ENGINEERING MATHEMATHICS II

010.141

PARTIAL DIFFERENTIAL EQUATIONS

MODULE 4

EXAMPLES OF PDE

A PDE is an equation involving one or more partial derivatives of a function.

One-D Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Two-D Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

One-D Heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Two-D Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Two-D Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Three-D Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

EXAMPLE OF WAVE EQUATION

- Problem, as a solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad u(t,x)$$

$$u(t,x) = \sin 9t \cdot \sin \tfrac{1}{4}x$$

$$\frac{\partial u}{\partial t} = (9 \cos 9t)(\sin \tfrac{1}{4}x)$$

$$\frac{\partial^2 u}{\partial t^2} = -81 \sin 9t \cdot \sin \tfrac{1}{4}x$$

$$\frac{\partial u}{\partial x} = \sin 9t \cdot (\tfrac{1}{4} \cos \tfrac{1}{4}x)$$

$$\frac{\partial^2 u}{\partial x^2} = \sin 9t \cdot (-\tfrac{1}{16} \cos \tfrac{1}{4}x)$$

$$-81 \sin 9t \cdot \sin \tfrac{1}{4}x = c^2 (-\tfrac{1}{16}) \sin 9t \cdot \sin \tfrac{1}{4}x \textit{ is } \text{identically true,}$$

$$\text{if } c^2 = (16)(81) \quad \rightarrow \quad c = \pm 36$$

EXAMPLE OF HEAT EQUATION

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u = e^{-4t} \cos 3x$$

$$\frac{\partial u}{\partial t} = -4e^{-4t} \cos 3x$$

$$\frac{\partial u}{\partial x} = -3e^{-4t} \sin 3x$$

$$\frac{\partial^2 u}{\partial x^2} = -9e^{-4t} \cos 3x$$

$$-4e^{-4t} \cos 3x = -9e^{-4t} \cos 3x \cdot c^2$$

$$\text{if } c^2 = \frac{4}{9}, \text{ an identity}$$

EXAMPLE OF LAPLACE EQUATION

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = e^x \cos y$$

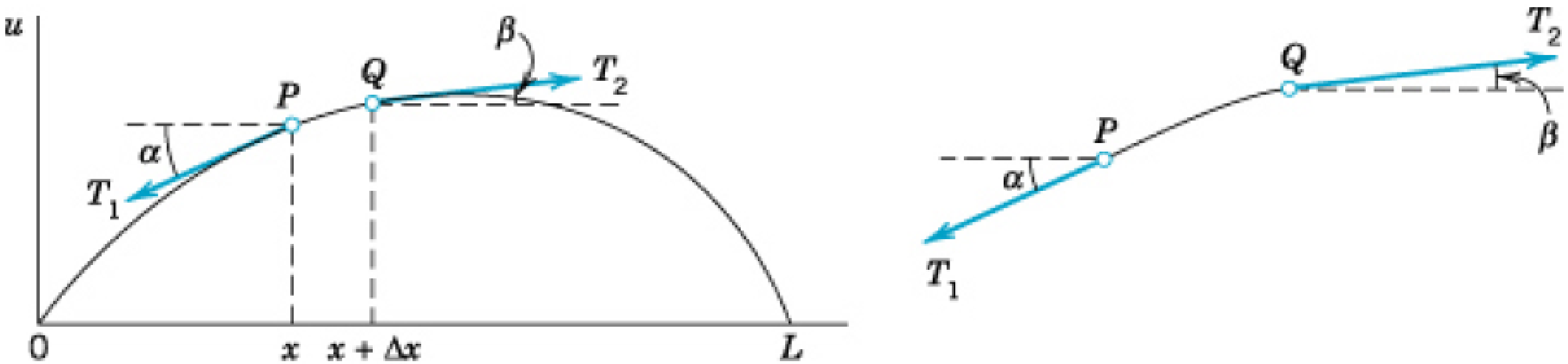
$$\frac{\partial^2 u}{\partial y^2} = -e^x \sin y$$

