# Chapter 8. Wave Forces

Potential flow approach

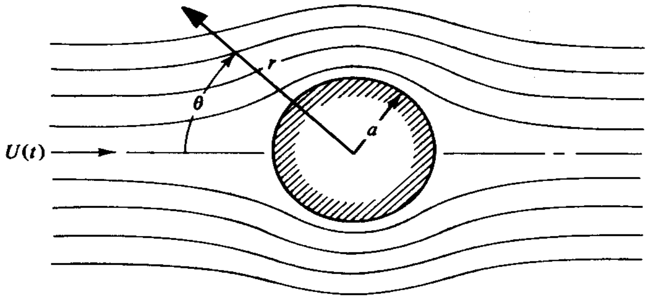



**Figure 8-1 Potential flow around a circular cylinder.**

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{r^2\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{with} \quad u_r = -\frac{\partial\phi}{\partial r}, \quad u_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, \quad u_z = -\frac{\partial\phi}{\partial z}$$

$$\phi(r,\theta,t) = U(t) r\left(1+\frac{a^2}{r^2}\right)\cos\theta$$

$$u_r(a,\theta,t) = -\frac{\partial\phi}{\partial r}\bigg|_{r=a} = -U(t)\left(1-\frac{a^2}{r^2}\right)\cos\theta\bigg|_{r=a} = 0$$

$$u_\theta(a,\theta,t) = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}\bigg|_{r=a} = U(t)\left(1+\frac{a^2}{r^2}\right)\sin\theta\bigg|_{r=a} = 2U(t)\sin\theta$$

Bernoulli equation at cylinder wall and far upstream:

$$\left[\frac{p(r,\theta)}{\rho} + gz + \frac{u_r^2+u_\theta^2}{2} - \frac{\partial\phi}{\partial t}\right]_{r=a} = \left[\frac{p(r,\theta)}{\rho} + gz + \frac{u_r^2+u_\theta^2}{2} - \frac{\partial\phi}{\partial t}\right]_{\substack{r=l>>a\\ \theta=0}}$$

$$p(a,\theta) - p(l,0) = \rho\left[\frac{U^2(t)}{2}(1-4\sin^2\theta) + 2a\frac{dU}{dt}\cos\theta - l\frac{dU}{dt}\right] + O(a^2/l^2)$$

↑ Steady flow term ↖ ↗ Inertial term

Steady flow term:

$$p(a,\theta) - p(l,0) = \frac{\rho U^2(t)}{2}(1 - 4\sin^2\theta)$$

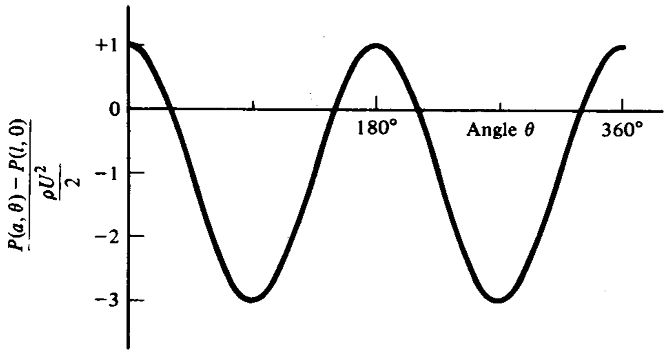



**Figure 8.2** Pressure distribution around cylinder for case of ideal flow. Note the low pressure at the sides, $\theta = 90°$, and the symmetry with respect to $\theta = 0°$ and $\theta = 90°$.

pressure at front face = pressure at rear face $\rightarrow$ net pressure force = 0

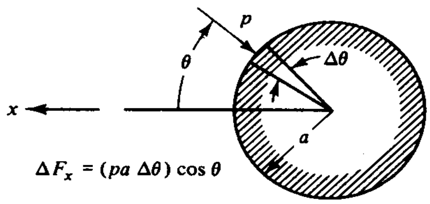



**Figure 8.3** Calculation of elemental force in $x$ direction. $\Delta F_x$ is positive in the downstream ($-x$) direction.

