Chapter 9. Waves over Real Seabeds

We have to consider the followings in real sea:

- viscosity (fluid)
- Roughness, rigidity, permeability (seabed)

<u>Effect of viscosity</u>

Inside the bottom boundary layer,

- Viscosity is important.
- No-slip condition at bottom:

$$u = -\left.\frac{\partial \phi}{\partial x}\right|_{z=-h} = 0$$

Linear laminar Navier-Stokes equation for viscous fluid:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) - g$$

To evaluate relative sizes of various terms, introduce nondimensional variables:

$$x' = kx$$

$$z' = kz$$

$$t' = \sigma t$$

$$p' = \frac{p}{\rho g a}$$

$$(u', w') = \frac{1}{a\sigma}(u, w)$$

By chain rule,

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial x'}\frac{\partial x'}{\partial x}=k\frac{\partial}{\partial x'}$$
$$\frac{\partial^2}{\partial x^2}=\frac{\partial}{\partial x}\left(k\frac{\partial}{\partial x'}\right)=k\frac{\partial}{\partial x'}\left(\frac{\partial}{\partial x}\right)=k\frac{\partial}{\partial x'}\left(k\frac{\partial}{\partial x'}\right)=k^2\frac{\partial^2}{\partial x'^2}$$
$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t'}\frac{\partial t'}{\partial t}=\sigma\frac{\partial}{\partial t'}$$

The x-momentum equation becomes

$$a\sigma^2\frac{\partial u'}{\partial t'}=-\frac{k}{\rho}\rho ga\frac{\partial p'}{\partial x'}+\nu k^2a\sigma\left(\frac{\partial^2u'}{\partial x'^2}+\frac{\partial^2u'}{\partial z'^2}\right)$$

$$\frac{\partial u'}{\partial t'}=-\frac{gk}{\sigma^2}\frac{\partial p'}{\partial x'}+\frac{\nu k^2}{\sigma}\left(\frac{\partial^2u'}{\partial x'^2}+\frac{\partial^2u'}{\partial z'^2}\right)$$

where

$$\frac{gk}{\sigma^2}=\frac{gk}{C^2k^2}=\frac{gk^{-1}}{C^2}=\frac{1}{\mathbf{F}^2}=O(1);\qquad \mathbf{F}=\frac{C}{\sqrt{gk^{-1}}}$$
$$\frac{\nu k^2}{\sigma}=\frac{\nu k^2}{Ck}=\frac{\nu k}{C}=\frac{1}{\mathbf{R}}=O(10^{-7}\sim 10^{-8});\qquad \mathbf{R}=\frac{Ck^{-1}}{\nu}\quad\text{and}\quad \nu=O(10^{-5})$$

which implies that frictional stresses are negligible so that there is a slip boundary condition at the bottom ($u\neq 0$ at $z=-h$). However, physically, there is no flow at the bottom in the viscous flow. Hence, our argument above must be modified.

Near the bottom, u varies rapidly with z. Therefore, the vertical length scale must be different from the horizontal length scale. Let us take the vertical length scale as the boundary layer thickness, δ, so that

$$z'=\frac{z}{\delta}$$

and

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}\frac{\partial z'}{\partial z} = \frac{1}{\delta}\frac{\partial}{\partial z'}\,; \qquad \frac{\partial^2}{\partial z^2} = \frac{1}{\delta^2}\frac{\partial^2}{\partial z'^2}$$

Using this scale but the same scales as before for other parameters,

$$a\sigma^2\frac{\partial u'}{\partial t'} = -\frac{k}{\rho}\rho g a\frac{\partial p'}{\partial x'} + \nu k^2 a\sigma\frac{\partial^2 u'}{\partial x'^2} + \nu\frac{1}{\delta^2}a\sigma\frac{\partial^2 u'}{\partial z'^2}$$

$$\frac{\partial u'}{\partial t'} = -\frac{1}{\mathbf{F}^2}\frac{\partial p'}{\partial x'} + \frac{1}{\mathbf{R}}\frac{\partial^2 u'}{\partial x'^2} + \frac{\nu}{\delta^2\sigma}\frac{\partial^2 u'}{\partial z'^2}$$

The second term on the RHS is again very small compared with other terms, but the last term is of $O(1)$ with $\delta \sim \sqrt{\nu/\sigma}$. Now, the x -momentum equation can be approximated to

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$

Assuming $u = u_p + u_r$, where u_p and u_r, respectively, represent the potential and rotational part of the flow, we can separate the equation into

$$\frac{\partial u_p}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial u_r}{\partial t} = \nu\frac{\partial^2 u_r}{\partial z^2}$$

each of which has a form of Euler equation and diffusion equation, respectively.

