of linearly independent row vectors is called the rank of the matrix.
- In terms of column vectors, the rank is the maximum number of linearly independent column vectors
- The rank of a matrix and the rank of its transpose are equal.

VECTOR SPACE

- A vector space is a non-empty set **V** of vectors such that with any two vectors **a** and **b** in **V**, all of their linear combinations:

$$\mathbf{a} + \mathbf{b}$$

  are also elements of **V**.

- A linearly independent set in **V** consisting of the maximum possible number of vectors in **V** is called a basis.

- The set of all linear combinations of the given vectors (with the same number of components), is the span.

MORE ON VECTOR SPACE

- Row-equivalent matrices have the same rank.
- Determination of rank:

Example

$$\begin{array}{cccccccccc} + & 3 & + & 0 & + & 2 & = & + & 2 \\ - & 6 & + & 42 & + & 24 & = & + & 54 \\ + & 21 & - & 21 & + & 0 & = & - & 15 \end{array} \rightarrow \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

$$\begin{array}{cccccccccc} + & 3 & + & 0 & + & 2 & = & + & 2 \\ + & 0 & + & 42 & + & 28 & = & + & 58 \\ + & 0 & - & 21 & - & 14 & = & - & 29 \end{array} \rightarrow \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}$$

$$\begin{array}{cccccccccc} + & 3 & + & 0 & + & 2 & = & + & 2 \\ + & 0 & + & 42 & + & 28 & = & + & 58 \\ + & 0 & + & 0 & = & 0 & = & + & 0 \end{array} \rightarrow \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\text{Rank} = 2$

LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

- p vectors $x_{(1)}$ … $x_{(p)}$ (with n components) are linearly independent if the matrix with rows $x_{(1)}$ … $x_{(p)}$ has rank p; they are linearly dependent if the rank is less than p.

- p vectors with $n < p$ components are always linearly dependent.

- The vectors space $\mathbf{R}^n$ consisting of all vectors with n components has rank n.

LINEAR INDEPENDENCE

- **Problem 8**

$$\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

the first row is subtracted from row 2 and also from row 3, the result is:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$$

Then subtract row 2 from row 3 and get:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

This is sometimes called canonical form, and is obviously of rank 3, hence given vectors are linearly Independent.

RANK

- **Problem 9**

$$\begin{matrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{matrix} \rightarrow \begin{matrix} 2 & -1 \\ 2 & -1 \\ 2 & -1 \end{matrix} \rightarrow \begin{matrix} 2 & -1 \\ 0 & 0 \\ 0 & 0 \end{matrix}$$

Therefore,

$$\text{Rank} = 1$$

RANK

➢ Problem 10

$$
\begin{array}{cccc} 3 & - 1 & = & 5 \\ 2 & - 4 & = & 6 \\ 10 & + 0 & = & 14 \end{array}
\rightarrow
\begin{array}{cccc} 1 & - \frac{1}{3} & = & \frac{5}{3} \\ 1 & - 2 & = & 3 \\ 1 & + 0 & = & \frac{7}{5} \end{array}
\rightarrow
\begin{array}{cccc} 1 & - \frac{1}{3} & = & \frac{5}{3} \\ 0 & - \frac{5}{3} & = & \frac{4}{3} \\ 0 & + \frac{1}{3} & = & \left(\frac{7}{5} - \frac{5}{3}\right) \end{array}
$$

$$
\begin{array}{cccc} 1 & - \frac{1}{3} & = & \frac{5}{3} \\ 0 & + 1 & = & -\frac{4}{5} \\ 0 & + \frac{1}{3} & = & -\frac{4}{15} \end{array}
\rightarrow
\begin{array}{cccc} 1 & - \frac{1}{3} & = & \frac{5}{3} \\ 0 & + 1 & = & -\frac{4}{5} \\ 0 & + 1 & = & -\frac{4}{5} \end{array}
\rightarrow
\begin{array}{cccc} 1 & + \frac{1}{3} & = & \frac{5}{3} \\ 0 & + 1 & = & -\frac{4}{5} \\ 0 & + 0 & = & 0 \end{array}
$$

Therefore,

$$R = 2$$

SOLUTIONS OF LINEAR SYSTEMS

- A linear system of m equations in n unknowns has solutions if and only if the coefficient matrix and the augmented matrix have the same rank. If the rank of the matrix and the augmented matrix is exactly n, the system has precisely one solution.