Drag force (=net force in *x*-direction):

$$dF_D = \int_0^{2\pi} p(a,\theta) a\cos\theta d\theta = \int_0^{2\pi}\left[\frac{\rho U^2(t)}{2}(1 - 4\sin^2\theta) + p(l,0)\right] a\cos\theta d\theta = 0$$

No force on cylinder in ideal steady flow (D'Alembert's paradox)

↑

∵ Potential flow assumption precludes the formation of boundary layers and a wake

Pressure distribution for real flow:

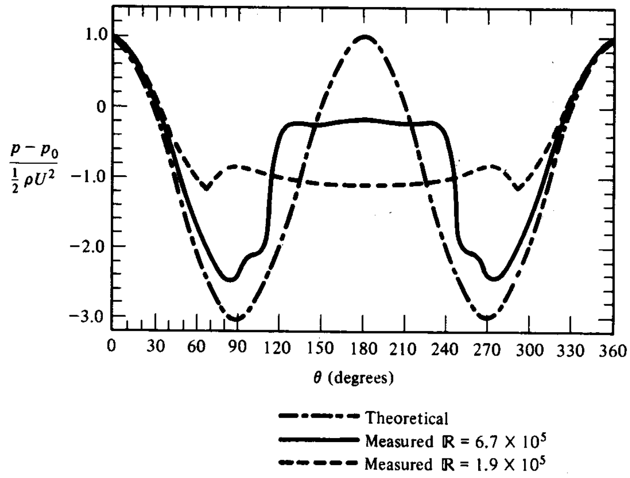



**Figure 8.4** Measured pressure distributions around cylinders. (From Goldstein, 1938.)

$$dF_D = 2\int_0^{\theta_s} \frac{\rho U^2(t)}{2}(1-4\sin^2\theta)a\cos\theta d\theta + 2\int_{\theta_s}^{\pi} p_{\text{wake}} a\cos\theta d\theta$$

$$= \rho U^2(t)a\left[\int_0^{\theta_s}(1-4\sin^2\theta)\cos\theta d\theta + \int_{\theta_s}^{\pi}\frac{p_{\text{wake}}}{\rho U^2(t)/2}\cos\theta d\theta\right]$$

$$= C_D(\mathbf{R})\frac{\rho D U^2(t)}{2}$$

$$= C_D(\mathbf{R})\frac{\rho A U^2(t)}{2}$$

where $D = 2a$ = diameter of cylinder, $A = 2a$ = projected area/unit elevation, $\mathbf{R} = UD/\nu$ = Reynolds number, $C_D$ = drag coefficient

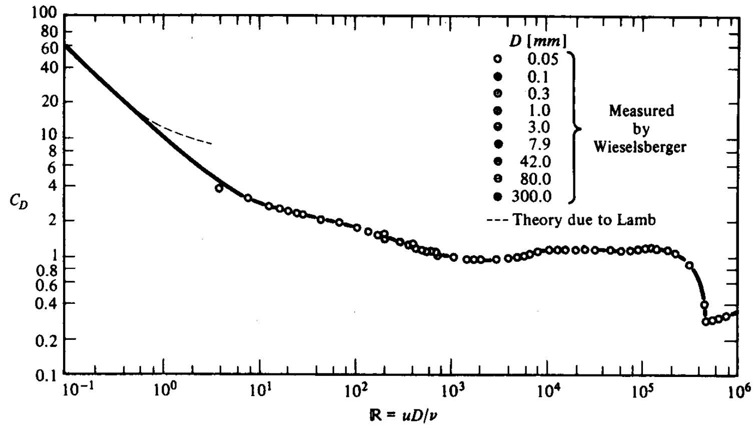



**Figure 8.5** Variation of drag coefficient, $C_D$ with Reynolds number ℝ for a smooth circular cylinder. (From H. Schlichting, *Boundary Layer Theory*. Copyright © 1968 by McGraw-Hill Book Company. Used with the permission of McGraw-Hill Book Company.)

