ADVANCED ENGINEERING MATHEMATICS

010.141

FOURIER SERIES, INTEGRALS, AND TRANSFORMS

MODULE 3

VERI LUX
TAS MEA

Fourier Series, Integrals, and Transform

- **Fourier series**: Infinite series designed to represent general periodic functions in terms of simple ones (e.g., sines and cosines).

- **Fourier series** is more general than Taylor series because many discontinuous periodic functions of practical interest can be developed in Fourier series.

- **Fourier integrals** and **Fourier Transforms** extend the ideas and techniques of Fourier series to non-periodic functions and have basic applications to PDEs.

PERIODIC FUNCTIONS

- A periodic function is a function such that $f(x + p) = f(x)$ where p is a period.
- The smallest period is called a **fundamental period**.

FOURIER SERIES

Assume that f(x) is periodic with period 2π and is integrable over a period.

f(x) can be represented by a trigonometric series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

i.e., the series converges and has sum f(x)

TRIGONOMETRIC SERIES WITH A PERIOD OF 2π

Euler Formulas

Fourier coefficients

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx\, dx$$

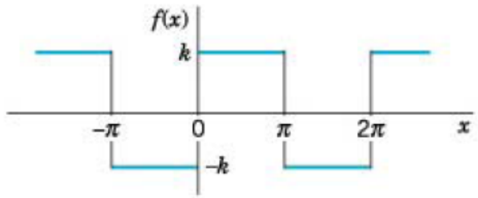
a_0, a_n, b_n are the **Fourier coefficients.**

The corresponding series is the Fourier series

FOURIER SERIES (cont)

Example: **Rectangular wave**

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$$



(a) The given function $f(x)$ (Periodic rectangular wave)

Fig. 257. Example 1

Fourier Coefficients

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} -k\,dx + \frac{1}{2\pi}\int_{0}^{\pi} k\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}[0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-k)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} k\sin nx\,dx$$

$$= \frac{k}{\pi}\left(\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \left[-\frac{\cos nx}{n}\right]_{0}^{\pi}\right)$$

$$= \frac{k}{\pi n}(1+1+1+1) = \frac{4k}{n\pi} \quad \textit{for} \text{ odd } n$$

EXAMPLE 1: RECTANGULAR WAVE

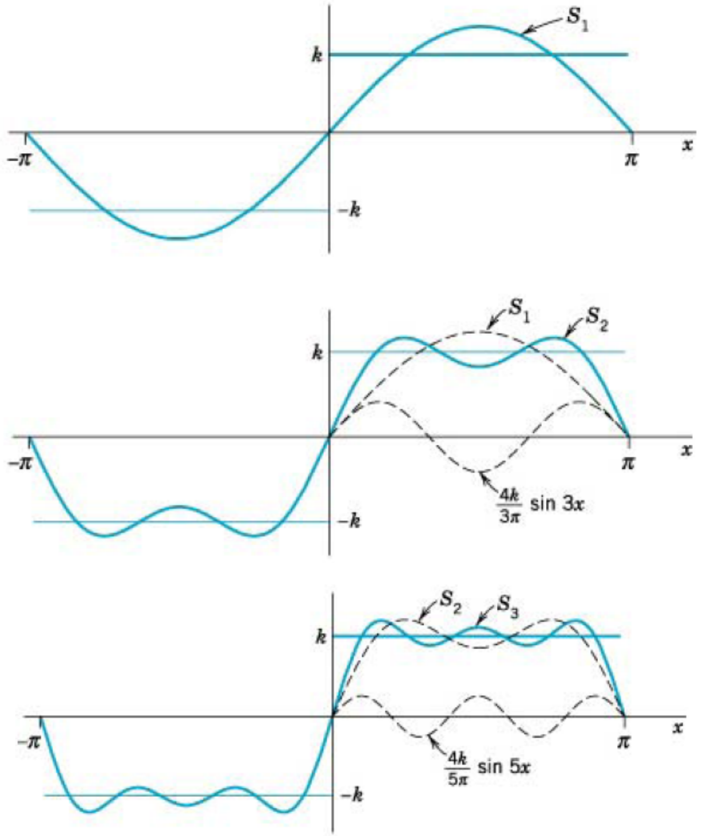
Fourier series:

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \cos nx$$

$$=\sum_{n=0}^{\infty} b_{2n+1} \sin(2n+1)x$$

$$=\sum_{n=0}^{\infty} \frac{4k}{(2n+1)\pi}\sin(2n+1)x$$

$$=\frac{4k}{\pi}\left(\sin x+\frac{1}{2\cdot 1+1}\sin 3x+\frac{1}{5}\sin 5x+\cdots\right)$$



(b) The first three partial sums of the corresponding Fourier series

Fig. 257. Example 1

ORTHOGONALITY OF SINE, COSINE FUNCTIONS

Two functions ϕ_m and ϕ_n of the same form are orthogonal if $\int \phi_m \phi_n \, dx = 0$ for all $m \neq n$ and $\int \phi_m \phi_n \, dx = \alpha$ for all $m = n$.