- If r < n, the system has infinitely many solutions. These are obtained by determining r suitable unknowns (whose sub-matrix of coefficients has rank r) in terms of the n – r remaining unknowns, to which arbitrary values can be assigned.

- Gauss elimination may be used to find the solutions.

HOMOGENEOUS LINEAR SYSTEM

- If a homogeneous linear system (right hand side = 0) has non-trivial solutions, then the rank of the coefficient matrix must be less than the number of unknowns, n.
- A homogeneous linear system with fewer independent equations (r) than unknowns (n) will always have non-trivial solutions.

CRAMER'S RULE

Define the determinant of a matrix: Example for a 2 × 2 matrix

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}$$

Example for a 3 × 3 matrix:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

CRAMER'S RULE (cont)

- **Example**

Consider this linear system of three equations with three unknowns:

$$
\begin{array}{ccccccc}
a_{11}\,x_1 & + & a_{12}\,x_2 & + & a_{13}\,x_3 & = & b_1 \\
a_{21}\,x_1 & + & a_{22}\,x_1 & + & a_{23}\,x_3 & = & b_2 \\
a_{31}\,x_1 & + & a_{32}\,x_1 & + & a_{33}\,x_3 & = & b_3
\end{array}
$$

Cramer's rule for x_1 is

$$
x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}
$$

DETERMINANT BY EXPANDING BY MINORS

- Equation 1, page 308

$$D = a_{j1} c_{j1} + a_{j2} c_{j2} + \cdots + a_{jn} c_{jn}$$

where

$$c_{jk} = (-1)^{j+k} \cdot m_{jk}$$

CRAMER'S THEOREM

If a linear system of n equations with n unknowns has precisely one solution, then the determinant of the coefficient matrix must be non-zero, and the unknown x_n is given by:

$$x_1 = \frac{D_1}{D}, \quad \cdots \quad , x_n = \frac{D_n}{D}$$

See page 312

EVALUATION OF DETERMINANTS

- **Problem 11**

$$\begin{vmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{vmatrix} = \cos^2 n\theta + \sin^2 n\theta = 1$$

- **Problem 12**

$$\begin{vmatrix} m & n & p \\ p & m & n \\ n & p & m \end{vmatrix} = m^3 + n^3 + p^3 - (-3mpn)$$

EVALUATION OF DETERMINANTS (cont)

- **Problem 13**

$$\begin{vmatrix} 1 & 0 & 3 & 7 \\ 4 & 2 & 0 & 1 \\ 7 & 7 & 3 & 0 \\ 5 & 0 & 6 & 8 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & 7 \\ 7 & 3 & 0 \\ 5 & 6 & 8 \end{vmatrix} - 7\begin{vmatrix} 1 & 3 & 7 \\ 4 & 0 & 1 \\ 5 & 6 & 8 \end{vmatrix}$$

$$= 2\left[24 + 294 - 105 - 168\right] - 7\left[15 + 168 - 96 - 6\right]$$

$$= 90 - 7(81) = 90 - 568 = -478$$

CRAMER'S RULE

- **Problem 14**

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 20 \\ 7x & + & 3y & + & z & = & 13 \\ x & + & 6y & + & 2z & = & 0 \end{array}$$

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 7 & 3 & 1 \\ 1 & 6 & 2 \end{vmatrix} = 6 + 2 + 126 - (9 + 29 + 6) = 134 - 43 = 91$$

CRAMER'S RULE (cont)

- **Problem 15**

$$x = \frac{\begin{matrix} 20 & 2 & 3 \\ 13 & 3 & 1 \\ 0 & 6 & 2 \end{matrix}}{91} = \frac{120 + 234 - (52 + 120)}{91} = \frac{354 - 172}{91} = \frac{182}{91} = 2$$

$$y = \frac{\begin{matrix} 1 & 20 & 3 \\ 7 & 13 & 1 \\ 1 & 0 & 2 \end{matrix}}{91} = \frac{26 + 20 - 39 - 280}{91} = \frac{-319 + 46}{91} = \frac{-273}{91} = -3$$

$$z = \frac{\begin{matrix} 1 & 2 & 20 \\ 7 & 3 & 13 \\ 1 & 6 & 0 \end{matrix}}{91} = \frac{26 + 840 - (60 + 78)}{91} = \frac{866 - 138}{91} = \frac{728}{91} = 8$$

Therefore,

$$x = 2, \quad y = -3, \quad z = 8$$