For the potential flow part, we have

$$u_p = \frac{gak}{\sigma}\frac{\cosh k(h+z)}{\cosh kh}e^{i(kx-\sigma t)}$$

For the rotational part, assume

$$u_r = Af(z)e^{i(kx-\sigma t)}$$

Substituting into the diffusion equation,

$$(-i\sigma Af - \nu Af'')e^{i(kx-\sigma t)} = 0$$

or

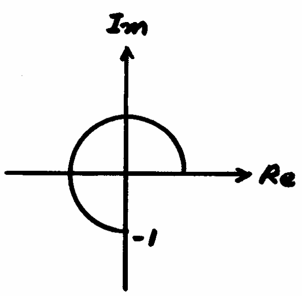
$$f'' + \frac{i\sigma}{\nu}f = 0$$

Solving this equation and keeping only the term that decays away from bed (or $f(z) \to 0$ as $(h+z) \to \infty$),

$$f(z) = Be^{\sqrt{-i\sigma/\nu}(h+z)} = Be^{-(1-i)\sqrt{\sigma/2\nu}(h+z)}$$

$\boxed{\text{Time-out}}$

De Moivre theorem: $z^{1/n} = \sqrt[n]{r}e^{i\theta/n}$



$$\left.\begin{array}{l} r = 1 \\ \theta = 3\pi/2 \\ n = 2 \end{array}\right\} \text{for } z = -i$$

$$\sqrt{-i} = \sqrt[2]{1e^{i\frac{3\pi}{2}/2}} = e^{i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}(1-i)$$

$$\sqrt{-i\sigma/\nu} = \sqrt{-i}\sqrt{\sigma/\nu} = -\frac{1}{\sqrt{2}}(1-i)\sqrt{\frac{\sigma}{\nu}} = -(1-i)\sqrt{\frac{\sigma}{2\nu}}$$

Now u_r becomes

$$\begin{aligned} u_r &= Ae^{-(1-i)\sqrt{\sigma/2\nu}(h+z)}e^{i(kx-\sigma t)} \\ &= Ae^{-\sqrt{\sigma/2\nu}(h+z)}e^{i(kx-\sigma t+\sqrt{\sigma/2\nu}(h+z))} \end{aligned}$$

Using $u = u_p + u_r = 0$ at $z = -h$,

$$\frac{gak}{\sigma}\frac{1}{\cosh kh}e^{i(kx-\sigma t)} + Ae^{i(kx-\sigma t)} = 0$$

or

$$A = -\frac{gak}{\sigma}\frac{1}{\cosh kh}$$

Therefore,

$$u_r = -\frac{gak}{\sigma}\frac{1}{\cosh kh}e^{-\sqrt{\sigma/2\nu}(h+z)}e^{i(kx-\sigma t+\sqrt{\sigma/2\nu}(h+z))}$$

and

$$u = \frac{gak}{\sigma\cosh kh}\left[\cosh k(h+z)\cos(kx-\sigma t) - e^{-\sqrt{\sigma/2\nu}(h+z)}\cos(kx-\sigma t+\sqrt{\sigma/2\nu}(h+z))\right]$$

See Figure 9.1 for the horizontal velocity profile near the bed.

The vertical velocity in the bottom boundary layer can be obtained from the continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \quad \rightarrow \quad \partial w=-\frac{\partial u}{\partial x}dz$$

$$w(z)=\left(\int_{-h}^{z}\partial w=w\Big|_{-h}^{z}=w(z)\right)=\int_{-h}^{z}\partial w=-\int_{-h}^{z}\frac{\partial u}{\partial x}dz=-\int_{0}^{h+z}\frac{\partial u}{\partial x}ds=\text{Eq. (9.12)}$$