$$\int_{-\pi}^{\pi} \cos mx \cos nx \ dx = 0 \qquad \text{for all } m \neq \pm n$$

$$\int_{-\pi}^{\pi} (\cos nx)^2 \ dx \neq 0 \qquad \text{for all } m = n$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \ dx = 0 \qquad \text{for all } m \neq n$$

$$\int_{-\pi}^{\pi} (\sin nx)^2 \ dx \neq 0 \qquad \text{for all } m = n$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \ dx = 0 \qquad \text{for all } m = n$$

CONVERGENCE AND SUM OF FOURIER SERIES

THEOREM 2

Representation by a Fourier Series

Let $f(x)$ be periodic with period 2π and piecewise continuous (see Sec. 6.1) *in the interval $-\pi \leqq x \leqq \pi$. Furthermore, let $f(x)$ have a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series* (5) *of $f(x)$* [with coefficients (6)] *converges. Its sum is $f(x)$, except at points x_0 where $f(x)$ is discontinuous. There the sum of the series is the average of the left- and right-hand limits*[2] *of $f(x)$ at x_0.*

The proof of convergence in the case of a piecewise continuous function $f(x)$ and the proof that under the assumptions in the theorem the Fourier series (5) with coefficients (6) represents $f(x)$ are substantially more complicated; see, for instance, Ref. [C12].

CONVERGENCE AND SUM OF FOURIER SERIES

Proof: For continuous f(x) with continuous first and second order derivatives.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\frac{f(x)\ \sin nx}{n}\right)' dx - \frac{1}{\pi}\int_{-\pi}^{\pi} f' \frac{\sin nx}{n} dx$$

$$= \underbrace{\frac{1}{\pi} \frac{f(x)\ \sin nx}{n}\bigg|_{-\pi}^{\pi}}_{0} - \frac{1}{\pi n}\int_{-\pi}^{\pi} f' \sin nx\, dx$$

$$= -\frac{1}{\pi n}\int_{-\pi}^{\pi} f' \sin nx\, dx$$

CONVERGENCE AND SUM OF FOURIER SERIES (cont)

Repeating the process:

$$a_n = -\frac{1}{\pi n^2} \int_{-\pi}^{\pi} f''(x) \cos nx \, dx$$

Since f'' is continuous on $[-\pi, \pi]$

$$|f''(x)| < M$$

$$|a_n| = \frac{1}{\pi n^2} \left| \int_{-\pi}^{\pi} f''(x) \cos nx \, dx \right| < \frac{1}{\pi n^2} \int_{-\pi}^{\pi} M \, dx$$

$$|a_n| < \frac{2\pi M}{\pi n^2} = \frac{2M}{n^2}$$

CONVERGENCE AND SUM OF FOURIER SERIES (cont)

Similarly for

$$|b_n| < \frac{2M}{n^2}$$

Hence

$$|f(x)| < |a_0| + 2M\left(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots\right)$$

which converges. (see Page 670)

FUNCTIONS OF ANY PERIOD P = 2L

we thus obtain from (1) the **Fourier series** of the function $f(x)$ of period $2L$

(5) $$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**

(6)

(a) $$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

(b) $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots$$

(c) $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots.$$

FUNCTIONS OF ANY PERIOD (cont)
P = 2L

Proof: Result obtained easily through change of scale.

$$v = \frac{\pi x}{L}$$

$$g(v) = a_0 + \sum_{n=1}^{\infty} a_n \cos nv + \sum_{n=1}^{\infty} b_n \sin nv$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\, dv = \frac{1}{\pi}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\cdot\frac{\pi}{L}dx = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

Same for a_0, b_n

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)dv = \frac{1}{2\pi}\int_{-L}^{L} f(x)\frac{\pi}{L}dx = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

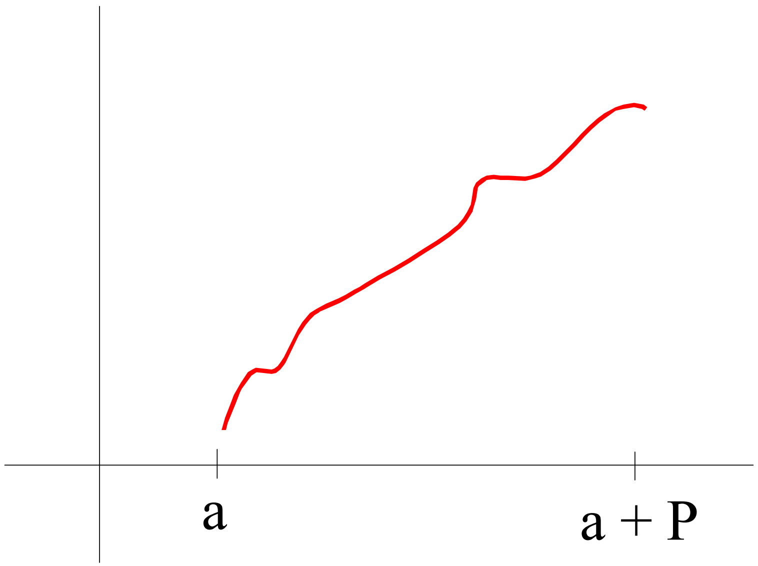
ANY INTERVAL (a, a + P)
P = PERIOD = 2L

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{P} + b_n \sin\frac{2n\pi x}{P}\right)$$