$$e^x \sin y - e^x \sin y = 0$$

$$0 = 0$$

VIBRATING STRING AND THE WAVE EQUATION

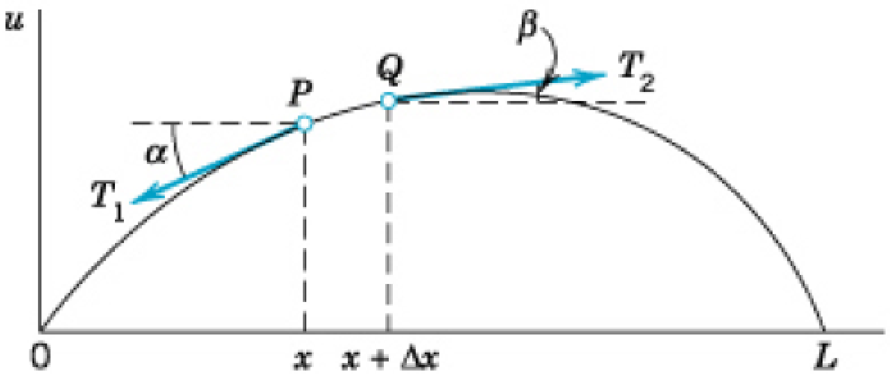
Deflected string at fixed time t

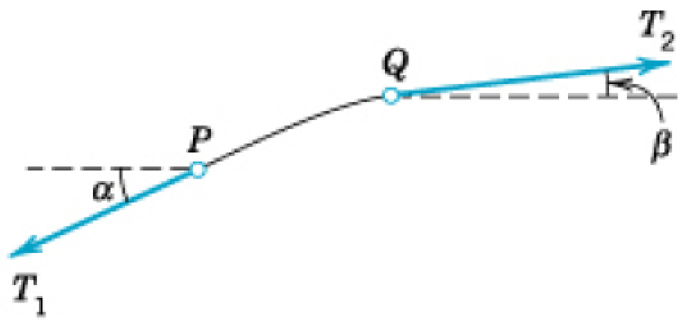


- General assumptions for vibrating string problem:
- mass per unit length is constant; string is perfectly elastic and no resistance to bending.
- tension in string is sufficiently large so that gravitational forces may be neglected.
- string oscillates in a plane; every particle of string moves vertically only; and, deflection and slope at every point are small in terms of absolute value.

DERIVATION OF WAVE EQUATION

Deflected string at fixed time t





T_1, T_2 = tension in string at point P and Q

$T_1 \cos\alpha = T_2 \cos\beta = T$, a constant (as string does not move in horizontal dir.)

Vertical components of tension:

$$-T_1 \sin\alpha \quad \text{and} \quad T_2 \sin\beta$$

DERIVATION OF WAVE EQUATION (Cont.)

Let Δx = length $\overline{PQ}$ and ρ = mass/unit length

Thus Δx has mass $\rho\Delta x$

Newtonns Law: $F = \text{mass} \times \text{acceleration}$

If u is the vertical position, $\dfrac{\partial^2 u}{\partial t^2}$ = acceleration

$$\underline{\underline{T_2 \sin\beta - T_1 \sin\alpha}} = (\rho\Delta x)\,\frac{\partial^2 u}{\partial t^2}$$

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha = \frac{\rho\Delta x}{T}\,\frac{\partial^2 u}{\partial t^2} \quad (\text{equation } 2)$$

At P (x is distance from origin), $\tan\alpha$ is slope $= \left.\dfrac{\partial u}{\partial x}\right|_x$

Likewise at Q, $\tan\beta = \left.\dfrac{\partial u}{\partial x}\right|_{x+\Delta x}$

DERIVATION OF WAVE EQUATION (Cont.)