<u>Bed shear stress</u>

$$\begin{aligned}\tau_{zx}(z=-h)&=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\cong\mu\left(\frac{\partial u_p}{\partial z}+\frac{\partial u_r}{\partial z}\right)\\&=\frac{\mu gak}{\sigma\cosh kh}\left(\sqrt{\frac{\sigma}{2\nu}}-i\sqrt{\frac{\sigma}{2\nu}}\right)e^{i(kx-\sigma t)}\\&=\frac{\mu gak}{\sigma\cosh kh}\sqrt{2}\sqrt{\frac{\sigma}{2\nu}}e^{i(kx-\sigma t-\pi/4)}\\&=\frac{\rho gak}{\cosh kh}\sqrt{\frac{\nu}{\sigma}}e^{i(kx-\sigma t-\pi/4)}\end{aligned}$$

where the relationship, $A(1-i)=\sqrt{2}Ae^{-i\pi/4}$, was used by De Moivre theorem. In terms of flow velocity, we have

$$\left(\tau_{zx}(z=-h)\right)_{\max}=\left(\frac{\rho f}{8}u_b|u_b|\right)_{\max}=\left(\frac{\rho f_w}{2}u_b|u_b|\right)_{\max}=\left(\frac{\rho gak}{\cosh kh}\sqrt{\frac{\nu}{\sigma}}e^{i(kx-\sigma t-\pi/4)}\right)_{\max}$$

where u_b = potential flow velocity outside boundary layer. Solving for the friction coefficient, f,

$$f=\frac{8}{u_{b_{\max}}^2}\frac{gak}{\cosh kh}\sqrt{\frac{\nu}{\sigma}}=\frac{8}{u_{b_{\max}}^2}\frac{gak\sigma}{\sigma\cosh kh}\sqrt{\frac{\nu}{\sigma}}=\frac{8}{\sqrt{u_{b_{\max}}^2/\sigma\nu}}=\frac{8}{\sqrt{u_{b_{\max}}\varsigma_b/\nu}}$$

or

$$f = \frac{8}{\mathbf{R}_b^{1/2}}; \quad \mathbf{R}_b = u_{b_{\max}} \varsigma_b / \nu = \varsigma_b^2 \sigma / \nu < 10^3 \sim 10^4 \text{ for smooth bottom}$$

See Figure 9.2 of textbook for linear relation between $\log f$ and $\log \mathbf{R}_b$ for laminar flow.

<u>Energy dissipation</u>

Mean rate of energy dissipation is given by (Fluid Dynamics, Batchelor)

$$\begin{aligned} \varepsilon_D &= \rho\nu \overline{\int_0^h \left\{ 2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 \right\} d(h+z)} \\ &\cong \rho\nu \overline{\int_0^h \left(\frac{\partial u_r}{\partial z}\right)^2 d(h+z)} \\ &= \frac{\nu k \sqrt{\sigma / 2\nu}}{\sinh 2kh} E \end{aligned}$$

$$\varepsilon_D = -\frac{\partial (E C_g)}{\partial x} = AE \quad \text{where} \quad A = \nu k \sqrt{\sigma / 2\nu} / \sinh 2kh .$$

Consider waves on a flat bottom. Since $C_g \neq f(x)$, we have

$$\frac{\partial E}{\partial x} = -\frac{A}{C_g} E$$

$$E = E_0 e^{-(A/C_g)x}$$

$$a^2 = a_0^2 e^{-(A/C_g)x}$$

$$a = a_0 e^{-(A/2C_g)x} = a_0 e^{-\left(\frac{\nu k \sqrt{\sigma / 2\nu}}{2C_g \sinh 2kh}\right)x}$$

which states that the wave amplitude decays exponentially with x.

Turbulent boundary layer

Large waves + Rough bottom → Turbulent boundary layer (most cases in nature)

In turbulent boundary layer,

$$f = f(k_e / \varsigma_b)$$

as shown in Figure 9.2 and where

$$k_e = 2d_{90}$$

considering only skin friction without form drag due to ripples.

The bed shear stress is given by

$$\tau_{zx}(z = -h) = \frac{\rho f}{8} u_b |u_b| = \frac{\rho f}{8} u_m^2 \cos\sigma t |\cos\sigma t|$$

where

$$u_b = u_m \cos\sigma t \quad \text{with} \quad u_m = \frac{gak}{\sigma \cosh kh} = \frac{a\sigma}{\sinh kh}$$