$$a_0 = \frac{1}{P}\int_a^{a+P} f(x)\,dx$$

$$a_n = \frac{2}{P}\int_a^{a+P} f(x)\cos\frac{2n\pi x}{P}\,dx$$

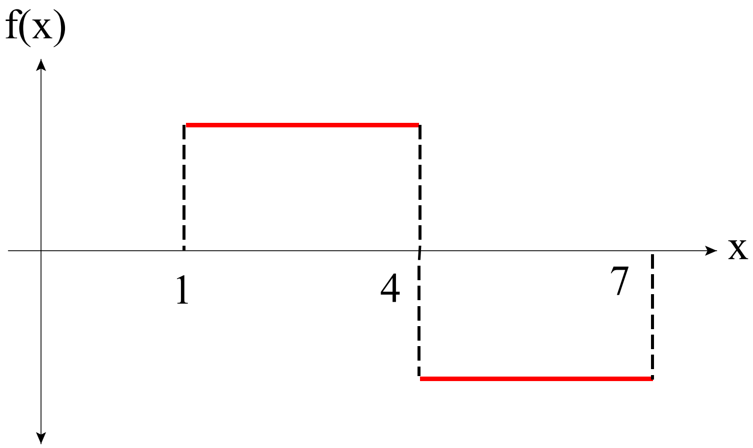
$$b_n = \frac{2}{P}\int_a^{a+P} f(x)\sin\frac{2n\pi x}{P}\,dx$$



EXAMPLE OF APPLICATION

Find the Fourier series of the function

$$f(x)=\begin{cases} 1 & 1<x<4 \\ & \\ -1 & 4<x<7 \end{cases} \qquad \begin{matrix} P=6 \\ a=1 \\ 1<x<7 \end{matrix}$$



Solution:

$$a_0=\frac{1}{6}\int_1^7 f(x)dx=\frac{1}{6}\left[\int_1^4 1dx+\int_4^7(-1)dx\right]=0$$

$$a_n=\frac{1}{3}\left[\int_1^4 \cos\frac{n\pi x}{3}dx-\int_4^7 \cos\frac{n\pi x}{3}dx\right]=\frac{1}{n\pi}\left[2\sin\frac{4n\pi}{3}-\sin\frac{n\pi}{3}-\sin\frac{7n\pi}{3}\right] \quad n\neq 0$$

$$b_n=\frac{1}{3}\int_1^7 f(x)\sin\frac{n\pi x}{3}dx=\frac{1}{3}\left[\int_1^4 \sin\frac{n\pi x}{3}dx-\int_4^7 \sin\frac{n\pi x}{3}dx\right]=-\frac{1}{n\pi}\left[2\cos\frac{4n\pi}{3}-\cos\frac{n\pi}{3}-\cos\frac{7n\pi}{3}\right]$$

EXAMPLE OF APPLICATION (cont)

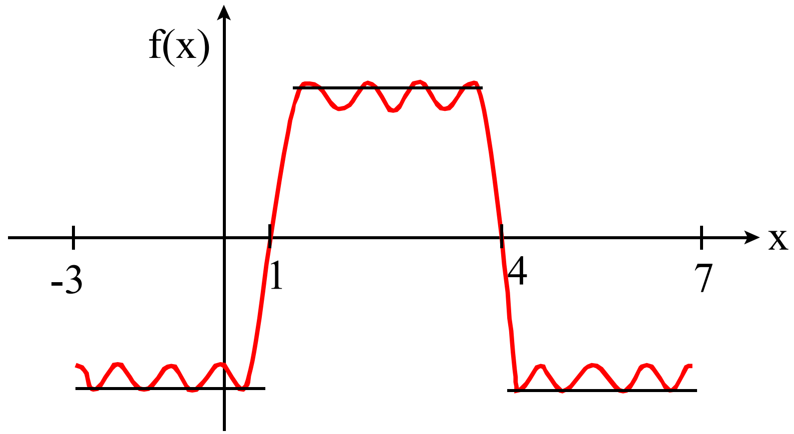
$$\sin\frac{4n\pi}{3} = \cos n\pi \sin\frac{n\pi}{3} \qquad \sin\frac{7n\pi}{3} = \sin\frac{n\pi}{3}$$

$$\cos\frac{4n\pi}{3} = \cos n\pi \cos\frac{n\pi}{3} \qquad \cos\frac{7n\pi}{3} = \cos\frac{n\pi}{3}$$

$$a_0 = 0$$

$$a_n = -2\left(\frac{1-\cos n\pi}{n\pi}\right)\sin\frac{n\pi}{3}$$

$$b_n = 2\left(\frac{1-\cos n\pi}{n\pi}\right)\cos\frac{n\pi}{3}$$



EVEN AND ODD FUNCTIONS

Examples: $\mathbf{x^4}$**, cos x**

f(x) is an ***even*** *function* of x, if f(-x) = f(x). For example, f(x) = x sin(x), then

$$f(-x) = -x \sin(-x) = f(x)$$

and so we can conclude that x sin (x) is an even function.