Unsteady flow:

$$p(a,\theta)-p(l,0)=\rho\left[\frac{U^2(t)}{2}(1-4\sin^2\theta)+2a\frac{dU}{dt}\cos\theta-l\frac{dU}{dt}\right]$$

$$\begin{aligned}dF_I &= \int_0^{2\pi}\rho\frac{dU(t)}{dt}2a^2\cos^2\theta d\theta-\int_0^{2\pi}\rho\frac{dU(t)}{dt}la\cos\theta d\theta \\ &= 2\rho\pi a^2\frac{dU}{dt} \\ &= C_M\rho V\frac{dU}{dt}\end{aligned}$$

where $V=\pi a^2$ = volume of cylinder per unit length

$C_M=1+k_m$ = inertia coefficient, $k_m$ = added mass coefficient

↑

1: pressure gradient to accelerate the fluid in the absence of cylinder

$k_m$ : additional local pressure gradient to accelerate the neighboring fluid around the cylinder (shape dependent, e.g. $k_m=1$ for circular cylinder)

In the case where a cylinder is accelerating in quiescent water, $C_M=k_m$ because there is no pressure gradient in quiescent water.

Read textbook for more detailed explanation for the added mass coefficient, which is briefly summarized as follows:

Vertical buoyancy force in hydrostatic (quiescent) fluid is

$F_B=\rho gV$ (i.e. Archimedes principle)

Vertical pressure gradient in hydrostatic fluid is

$$\frac{\partial p}{\partial z}=-\rho g$$

Therefore

$$F_B=-\frac{\partial p}{\partial z}V$$

The buoyancy force in hydrostatic fluid is caused by the pressure difference in vertical direction (i.e. larger pressure on the lower side of a body than on the upper side). There is no horizontal buoyancy force in hydrostatic fluid because $\partial p / \partial x = 0$. However, $\partial p / \partial x \neq 0$ in an accelerating fluid. In the form similar to the vertical buoyancy force in hydrostatic fluid, the horizontal buoyancy-like force in accelerating fluid is

$$F_{BH} = -\frac{\partial p}{\partial x} V = \rho \frac{du}{dt} V \qquad (\because \frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \text{ Euler equation})$$

Comparing with the inertia force,

$$dF_I = C_M \rho V \frac{du}{dt} = (1 + k_m) \rho V \frac{du}{dt}$$

↑

'1' indicates the horizontal buoyancy-like force in accelerating fluid

↑

This force exists even if there is no structure. If there is a structure, additional acceleration will occur around the structure, which is represented by $k_m$ (added mass coefficient)

$$(1 + k_m) \rho V = \rho V + k_m \rho V$$

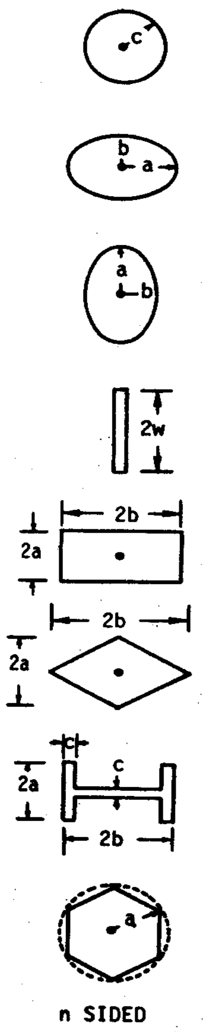
↓ mass of the structure ↘ added mass

If a structure is accelerating through a quiescent ideal fluid, since there is no pressure gradient in the fluid, the force would only be due to the added mass coefficient, i.e. $dF_I = C_M \rho V \frac{du}{dt} = k_m \rho V \frac{du}{dt}$