Note that the time average of shear stress is zero. The mean rate of energy dissipation is not zero and is given by

$$\varepsilon_D = \overline{\tau_{zx}(-h)u_b} = \overline{\frac{\rho f}{8} u_b^2 |u_b|} = \frac{\rho f}{8} u_m^3 \frac{1}{T/4} \int_0^{T/4} \cos^3 \sigma t dt$$
$$= \frac{\rho f}{8} u_m^3 \frac{1}{\sigma(T/4)} \int_0^{T/4} \sigma \cos^3 \sigma t dt$$
$$= \frac{\rho f}{8} u_m^3 \frac{4}{2\pi} \frac{2}{3} = \frac{\rho f}{6\pi} u_m^3$$
$$= \frac{\rho f}{6\pi} \left(\frac{a\sigma}{\sinh kh} \right)^3$$

Energy loss with distance

$$\frac{d(EC_g)}{dx} = -\varepsilon_D = -\frac{\rho f}{6\pi} \frac{a^3 \sigma^3}{\sinh^3 kh}$$

On flat bottom,

$$\frac{1}{2} \rho g C_g \frac{da^2}{dx} = -\frac{\rho f}{6\pi} \frac{a^3 \sigma^3}{\sinh^3 kh}$$
$$a g C_g \frac{da}{dx} = -\frac{f}{6\pi} \frac{a^3 \sigma^3}{\sinh^3 kh}$$
$$\frac{da}{a^2} = -\frac{f \sigma^3}{6\pi g C_g \sinh^3 kh} dx$$

Defining $A = f\sigma^3 / (6\pi g C_g \sinh^3 kh)$ and integration give

$$-\frac{1}{a} = -Ax + C_1$$

Using the boundary condition $a = a_0$ at $x = 0$, we get $C_1 = -1/a_0$. Therefore

$$-\frac{1}{a} = -Ax - \frac{1}{a_0}$$

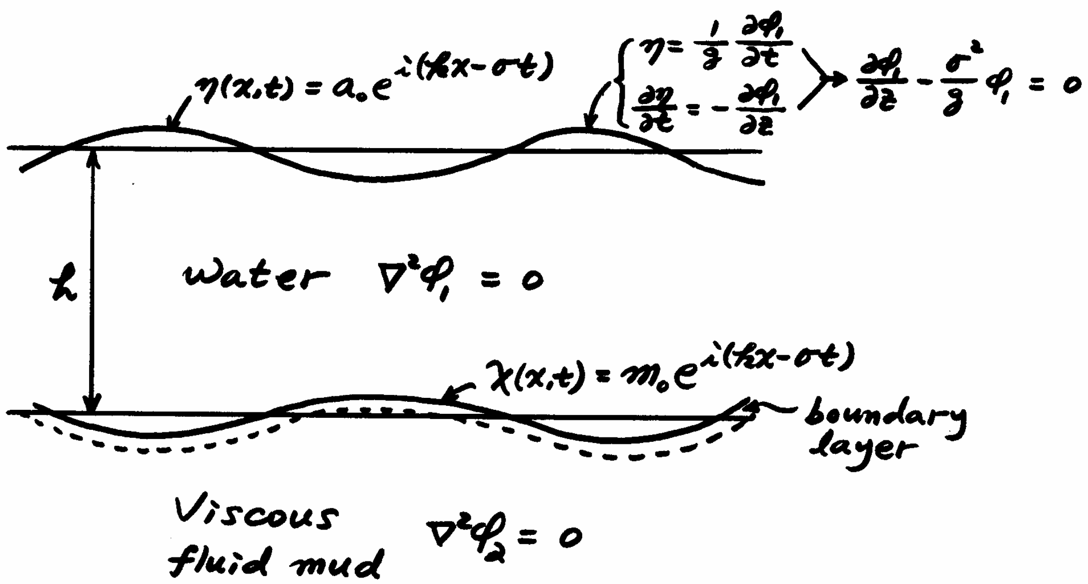
Finally we have

$$a = \frac{a_0}{1 + A a_0 x}$$

which is not exponential decay with x. For small Aa_0, we have

$$a \cong a_0 (1 - A a_0 x) \cong a_0 e^{-A a_0 x} \leftarrow \text{exponential decay}$$