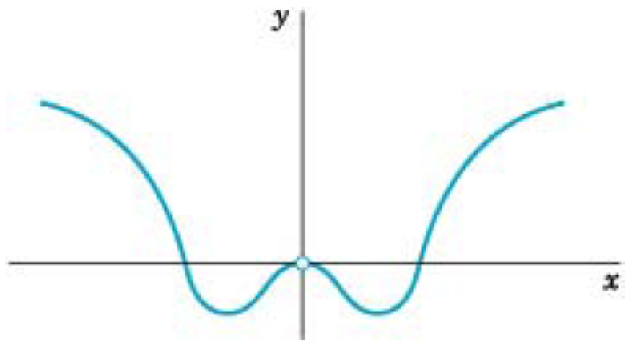


Fig. 262. Even function

EVEN AND ODD FUNCTIONS

Properties

1. If g(x) is an even function

$$\int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx$$

2. If h(x) is an odd function

$$\int_{-L}^{L} h(x)\,dx = 0$$

3. The product of an even and odd function is odd

EVEN AND ODD FUNCTIONS

THEOREM 1

Fourier Cosine Series, Fourier Sine Series

The Fourier series of an ***even*** *function of period 2L is a* **"Fourier cosine series"**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \qquad (f \text{ even}) \tag{1}$$

with coefficients (note: integration from 0 to L only!)

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \cdots . \tag{2}$$

EVEN AND ODD FUNCTIONS

The Fourier series of an **odd** *function of period 2L is a* **"Fourier sine series"**

(3) $$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \qquad (f \text{ odd})$$

with coefficients

(4) $$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

SUM

THEOREM 2

Sum and Scalar Multiple

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

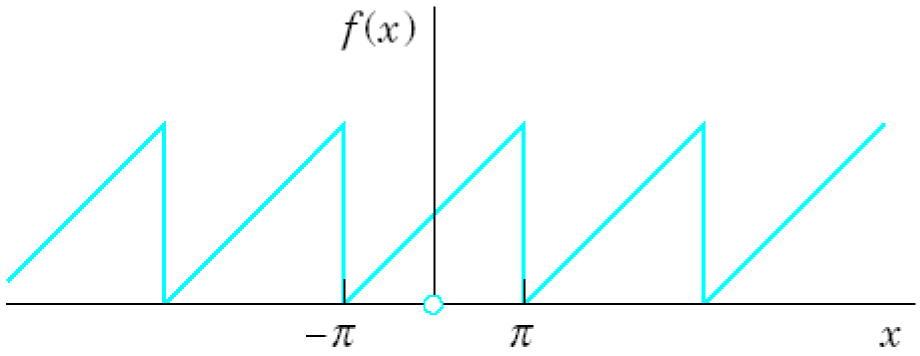
SUM

EXAMPLE 3

Sawtooth Wave

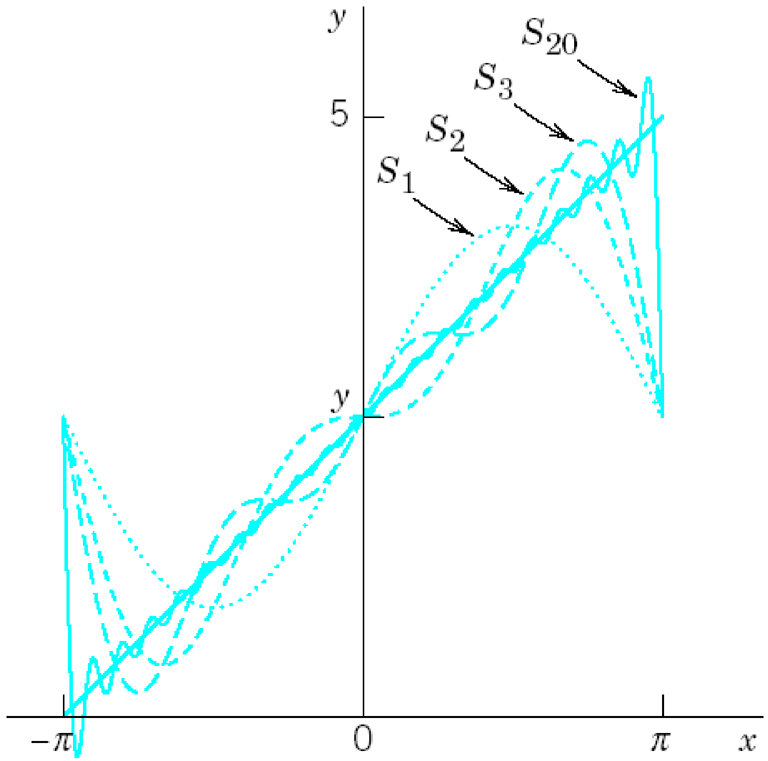
Find the Fourier series of the function (Fig. 266)

$$f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$



(a) The function $f(x)$

SUM



(b) Partial sums S_1, S_2, S_3, S_{20}

Fig. 266. Example 3

SUM

Solution. We have $f = f_1 + f_2$, where $f_1 = x$ and $f_2 = \pi$. The Fourier coefficients of f_2 are zero, except for the first one (the constant term), which is π. Hence, by Theorem 2, the Fourier coefficients a_n, b_n are those of f_1, except for a_0, which is π. Since f_1 is odd, $a_n = 0$ for $n = 1, 2, \cdots$, and

$$b_n = \frac{2}{\pi} \int_0^{\pi} f_1(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx.$$

Integrating by parts, we obtain

$$b_n = \frac{2}{\pi} \left[\frac{-x \cos nx}{n} \bigg|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right] = -\frac{2}{n} \cos n\pi.$$