Sarpkaya and Isaacson (1981), Mechanics of Wave Forces on Offshore Structures, gives

## APPENDIX A

### Table 2.3 Added Masses of Various Bodies




ADDED MASS PER UNIT LENGTH $= \rho \forall C_a$ ) e.g.

| SHAPE (MOTION ⟷) | a/b | ADDED MASS PER UNIT LENGTH | Reference |
|---|---|---|---|
| CIRCLE | | $\rho\pi c^2$ ← $C_a = 1$ | |
| ELLIPSE (⟷) | | $\rho\pi b^2$ | |
| ELLIPSE (⟷) | | $\rho\pi a^2$ | |
| PLATE (⟷) | | $\rho\pi w^2$ | |
| RECTANGLE (⟷) | ∞ | 1.00 $\rho\pi a^2$ | |
| | 10 | 1.14 " | |
| | 5 | 1.21 " | |
| | 2 | 1.36 " | |
| | 1 | 1.51 $\rho\pi a^2$ | |
| | 0.5 | 1.70 " | |
| | 0.2 | 1.98 " | |
| | 0.1 | 2.23 " | (Wendel 1950) |
| DIAMOND (⟷) | 2 | 0.85 " | |
| | 1 | 0.76 " | |
| | 0.5 | 0.67 " | |
| | 0.2 | 0.61 " | (Wendel 1950) |
| I-BEAM (⟷), a/c = 2.6, b/c = 3.6 | | 2.11 $\rho\pi a^2$ | (Patton 1965) |
| REGULAR POLYGON (⟷ OR ↕) | n = 3 | 0.654 $\rho\pi a^2$ | |
| | 4 | 0.787 " | |
| | 5 | 0.823 " | |
| | 6 | 0.867 " | |
| | ∞ | 1.000 " | (Wendel 1950) |

Forces due to real fluids

Morison equation:

$$dF = dF_D + dF_I = \frac{1}{2} C_D \rho A u |u| + C_M \rho V \frac{du}{dt}$$

Assuming constant $C_D$ and $C_M$ over the depth and using linear wave theory,

Total force $= F = \int_{-h}^{\eta} dF$ = Eq. (8.36)

Moment about seabed $= M = \int_{-h}^{\eta} (h+z) dF$ = Eq. (8.38)

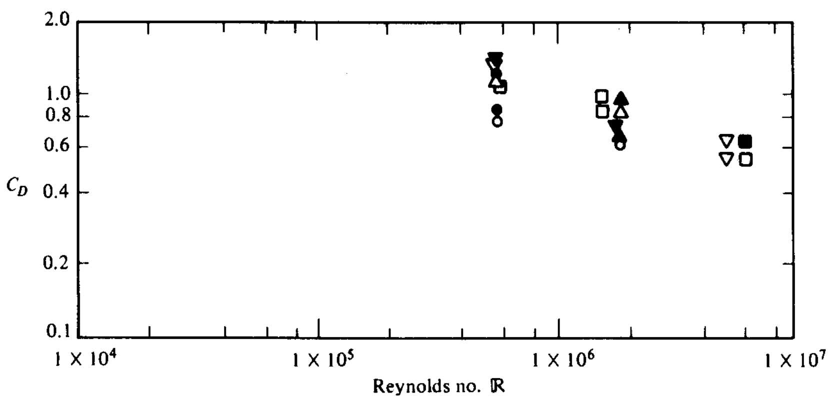
To determine $C_D$ and $C_M$ for real fluid, minimize

$$\varepsilon^2 = \frac{1}{I} \sum_{i=1}^{I} (F_{mi} - F_{pi})^2$$

with respect to $C_D$ and $C_M$ for the data groups of approximately same Reynolds number.