<u>Waves over viscous mud bottom</u>



At the bottom, we need the kinematic and dynamic matching conditions:

$$\frac{\partial \chi}{\partial t} = -\frac{\partial \phi_2}{\partial z} = -\frac{\partial \phi_1}{\partial z} \qquad \text{on} \quad z = -h + \chi$$

$$p_1 = p_2 \qquad \text{on} \quad z = -h + \chi$$

In Region 1, we can assume the solution as

$$\phi_1 = \left(A \cosh k(h+z) + B \sinh k(h+z)\right) e^{i(kx - \sigma t)}$$

where the second term is included because the vertical velocity is no longer zero at the bottom. Using LDFSBC,

$$a_0 = -\frac{i\sigma}{g}(A\cosh kh + B\sinh kh)$$

Using LCFSBC,

$$k(A\sinh kh + B\cosh kh) - \frac{\sigma^2}{g}(A\cosh kh + B\sinh kh) = 0$$

Multiplying $g/\cosh kh$ on both sides,

$$A(gk\tanh kh - \sigma^2) + B(gk - \sigma^2\tanh kh) = 0$$

From LDFSBC,

$$A = \frac{-\frac{ga_0}{i\sigma} - B\sinh kh}{\cosh kh} = \frac{iga_0}{\sigma\cosh kh} - B\tanh kh$$

Plug in LCFSBC,

$$(gk - \sigma^2\tanh kh)B = \left(\frac{-iga_0}{\sigma\cosh kh} + B\tanh kh\right)(gk\tanh kh - \sigma^2)$$

$$B(gk - \sigma^2\tanh kh - gk\tanh^2 kh + \sigma^2\tanh kh) = \frac{iga_0}{\sigma\cosh kh}(\sigma^2 - gk\tanh kh)$$

$$Bgk\frac{1}{\cosh^2 kh} = \frac{iga_0}{\sigma\cosh kh}(\sigma^2 - gk\tanh kh)$$

$$B = \frac{\cosh^2 kh}{gk}\frac{iga_0}{\sigma\cosh kh}(\sigma^2 - gk\tanh kh) = \frac{ia_0\cosh kh}{\sigma k}(\sigma^2 - gk\tanh kh)$$

Now,

$$
\begin{aligned}
A &= \frac{iga_0}{\sigma \cosh kh} - \frac{ia_0 \cosh kh}{\sigma k}(\sigma^2 - gk \tanh kh)\frac{\sinh kh}{\cosh kh} \\
&= \frac{ia_0 \cosh kh}{\sigma k}\left(\frac{gk}{\cosh^2 kh} - \sigma^2 \tanh kh + gk \tanh^2 kh\right) \\
&= \frac{ia_0 \cosh kh}{\sigma k}(gk - \sigma^2 \tanh kh)
\end{aligned}
$$

Note: On rigid bottom, $B = 0$ or $\sigma^2 = gk \tanh kh$.

In Region 2, assuming infinite depth,

$$\phi_2 = de^{k(h+z)} e^{i(kx-\sigma t)}$$

which indicates exponential decay as $h + z \to -\infty$. The horizontal velocity is given by

$$u_2 = u_{2_p} + u_{2_r} = -\frac{\partial \phi_2}{\partial x} + u_{2_r}$$

where u_{2_r} is the boundary layer correction given by (cf. Eq. 9.9)

$$u_{2_r} = fe^{(1-i)\sqrt{\sigma/2\nu_2}(h+z)} e^{i(kx-\sigma t)}$$

Assuming w_r is small and using the linearized kinematic matching condition at the boundary:

$$\frac{\partial \chi}{\partial t} = -\frac{\partial \phi_2}{\partial z} = -\frac{\partial \phi_1}{\partial z} \qquad \text{on} \quad z = -h$$

$$-i\sigma m_0 = -dk = -Bk$$

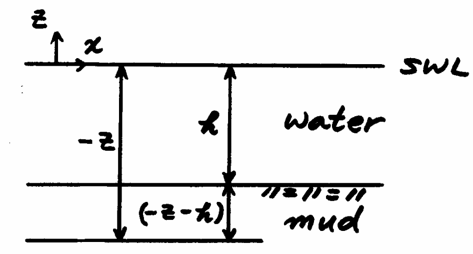
$$\therefore\ d = B\,,\quad m_0 = \frac{Bk}{i\sigma} = -\frac{iBk}{\sigma} = -\frac{idk}{\sigma}$$

Applying dynamic matching condition:

$$p_1 = p_2 \qquad \text{on} \quad z = -h + \chi$$

$$\rho_1 \frac{\partial \phi_1}{\partial t} - \rho_1 g z = \rho_2 \frac{\partial \phi_2}{\partial t} + \rho_1 g h + \rho_2 g(-z - h) \qquad \text{on} \quad z = -h + \chi$$

where the last two terms represent hydrostatic pressures due to water and mud, respectively.