Hence $b_1 = 2$, $b_2 = -2/2$, $b_3 = 2/3$, $b_4 = -2/4$, $\cdots$, and the Fourier series of $f(x)$ is

$$f(x) = \pi + 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - + \cdots \right).$$

Forced Oscillations

EXAMPLE 1

Forced Oscillations under a Nonsinusoidal Periodic Driving Force

In (1), let $m = 1$ (gm), $c = 0.05$ (gm/sec), and $k = 25$ (gm/sec^2), so that (1) becomes

$$y'' + 0.05y' + 25y = r(t) \tag{2}$$

where $r(t)$ is measured in gm · cm/sec^2. Let (Fig. 273)

$$r(t) = \begin{cases} t + \dfrac{\pi}{2} & \text{if } -\pi < t < 0, \\ -t + \dfrac{\pi}{2} & \text{if } \quad 0 < t < \pi, \end{cases} \qquad r(t + 2\pi) = r(t).$$

Find the steady-state solution $y(t)$.

Forced Oscillations

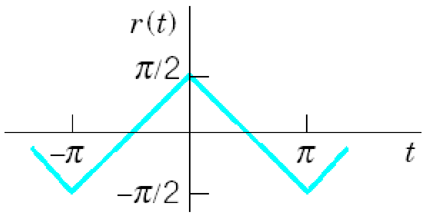


Fig. 273. Force in Example 1

Solution. We represent $r(t)$ by a Fourier series, finding

$$(3) \qquad r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

(take the answer to Prob. 11 in Problem Set 11.3 minus $\frac{1}{2}\pi$ and write t for x). Then we consider the ODE

$$(4) \qquad y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \qquad (n = 1, 3, \cdots)$$

whose right side is a single term of the series (3). From Sec. 2.8 we know that the steady-state solution $y_n(t)$ of (4) is of the form

$$(5) \qquad y_n = A_n \cos nt + B_n \sin nt.$$

Forced Oscillations

By substituting this into (4) we find that

$$(6) \qquad A_n = \frac{4(25 - n^2)}{n^2 \pi D_n}, \qquad B_n = \frac{0.2}{n\pi D_n}, \qquad \text{where} \qquad D_n = (25 - n^2)^2 + (0.05n)^2.$$

Since the ODE (2) is linear, we may expect the steady-state solution to be

$$(7) \qquad y = y_1 + y_3 + y_5 + \cdots$$

where y_n is given by (5) and (6). In fact, this follows readily by substituting (7) into (2) and using the Fourier series of $r(t)$, provided that termwise differentiation of (7) is permissible. (Readers already familiar with the notion of uniform convergence [Sec. 15.5] may prove that (7) may be differentiated term by term.)

From (6) we find that the amplitude of (5) is (a factor $\sqrt{D_n}$ cancels out)

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2 \pi \sqrt{D_n}}.$$

Forced Oscillations

Numeric values are

$$C_1 = 0.0531$$

$$C_3 = 0.0088$$

$$C_5 = 0.2037$$

$$C_7 = 0.0011$$

$$C_9 = 0.0003.$$

Figure 274 shows the input (multiplied by 0.1) and the output. For $n = 5$ the quantity D_n is very small, the denominator of C_5 is small, and C_5 is so large that y_5 is the dominating term in (7). Hence the output is almost a harmonic oscillation of five times the frequency of the driving force, a little distorted due to the term y_1, whose amplitude is about 25% of that of y_5. You could make the situation still more extreme by decreasing the damping constant c. Try it.