Substituting and $\div$ Δx: $\dfrac{1}{\Delta x}\left[\left.\dfrac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\dfrac{\partial u}{\partial x}\right|_{x} \right] = \dfrac{\rho}{T}\dfrac{\partial^2 u}{\partial t^2}$

as $\Delta x \rightarrow 0$, L.H.S. becomes $\dfrac{\partial^2 u}{\partial x^2}$

Let $c^2 = \dfrac{T}{\rho}$, so that

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

This is the 1-D Wave equation

If $T\uparrow$ or $\rho\downarrow$ $c^2 \uparrow$

Solution by Separating Variables

one-dimensional wave equation

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T}{\rho}$$

for the unknown deflection $u(x, t)$ of the string.

Since the string is fastened at the ends $x = 0$ and $x = L$ (see Sec. 12.2), we have the two **boundary conditions**

(2) $$(a)\quad u(0,t) = 0, \qquad (b)\quad u(L,t) = 0 \qquad \text{for all } t.$$

Furthermore, the form of the motion of the string will depend on its *initial deflection* (deflection at time $t = 0$), call it $f(x)$, and on its *initial velocity* (velocity at $t = 0$), call it $g(x)$. We thus have the two **initial conditions**

(3) $$(a)\quad u(x,0) = f(x), \qquad (b)\quad u_t(x,0) = g(x) \qquad (0 \leqq x \leqq L)$$

where $u_t = \partial u/\partial t$. We now have to find a solution of the PDE (1) satisfying the conditions (2) and (3) .

Solution by Separating Variables

Step 1. By the **"method of separating variables"** or *product method*, setting $u(x, t) = F(x)G(t)$, we obtain from (1) two ODEs, one for $F(x)$ and the other one for $G(t)$.

Step 2. We determine solutions of these ODEs that satisfy the boundary conditions (2) .

Step 3. Finally, using **Fourier series**, we compose the solutions gained in Step 2 to obtain a solution of (1) satisfying both (2) and (3) , that is, the solution of our model of the vibrating string.

Step 1. Two ODEs from the Wave Equation (1)

(4) $$u(x,t) = F(x)G(t)$$

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \qquad \frac{\partial^2 u}{\partial x^2} = F''G \quad \Rightarrow \quad F\ddot{G} = c^2 F''G .$$

Dividing by c^2FG and simplifying gives

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k .$$

(5) $$F'' - kF = 0$$

(6) $$\ddot{G} - c^2 kG = 0.$$

Solution by Separating Variables

Step 2. Satisfying the Boundary Conditions (2)

$u = FG$ satisfies the boundary conditions (2) ,

(7) $$u(0,t) = F(0)G(t) = 0, \qquad u(L,t) = F(L)G(t) = 0 \qquad \text{for all } t.$$

We first solve (5) . If $G \equiv 0$, then $u = FG \equiv 0$, which is of no interest. Hence $G \not\equiv 0$ and then by (7) ,

(8) $$(a) \quad F(0) = 0, \qquad (b) \quad F(L) = 0.$$

We show that k must be negative. For $k = 0$ the general solution of (5) is $F = ax + b$, and from (8) we obtain $a = b = 0$, so that $F \equiv 0$ and $u = FG \equiv 0$, which is of no interest. For positive $k = \mu^2$ a general solution of (5) is

$$F = Ae^{\mu x} + Be^{-\mu x}$$

and from (8) we obtain $F \equiv 0$ as before (verify!). Hence we are left with the possibility of choosing k negative, say, $k = -p^2$. Then (5) becomes $F'' + p^2F = 0$ and has as a general solution

$$F(x) = A\cos px + B\sin px .$$

Solution by Separating Variables

From this and (8) we have

$$F(0) = A = 0 \quad \text{and then} \quad F(L) = B \sin pL = 0.$$

We must take $B \neq 0$ since otherwise $F \equiv 0$. Hence $\sin pL = 0$. Thus

(9) $$pL = n\pi, \qquad \text{so that} \qquad p = \frac{n\pi}{L} \qquad (n \text{ integer}).$$