Linearizing,

$$\rho_1 \frac{\partial \phi_1}{\partial t} - \rho_1 g \chi = \rho_2 \frac{\partial \phi_2}{\partial t} - \rho_2 g \chi \qquad \text{on} \quad z = -h$$

$$-i\sigma\rho_1 A - \rho_1 g m_0 = -i\sigma\rho_2 d - \rho_2 g m_0$$

$$i\sigma\rho_1 A = i\sigma\rho_2 B + (\rho_2 - \rho_1) g\left(-\frac{iBk}{\sigma}\right)$$

$$A = \frac{\rho_2}{\rho_1} B - \frac{(\rho_2 - \rho_1) gkB}{\rho_1 \sigma^2} = B\left[\frac{\rho_2}{\rho_1}\left(1 - \frac{gk}{\sigma^2}\right) + \frac{gk}{\sigma^2}\right]$$

Now,

$$\frac{A}{B}=\frac{\rho_2}{\rho_1}\left(1-\frac{gk}{\sigma^2}\right)+\frac{gk}{\sigma^2}=\frac{gk-\sigma^2\tanh kh}{\sigma^2-gk\tanh kh}$$

where Eqs. (9.51) and (9.52) were used. Rearranging gives the dispersion relationship:

$$\frac{\rho_2}{\rho_1}\left(1-\frac{gk}{\sigma^2}\right)(\sigma^2-gk\tanh kh)+\frac{gk}{\sigma^2}(\sigma^2-gk\tanh kh)=gk-\sigma^2\tanh kh$$

$$\frac{\rho_2}{\rho_1}(\sigma^2-gk)(\sigma^2-gk\tanh kh)+gk(\sigma^2-gk\tanh kh)=gk\sigma^2-\sigma^4\tanh kh$$

$$\frac{\rho_2}{\rho_1}(1+\tanh kh)\sigma^4-\frac{\rho_2}{\rho_1}gk(1+\tanh kh)\sigma^2+\left(\frac{\rho_2}{\rho_1}-1\right)(gk)^2\tanh kh=0$$

$$(\sigma^2-gk)\left[\sigma^2\left(\frac{\rho_2}{\rho_1}+\tanh kh\right)-\left(\frac{\rho_2}{\rho_1}-1\right)gk\tanh kh\right]=0$$

We have two possible cases:

$$\sigma^2=gk \quad \leftarrow \text{surface wave case}$$

or

$$\sigma^2=\frac{gk\left(\frac{\rho_2}{\rho_1}-1\right)\tanh kh}{\frac{\rho_2}{\rho_1}+\tanh kh} \quad \leftarrow \text{interfacial wave case}$$

Using Eqs. (9.52) and (9.59),

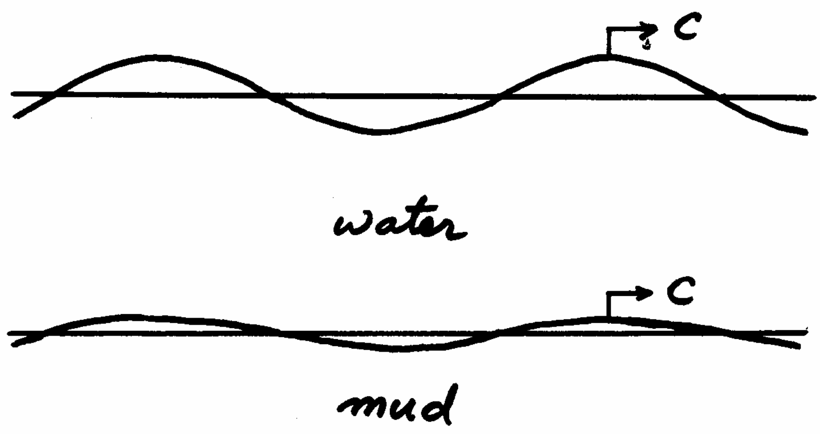
$$\frac{a_0}{m_0}=\frac{\sigma kB}{i\cosh kh(\sigma^2-gk\tanh kh)\left(-\frac{iBk}{\sigma}\right)}$$

$$=\frac{\sigma^2}{\cosh kh(\sigma^2-gk\tanh kh)}$$

For surface wave case,

$$\frac{a_0}{m_0} = \frac{gk}{gk\cosh kh(1-\tanh kh)} = \frac{1}{\cosh kh - \sinh kh} = e^{kh} > 1$$