Forced Oscillations

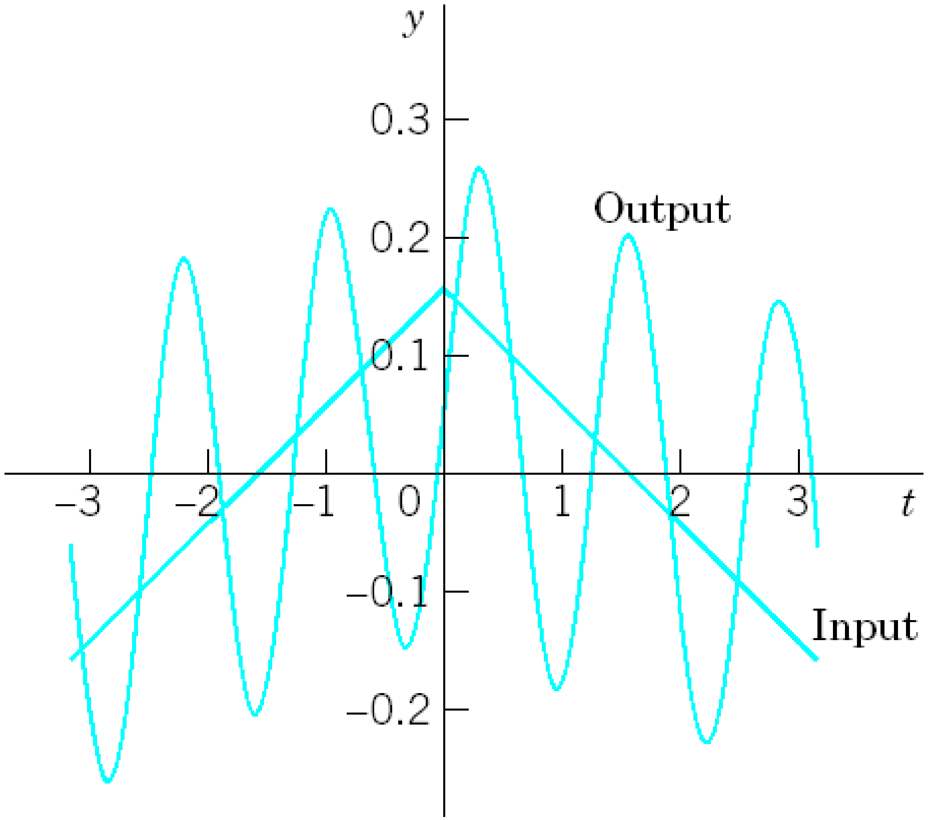


Fig. 274. Input and steady-state output in Example 1

APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS

Consider a function f(x), periodic of period 2π. Consider an approximation of f(x),

$$f(x) \approx F(\mathrm{x}) = A_0 + \sum_{\mathrm{n}=1}^{N} A_n \cos nx + B_n \sin nx$$

The **total square error of F**

$$\mathrm{E} = \int_{-\pi}^{\pi} (\mathrm{f} - \mathrm{F})^2 \, \mathrm{dx}$$

is minimum when F's coefficients are the Fourier coefficients.

Read Page 503 to see the derivation procedure of Eq. (6)

PARSEVAL'S THEOREM

The square error, call it *E, is**

(6) $$E^* = \int_{-\pi}^{\pi} f^2\,dx - \pi\left[2a_0^{\,2} + \sum_{n=1}^{N}\left(a_n^{\,2} + b_n^{\,2}\right)\right].$$

Where a_n, b_n are the Fourier coefficients of *f.*

$$2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx$$

FOURIER INTEGRALS

Since many problems involve functions that are nonperiodic and are of interest on the whole x-axis, we ask what can be done to extend the method of Fourier series to such functions. This idea will lead to "Fourier integrals."

we start from a special function f_L of period $2L$ and see what happens to its Fourier series if we let $L \to \infty$.

EXAMPLE 1 Rectangular Wave

Consider the periodic rectangular wave $f_L(x)$ of period $2L > 2$ given by

$$f_L(x) = \begin{cases} 0 & \text{if} \quad -L < x < -1 \\ 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < L. \end{cases}$$

FOURIER INTEGRALS

The left part of Fig. 277 shows this function for $2L = 4, 8, 16$ as well as the nonperiodic function $f(x)$, we obtain from f_L if we let $L \to \infty$,

$$f(x) = \lim_{L \to \infty} f_L(x) = \begin{cases} 1 & \text{if} -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We now explore what happens to the Fourier coefficients of f_L as L increases.

Since f_L is even, $b_n = 0$ for n. For a_n the Euler formulas (6) , Sec. 11.2, give

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}, \quad a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L} dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi x}{L} dx = \frac{2}{L}\frac{\sin(n\pi / L)}{n\pi / L}.$$