Setting $B = 1$, we thus obtain infinitely many solutions $F(x) = F_n(x)$, where

(10) $$F_n(x) = \sin \frac{n\pi}{L} x \qquad (\mathrm{n} = 1, 2, \cdots).$$

We now solve (6) with $k = -p^2 = -(n\pi/L)^2$ resulting from (9), that is,

(11*) $$\ddot{G} + \lambda_n^2 G = 0 \qquad \text{where} \qquad \lambda_n = cp = \frac{cn\pi}{L}.$$

A general solution is

$$G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t.$$

Hence solutions of (1) satisfying (2) are $u_n(x, t) = F_n(x)G_n(t) = G_n(t)F_n(x)$, written out

(11) $$u_n(x, t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x \qquad (n = 1, 2, \cdots).$$

These functions are called the **eigenfunctions**, or *characteristic functions*, and the values $\lambda_n = cn\pi/L$ are called the **eigenvalues**, or *characteristic values*, of the vibrating string. The set $\{\lambda_1, \lambda_2, \cdots\}$ is called the **spectrum**.

Solution by Separating Variables

Discussion of Eigenfunctions. We see that each u_n represents a harmonic motion having the **frequency** $\lambda_n/2\pi = cn/2L$ cycles per unit time. This motion is called the nth **normal mode** of the string. The first normal mode is known as the *fundamental mode* ($n = 1$), and the others are known as *overtones;* musically they give the octave, octave plus fifth, etc. Since in (11)

$$\sin\frac{n\pi x}{L} = 0 \qquad \text{at} \qquad x = \frac{L}{n}, \frac{2L}{n}, \cdots, \frac{n-1}{n}L,$$

the nth normal mode has $n - 1$ **nodes**, that is, points of the string that do not move (in addition to the fixed endpoints); see Fig. 284.

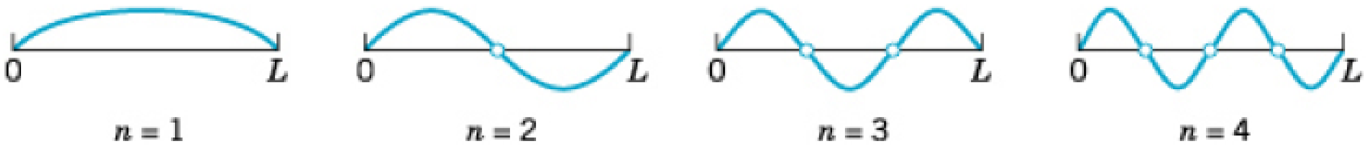


Fig. 284. Normal modes of the vibrating string

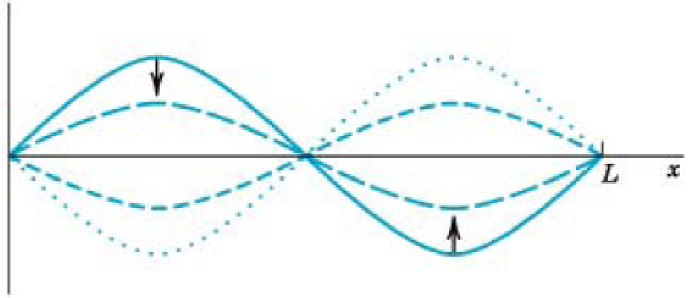


Fig. 285. Second normal mode for various values of t

Solution by Separating Variables

Step 3. Solution of the Entire Problem. Fourier Series

The eigenfunctions (11) satisfy the wave equation (1) and the boundary conditions (2) (string fixed at the ends). A single u_n will generally not satisfy the initial conditions (3) .