$C_D = f(\mathbf{R})$ as shown in Figure 8.9

$C_M = 1.33$ as shown in Figure 8.10 ($k_m$ reduces to 0.33 from 1.0 for potential flow)




| Dia. (ft.) | Symbol Inline | Symbol Resultant |
|---|---|---|
| 2* | ○ | ● |
| 3* | △ | ▲ |
| 3.71° | □ | ■ |
| 4* | ▽ | ▼ |

*Water depth ~ 33 ft

°Water depth ~ 100 ft

**Figure 8.9** Drag coefficient variation with Reynolds number as determined by Dean and Aagaard (1970). Copyright 1970 SPE-AIME.

Relative importance of inertia and drag forces:

$$\frac{(dF_I)_{\max}}{(dF_D)_{\max}} = \frac{C_M \pi kh}{C_D} \frac{D}{H}$$

Therefore, large $D/H \rightarrow$ inertia-dominant

small $D/H \rightarrow$ drag-dominant

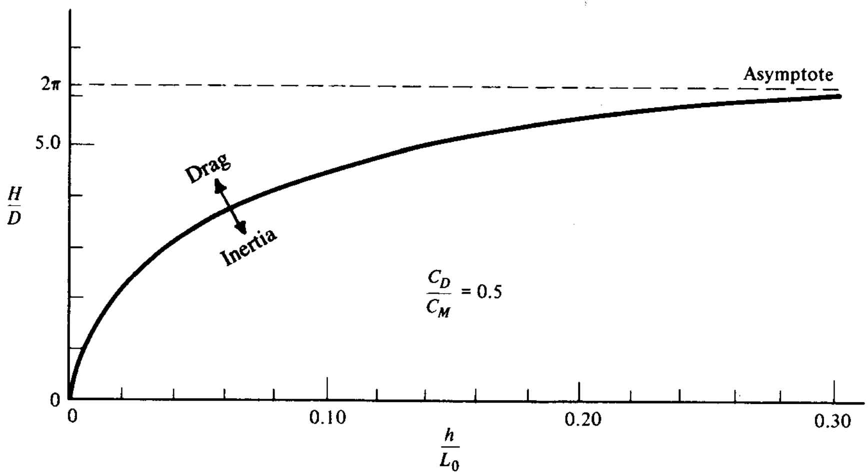
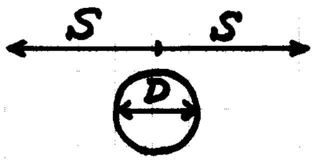



**Figure 8.13** *H/D* versus *h/L₀* for condition of equal maximum drag and inertia force components.

$$\frac{(dF_I)_{\max}}{(dF_D)_{\max}} = \pi^2 \frac{C_M}{C_D} \frac{1}{u_m T / D} = \frac{\pi}{2} \frac{C_M}{C_D} \frac{1}{S/D}$$

where $u_m T / D$ = Keulegan-Carpenter number, and $S/D$ = displacement parameter




$u = u_m \cos \sigma t$ ; $S = \int_0^{T/4} u_m \cos \sigma t dt = \frac{u_m T}{2\pi}$

large $u_m T / D$ or $S/D \rightarrow$ drag-dominant

small $u_m T / D$ or $S/D \rightarrow$ inertia-dominant

Maximum total force:

$$F_T = F_D \cos\sigma t\,|\cos\sigma t| - F_I \sin\sigma t$$

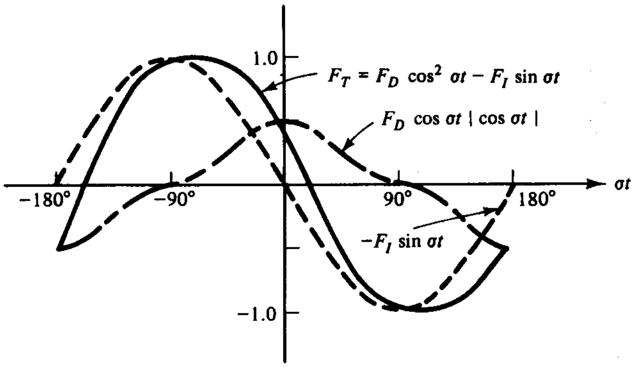



**Figure 8.18** **Illustration of force component combination for the case of $|F_I| = 2|F_D|$.**

$(F_T)_m$ occurs somewhere between $\sigma t = -90°$ and $\sigma t = 0°$.