FOURIER INTEGRALS

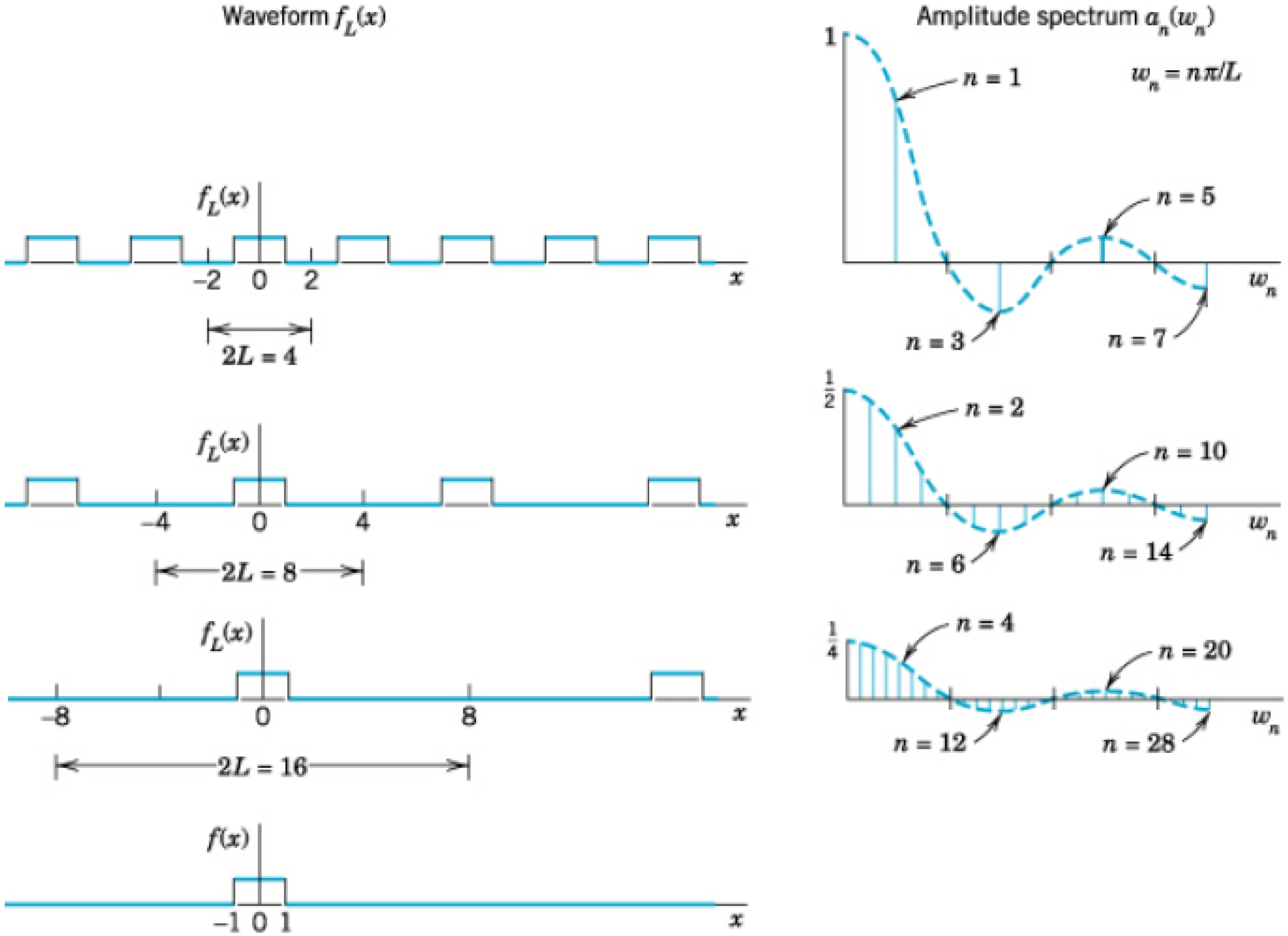


Fig. 277. Waveforms and amplitude spectra in Example 1

FOURIER COSINE AND SINE INTEGRALS

We now consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$$

and find out what happens if we let $L \to \infty$. Together with Example 1 the present calculation will

If we insert a_n and b_n from the Euler formulas (6) , Sec. 11.2, and denote the variable of integration by v, the Fourier series of $f_L(x)$ becomes

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v dv + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v dv\right].$$

We now set

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$

Then $1/L = \Delta w/\pi$, and we may write the Fourier series in the form

$$(1) \quad f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\Delta w \int_{-L}^{L} f_L(v)\cos w_n v dv + (\sin w_n x)\Delta w \int_{-L}^{L} f_L(v)\sin w_n v dv\right]$$

This representation is valid for any fixed L, arbitrarily large, but finite.

FOURIER COSINE AND SINE INTEGRALS

We now let $L \to \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L \to \infty} f_L(x)$$

is **absolutely integrable** on the x-axis; that is, the following (finite!) limits exist:

(2) $$\lim_{a \to -\infty} \int_a^0 |f(x)|\,dx + \lim_{b \to \infty} \int_0^b |f(x)|\,dx \left(\text{written} \int_{-\infty}^{\infty} |f(x)|\,dx\right).$$

Then $1/L \to 0$,

(3) $$f(x) = \frac{1}{\pi}\int_0^{\infty} \left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\,dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv\,dv\right] dw\,.$$

If we introduce the notations

(4) $$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv, \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

we can write this in the form

(5) $$f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\,dw\,.$$

This is called a representation of $f(x)$ by a **Fourier integral**.

FOURIER COSINE AND SINE INTEGRALS

$$f(x) = \int_0^{\infty} \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x \right] d\omega$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v)\, \cos(\omega v)\, dv$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v)\, \sin(\omega v)\, dv$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx\, dx$$

EXISTENCE

If f(x) is piecewise continuous in every finite interval and has a right hand and left hand derivative at every point and if

$$\lim_{a \to -\infty} \int_a^0 |f(x)|\,dx + \lim_{b \to \infty} \int_0^b |f(x)|\,dx$$

exists, then f(x) can be represented by the Fourier integral.

Where f(x) is discontinuous, the F.I. equals the average of the left-hand and right-hand limit of f(x).

FOURIER COSINE AND SINE INTEGRALS

For an **even** or **odd** function the F.I. becomes much simpler.

If f(x) is **even**

$$A(\omega)=\frac{2}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv$$

$$f(x)=\int_{0}^{\infty} A(\omega)\cos(\omega x)\,d\omega \qquad \text{Fourier cosine integral}$$

If f(x) is **odd**

$$B(\omega)=\frac{2}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv$$

$$f(x)=\int_{0}^{\infty} B(\omega)\sin(\omega x)\,d\omega \qquad \text{Fourier sine integral}$$

EXAMPLE

Consider $f(x) = e^{-kx}$ $x > 0,\ k > 0$

Evaluate the Fourier cosine integral A(ω) and sine integral B(ω).