(12) $$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x .$$

Satisfying Initial Condition (3a) (Given Initial Displacement). From (12) and (3a) we obtain

(13) $$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x) .$$

(14) $$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, \cdots .$$

Satisfying Initial Condition (3b) (Given Initial Velocity). Similarly, by differentiating (12) with respect to t and using (3b) , we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty} (-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t) \sin \frac{n\pi x}{L}\right]_{t=0}$$
$$= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L} = g(x) .$$

(15) $$B_n{}^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, \cdots .$$

HEAT EQUATION

From prior work the heat equation is:

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u \qquad c^2 = \frac{k}{\sigma \rho}$$

Fig. 291. Bar under consideration

In one dimension (laterally insulated):

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Some boundary conditions at each end:

$$u(0, t) = u(L, t) = 0,$$

for all t

Initial Condition:

$$u(x, 0) = f(x),$$

specified $\quad 0 \le x \le L$

CONVERT TO ORDINARY DIFFERENTIAL EQUATIONS

$$U(x,t) = F(x)G(t)$$
$$FG' = c^2F''G$$
$$\div\ FGc^2$$
$$G'/(c^2G) = F''/F = -p^2$$
$$F'' + p^2F = 0,$$

and

$$G' + (cp)^2G = 0$$

Solution for F is:

$$F = A \cos px + B \sin px$$

Applying boundary conditions at $x = 0$ and $x = L$ gives $A = 0$ and $\sin pL = 0$, which results in $pL = n\pi$, or:

$$p = (n\pi)/L.$$

Now

$$F_n = \sin(n\pi x/L)$$

let $\lambda_n = (cn\pi/L)$, and proceed to time solution

TIME-DEPENDENT SOLUTION

$$G_n = B_n \exp(-\lambda_n{}^2 t)$$

Combined solution in terms of space and time:

$$u_n(x,\ t) \ = \ B_n \sin\frac{n\pi x}{L}\ e^{-\lambda_n^2 t}$$

This solution must satisfy the initial condition that u(x,0) equals f(x)

Thus:

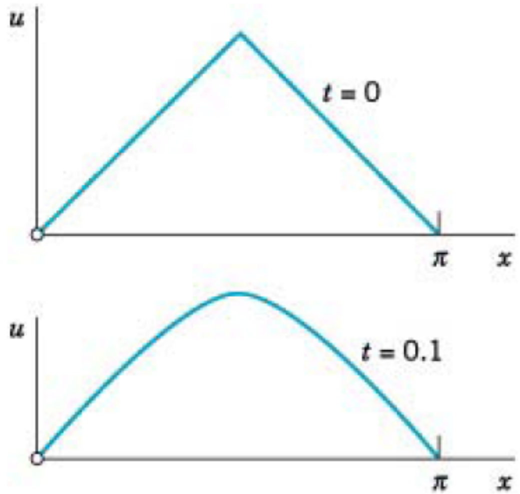
$$u(x,\ 0) \ = \ \sum_{n\,=\,1}^{\infty} B_n \sin\frac{n\pi x}{L} \ = \ f(x)$$

As before, the constants B_n must be the coefficients of the Fourier sine series:

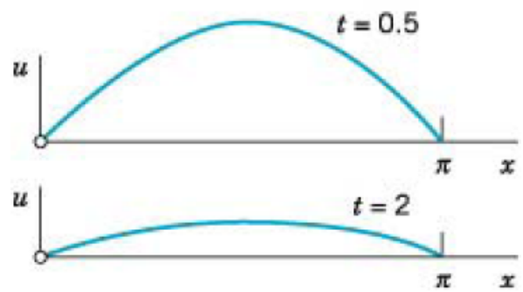
$$B_n \ = \ \frac{2}{L}\int_0^L f(x)\sin dx$$

SOME OBSERVATIONS

- The exponential terms in u(x,t) approach zero for increasing time, and this should be expected, as the edge conditions (x = 0 and x = L) are at zero, and there is no internal heat generation.
- Since exp(-0.001785t) is 0.5, whenever the exponent for the nth term, $\lambda_n t$, is -0.693, the temperature has decreased by 1/2.
- If the ends of the bar were perfectly insulated, then over time the bar temperature will approach some uniform value



$$f(x) = \begin{cases} x & \text{if } 0 < x < L/2, \\ L - x & \text{if } L/2 < x < L. \end{cases}$$