At $(F_T)_m$, $dF_T / dt = 0 = -2F_D\sigma\cos(\sigma t)_m \sin(\sigma t)_m - F_I\sigma\cos(\sigma t)_m$ (8.67)

Two possible roots:

1) Dividing Eq. (8.67) by $\sigma\cos(\sigma t)_m$, $\sin(\sigma t)_m = -\dfrac{F_I}{2F_D}$ $\rightarrow$ $(F_T)_m = F_D + \dfrac{F_I^2}{4F_D}$

2) $\cos(\sigma t)_m = 0$ $\rightarrow$ $(F_T)_m = F_I$ $\leftarrow$ This solution must be taken if $F_I / 2F_D > 1$

Inertia force dominant case

Large structure $\rightarrow$ Drag force is negligible $\rightarrow$ Potential flow theory

MacCamy and Fuchs solution for circular cylinder:

Incident wave propagating in $+x$ direction is

$$\phi_I = -\frac{gH}{2\sigma}\frac{\cosh k(h+z)}{\cosh kh}e^{i(kx-\sigma t)}$$
$$= -\frac{gH}{2\sigma}\frac{\cosh k(h+z)}{\cosh kh}\left[J_0(kr)+\sum_{m=1}^{\infty}2i^m\cos m\theta J_m(kr)\right]e^{-i\sigma t}$$

which satisfies the Laplace equation, BBC, LKFSBC and LDFSBC.

Reflected wave from the cylinder is

$$\phi_R = \sum_{m=0}^{\infty}A_m\cos m\theta[J_m(kr)+iY_m(kr)]e^{-i\sigma t}\frac{\cosh k(h+z)}{\cosh kh}$$

which satisfies the Laplace equation, BBC, and radiation boundary condition for large $kr$.

No-flow condition at the cylinder, $\partial(\phi_I+\phi_R)/\partial r = 0$ at $r = a$, gives

$$\phi_{I+R} = \mathrm{Re}\left\{\frac{gH\cosh k(h+z)}{2\sigma\cosh kh}e^{-i\sigma t}\left\{\left[J_0(kr)-\frac{J_1'(ka)}{J_0'(ka)-iY_0'(ka)}(J_0(kr)+iY_0(kr))\right]\right.\right.$$
$$\left.\left.+2\sum_{m=1}^{\infty}i^m\left[J_m(kr)-\frac{J_m'(ka)}{J_m'(ka)-iY_m'(ka)}(J_m(kr)+iY_m(kr))\right]\right\}\cos m\theta\right\}$$

Using the unsteady form of Bernoulli equation to obtain the pressure (see Eq. 8.5), force per unit length of the cylinder is obtained:

$$dF_I = \frac{2\rho gH}{k}\frac{\cosh k(h+z)}{\cosh kh}G\left(\frac{D}{L}\right)\cos(\sigma t-\alpha)$$

where

$$\tan\alpha = \frac{J_1'(ka)}{Y_1'(ka)}; \quad G\left(\frac{D}{L}\right) = \frac{1}{\sqrt{J_1'(ka)^2+Y_1'(ka)^2}}$$