For Fourier cosine integral, $A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\, dv$

Integration by parts gives $\int e^{-kv}\cos wv\, dv = -\frac{k}{k^2+w^2}e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right)$

$$A(w) = \frac{2k/\pi}{k^2+w^2}$$

Fourier cosine integral is

$$f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}dw \quad (x > 0,\ k > 0)$$

EXAMPLE

For Fourier sine integral, $B(w) = \frac{2}{\pi}\int_0^{\infty} e^{-kv} \sin wv \, dv$

Integration by parts gives $\int e^{-kv} \sin wv \, dv = -\frac{w}{k^2+w^2} e^{-kv}\left(\frac{k}{w}\sin wv + \cos wv\right)$

$$B(w) = \frac{2w/\pi}{k^2+w^2}$$

Fourier sine integral is

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^{\infty} \frac{w \sin wx}{k^2+w^2} dw$$

Laplace integrals

$$\int_0^{\infty} \frac{\cos wx}{k^2+w^2} dw = \frac{\pi}{2k} e^{-kx} \quad (x > 0, k > 0)$$

$$\int_0^{\infty} \frac{w \sin wx}{k^2+w^2} dw = \frac{\pi}{2} e^{-kx} \quad (x > 0, k > 0)$$

FOURIER SINE AND COSINE TRANSFORMS

For an **even** function, the Fourier integral is the Fourier cosine integral

$$f(x) = \int_0^{\infty} A(\omega)\cos(\omega x)\,d\omega \qquad A(\omega) = \frac{2}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv = \sqrt{\frac{2}{\pi}}\,\hat{f}_c(\omega)$$

Then

$$\hat{f}_c(\omega) = \sqrt{\frac{\pi}{2}}\,A(\omega) = \sqrt{\frac{\pi}{2}}\,\frac{2}{\pi}\int_0^{\infty} f(v)\cos\omega v\,dv = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(v)\cos\omega v\,dv$$

$$f(x) = \int_0^{\infty}\sqrt{\frac{2}{\pi}}\,\hat{f}_c(\omega)\cos\omega x\,d\omega = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}_c(\omega)\cos\omega x\,d\omega$$

$\hat{f}_c(\omega)$ is defined as the Fourier cosine transform of f.

f is the inverse Fourier cosine transform of $\hat{f}_c$.

FOURIER SINE AND COSINE TRANSFORMS (cont)

Similarly for **odd** function

$$\mathrm{F\,sine\,T} \rightarrow \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x \, dx$$

$$\mathrm{Inverse\,F\,sine\,T} \rightarrow f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(\omega) \sin \omega x \, d\omega$$

NOTATION AND PROPERTIES

$$\mathsf{F}_c\{f\} = \hat{f}_c, \quad \mathsf{F}_s\{f\} = \hat{f}_s$$

$$\mathsf{F}_c^{-1}\{\hat{f}_c\} = f, \quad \mathsf{F}_s^{-1}\{\hat{f}_s\} = f$$

$$(1) \quad F_c\{\mathrm{af} + \mathrm{bg}\} = \mathrm{a}\,F_c\{\mathrm{f}\} + \mathrm{b}\,F_c\{\mathrm{g}\}$$

$$(2) \quad F_s\{\mathrm{af} + \mathrm{bg}\} = \mathrm{a}\,F_s\{\mathrm{f}\} + \mathrm{b}\,F_s\{\mathrm{g}\}$$

$$(3) \quad \mathsf{F}_c\{\mathrm{f}'(\mathrm{x})\} = \omega \mathsf{F}_s\{\mathrm{f}\} - \sqrt{\frac{2}{\pi}}\,\mathrm{f}(0)$$

$$(4) \quad \mathsf{F}_s\{\mathrm{f}'(\mathrm{x})\} = -\omega \mathsf{F}_c\{\mathrm{f}\}$$

$$(5) \quad \mathsf{F}_c\{\mathrm{f}''\} = -\omega^2 \mathsf{F}_c\{\mathrm{f}\} - \sqrt{\frac{2}{\pi}}\,\mathrm{f}'(0)$$

$$(6) \quad \mathsf{F}_s\{\mathrm{f}''\} = -\omega^2 \mathsf{F}_s\{\mathrm{f}\} + \sqrt{\frac{2}{\pi}}\,\omega \mathrm{f}'(0)$$

FOURIER TRANSFORM

The F.I. is:

$$f(x)=\int_{0}^{\infty}\left[A(\omega)\cos\omega x+B(\omega)\sin\omega x\right]dx$$

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\cos\omega v\,dv$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\sin\omega v\,dv$$

Replacing

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}d\omega\int_{-\infty}^{\infty}\left[f(v)\cos\omega v\cos\omega x+f(v)\sin\omega v\sin\omega x\right]dv$$

$$=\frac{1}{\pi}\int_{0}^{\infty}d\omega\underbrace{\int_{-\infty}^{\infty}f(v)\cos(\omega v-\omega x)dv}_{\text{Even in }\omega}$$

FOURIER TRANSFORM

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}f(v)\cos\omega(x-v)dv$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\underbrace{\int_{-\infty}^{\infty}f(v)\sin\omega(x-v)dv1}_{\text{odd in }\omega}$$

Now,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)+i\,G(\omega)\,d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}f(v)e^{i\omega(x-v)}dv \qquad \text{Complex Fourier Integral}$$

FOURIER TRANSFORM (cont)

$$f(x)=\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}e^{i\omega x}d\omega\;\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(v)e^{-i\omega v}dv}_{\hat{f}(\omega)\equiv \text{Fourier Transform of } f}$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$

NOTATION AND PROPERTIES

$$F\{af + bg\} = aF\{f\} + bF\{g\}$$

$$F\{f'\} = i\omega F\{f\}$$

$$F\{f''\} = -\omega^2 F\{f\}$$

$$F\{f * g\} = \sqrt{2\pi}F\{f\} + F\{g\}$$