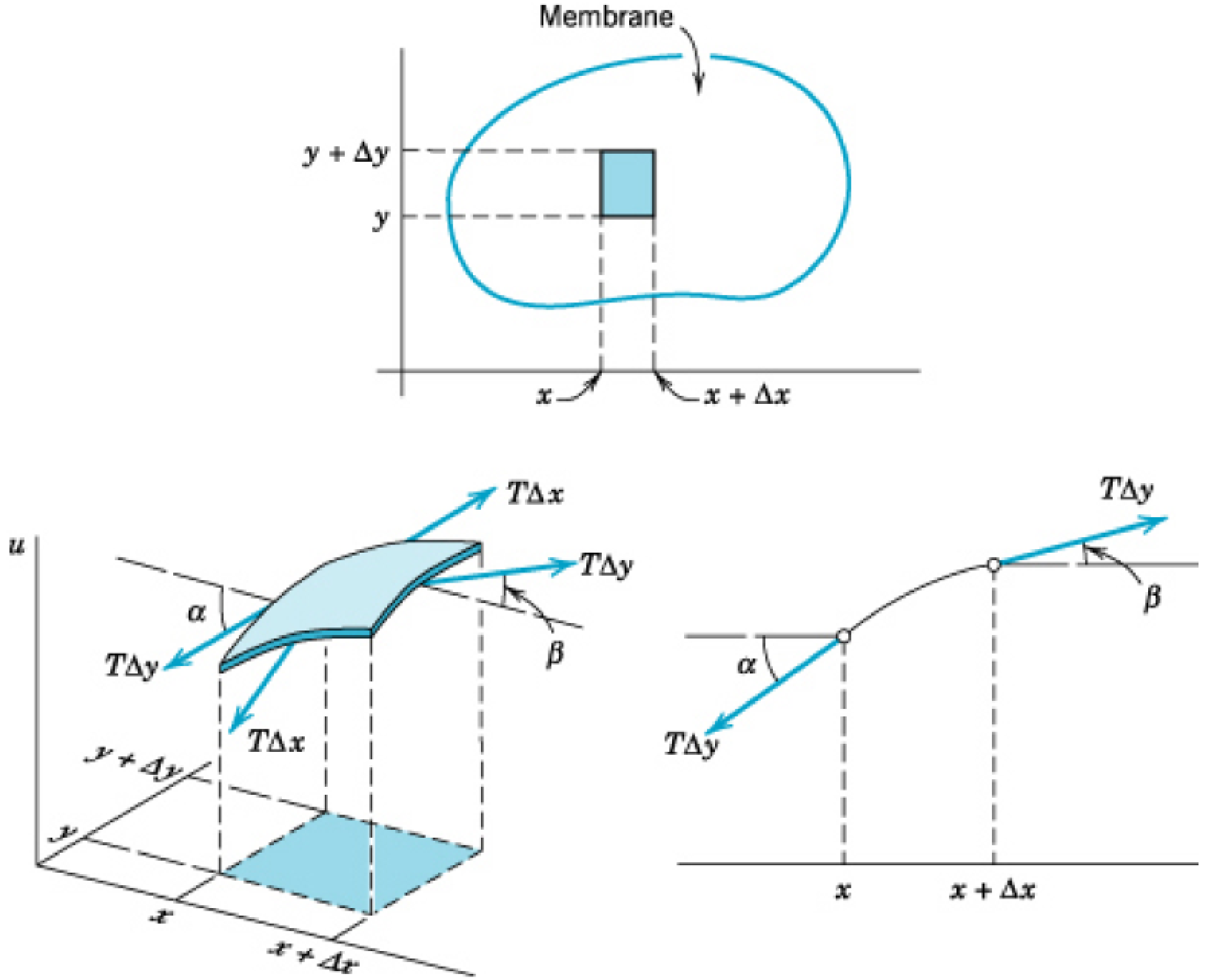
VIBRATING MEMBRANE AND THE TWO-DIMENSIONAL WAVE EQUATION

Three Assumptions:

- The mass of the membrane is constant, the membrane is perfectly flexible, and offers no resistance to bending;
- The membrane is stretched and then fixed along its entire boundary in the x-y plane. Tension per unit length (T) which is caused by stretching is the same at all points and in the plane and does not change during the motion;
- The deflection of the membrane u(x,y,t) during vibratory motion is small compared to the size of the membrane, and all angles of inclination are small

GEOMETRY OF VIBRATING "DRUM"

Vibrating Membrane



NOW FOR NEWTON'S LAW

$$T\Delta y(\sin\beta - \sin\alpha) \approx T\Delta y(\tan\beta - \tan\alpha)$$
$$= T\Delta y\left[u_x(x+\Delta x, y_1) - u_x(x, y_2)\right]$$

$$(\rho\Delta x\Delta y)\frac{\partial^2 u}{\partial t^2} = T\Delta y\left[u_x(x+\Delta x, y_1) - u_x(x, y_2)\right] + T\Delta x\left[u_y(x_1, y+\Delta y) - u_y(x_2, y)\right]$$

Divide by $\rho\Delta x\Delta y$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left[\frac{u_x(x+\Delta x, y_1) - u_x(x, y_2)}{\Delta x} + \frac{u_y(x_1, y+\Delta y) - u_y(x_2, y)}{\Delta y}\right]$$

Take limit as $\Delta x \to 0, \Delta y \to 0$

$$\frac{T}{\rho} = c^2$$

This is the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

WAVE EQUATION

- In Laplacian form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

where

$$c^2 = T/\rho$$

  - Some boundary conditions: u = 0 along all edges of the boundary.
  - Initial conditions could be the initial position and the initial velocity of the membrane
  - As before, the solution will be broken into separate functions of x,y, and t.
  - Subscripts will indicate variable for which derivatives are taken.

SOLUTION OF 2-D WAVE EQUATION

Let

$$u(x, y, t) = F(x, y)G(t)$$

substitute into wave equation:

$$F\ddot{G} = c^2\left(F_{xx}G + F_{yy}G\right);$$

divide by c^2FG, to get:

$$\frac{\ddot{G}}{c^2G} = \frac{1}{F}\left(F_{xx} + F_{yy}\right) = -\nu^2$$

This gives two equations, one in time and one in space. For time,

$$\ddot{G} + \lambda^2 G = 0$$

where $\lambda = c\nu$, and, what is called the amplitude function:

$$F_{xx} + F_{yy} + \nu^2 F = 0$$

also known as the Helmholtz equation.

SEPARATION OF THE HELMHOLTZ EQUATIONS

Let $F(x, y)=H(x)Q(y)$

and, substituting into the Helmholtz:

$$\frac{d^2H}{dx^2}\mathrm{Q} = -\left(H\frac{d^2Q}{dy^2} + \nu^2 HQ\right)$$

Here the variables may be separated by dividing by HQ:

$$\frac{1}{\mathrm{H}}\frac{\mathrm{d}^2\mathrm{H}}{\mathrm{dx}^2} = -\frac{1}{\mathrm{Q}}\left(\frac{\mathrm{d}^2\mathrm{Q}}{\mathrm{dy}^2} + \nu^2\mathrm{Q}\right) = -\mathrm{k}^2$$

Note: $p^2 = \nu^2 - k^2$

As usual, set each side equal to a constant, $-k^2$. This leads to two ordinary linear differential equations:

$$\frac{\mathrm{d}^2\mathrm{H}}{\mathrm{dx}^2} + \mathrm{k}^2\mathrm{H} = 0, \qquad \frac{\mathrm{d}^2\mathrm{Q}}{\mathrm{dy}^2} + \mathrm{p}^2\mathrm{Q} = 0$$