Comparing with the general formula for inertial force,

$$dF_I = C_M \rho V \frac{\partial u}{\partial t}$$

we find $C_M = 4G(D/L)/\pi^3(D/L)^2$

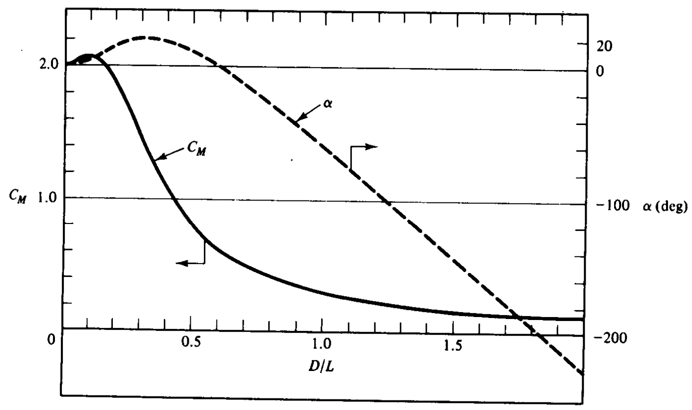



**Figure 8.19** Variation of inertia coefficient $C_M$ and phase angle $\alpha$ of maximum force with parameter $D/L$.

$C_M$ and $\alpha$ reduce to 2.0 and 0, respectively, for small values of $D/L$, as predicted from potential flow theory.

Wave force on large rectangular objects:

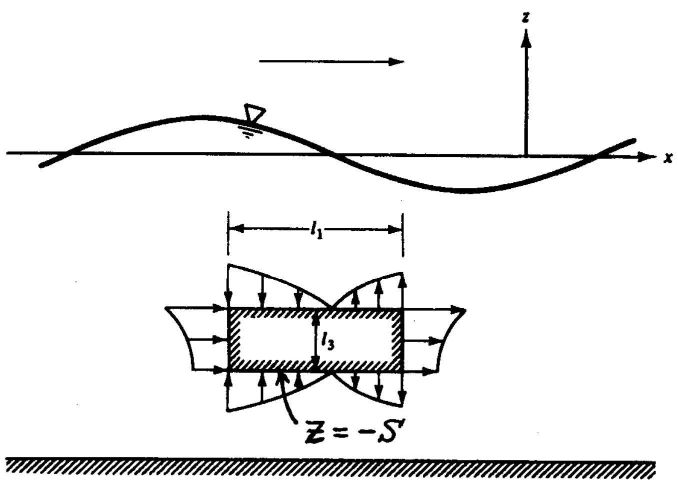



**Figure 8.20** Dynamic wave pressures on rectangular object.

$l_1$, $l_2$, and $l_3$ = length in $x$, $y$, and $z$ directions

$z = -S$ at the bottom of the object

Calculate the horizontal inertia force $F_x$ due to wave propagating in the $x$ direction

In the absence of the structure,

$$p(x,z,t) = \rho g \eta K_p(z) = \frac{\rho g H}{2} \frac{\cosh k(h+z)}{\cosh kh} \cos(kx - \sigma t)$$

$$P_1(x_1,t) = \int_{-S}^{-S+l_3} l_2 p(x_1,z,t)dz = \frac{l_2 l_3 \rho g H \cos(kx_1 - \sigma t)}{2\cosh kh} \cosh k\left(h - S + \frac{l_3}{2}\right) \frac{\sinh\left(\frac{1}{2}kl_3\right)}{\frac{1}{2}kl_3}$$

$$P_2(x_1+l_1,t) = \int_{-S}^{-S+l_3} l_2 p(x_1+l_1,z,t)dz$$

$$= \frac{l_2 l_3 \rho g H}{2\cosh kh} \cos[k(x_1+l_1) - \sigma t] \cosh k\left(h - S + \frac{l_3}{2}\right) \frac{\sinh\left(\frac{1}{2}kl_3\right)}{\frac{1}{2}kl_3}$$

$$F_x = P_1 - P_2 = \frac{l_1 l_2 l_3 \rho g H k \cosh k(h - S + l_3/2)}{2\cosh kh} \frac{\sinh(kl_3/2)}{kl_3/2} \frac{\sin(kl_1/2)}{kl_1/2} \sin\left(k\left(x_1 + \frac{l_1}{2}\right) - \sigma t\right)$$

This can be written in a more familiar form:

$$F_x = \rho V \frac{\sinh(kl_3/2)}{kl_3/2} \frac{\sin(kl_1/2)}{kl_1/2} \frac{\partial u}{\partial t}$$

where $\partial u / \partial t$ is evaluated at the center of the rectangular object.

For an object occupying from bottom to water surface, $S = l_3 = h$ gives

$$F_x = \frac{l_1 l_2 h \rho g H k \cosh(kh/2)}{2\cosh kh} \frac{\sinh(kh/2)}{kh/2} \frac{\sin(kl_1/2)}{kl_1/2} \sin\left(k\left(x_1 + \frac{l_1}{2}\right) - \sigma t\right)$$

Accounting for the interaction of the structure with the waves,

$$F_x' = (1 + k_m) F_x$$

Added mass coefficient, $k_m$, should be determined by experiments or Table 2.3 of

Sarpkaya and Isaacson (1981)

The vertical force can be calculated in a similar manner:

$$F_z = \rho V \frac{\sinh(kl_3/2)}{kl_3/2} \frac{\sin(kl_1/2)}{kl_1/2} \frac{\partial w}{\partial t}$$

where $\partial w/\partial t$ is again evaluated at the center of the object.

If the tank is situated on the bottom, such that the wave-induced pressure is not transmitted to the bottom of the tank,

$$F_z = -\rho V \frac{\coth(kl_3)}{kl_3} \frac{\sin(kl_1/2)}{kl_1/2} \frac{\partial w}{\partial t}$$

where $\partial w/\partial t$ is now calculated at the center of the top of the object.

Spectral approach for irregular waves

For inertia-dominant case,

Total force $(F_I)_{\max} = G(\sigma)H; \quad G(\sigma) = C_M \dfrac{\rho\pi D^2}{8k}\sigma^2$

For the case in which both drag and inertia forces are important, the force on an element length $ds$ at a distance $s$ above the seabed is

$$dF = \left( \frac{C_D \rho D}{2} u_m^2 \cos\sigma t \,|\cos\sigma t| - \frac{C_M \rho\pi D^2}{4} u_m \sigma \sin\sigma t \right) ds$$

Linearizing,

$$dF = \left( \frac{C_{D_L} \rho D}{2} u_m \cos\sigma t - \frac{C_M \rho\pi D^2}{4} u_m \sigma \sin\sigma t \right) ds$$

where

$$C_{D_L} = C_D \sqrt{\frac{8}{\pi}} U_{rms}$$

Force spectrum is related to surface wave spectrum by

$$S_{dF}(\sigma) = | X_{dF}(\sigma) |^2 \, S_\eta(\sigma)$$

where

$$| X_{dF}(\sigma) |^2 = \left[ \frac{C_D \rho D}{2} \sqrt{\frac{8}{\pi}} U_{rms} X_u(\sigma, s) \right]^2 + \left[ \frac{C_M \rho\pi D^2}{4} X_u(\sigma, s)\sigma \right]^2$$

$$X_u(\sigma, s) = \frac{u_m}{|\eta|} = \sigma \frac{\cosh ks}{\sinh kh}$$

$$U_{rms} = \sqrt{\int_0^\infty | X_u(\sigma, s) |^2 \, S_\eta(\sigma) d\sigma}$$

Total wave force over the entire water depth is

$$S_F(\sigma) = \left\{ \left( \frac{C_D \rho D}{2} \right)^2 \sigma \frac{8}{\pi} G_1(\sigma) + \left( \frac{C_M \rho \pi D^2}{4} \right)^2 G_2(\sigma) \right\} S_\eta(\sigma)$$

where

$$G_1(\sigma) = \frac{\int_0^h U_{rms}(s) \cosh ksds}{\sinh kh}$$

$$G_2(\sigma) = \frac{\sigma^2}{k}$$