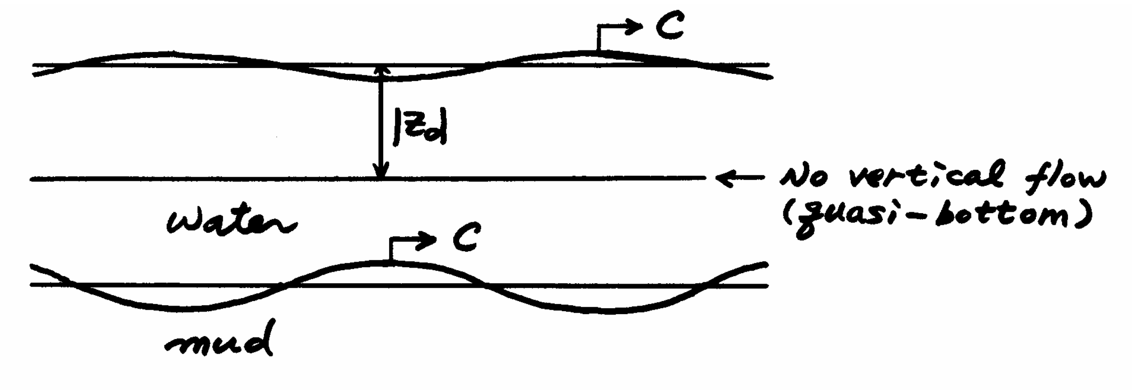
The free surface and interface are in phase, and the free surface amplitude is greater than the interface amplitude.



For interfacial wave case,

$$\frac{a_0}{m_0} = -\left(\frac{\rho_2}{\rho_1} - 1\right)e^{-kh}$$

$|a_0 / m_0|$ depends of ρ_2 / ρ_1 and kh. $|a_0 / m_0| < 1$ if $\rho_2 / \rho_1 < 2$ for $kh = 0$. ρ_2 / ρ_1 can be larger for larger kh to satisfy $|a_0 / m_0| < 1$. Also the free surface and interface are $180°$ out of phase because $a_0 / m_0 < 0$ always.



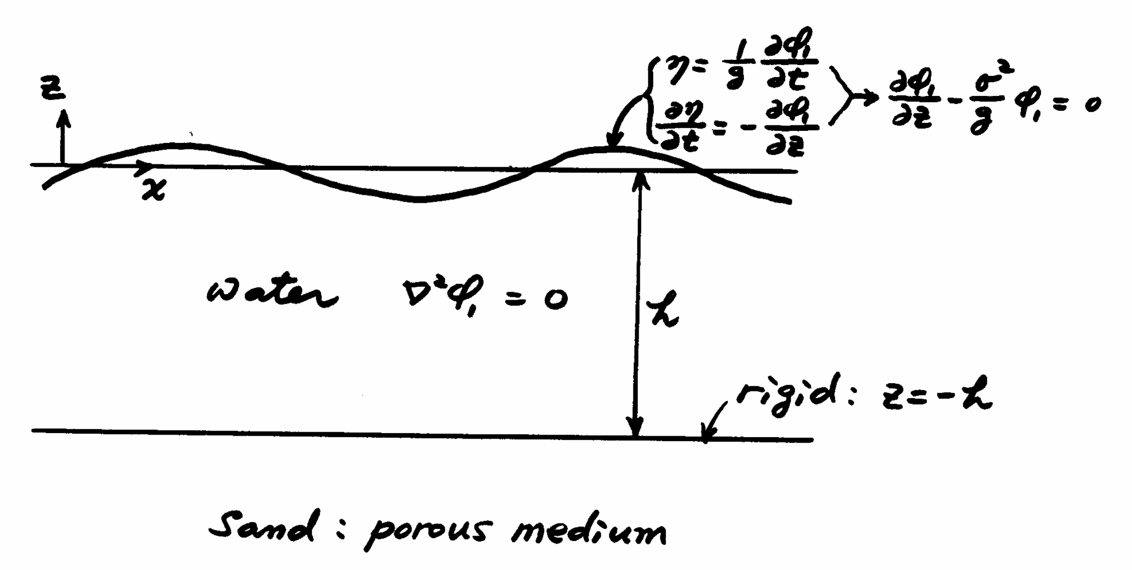
Because the free surface and interface are $180°$ out of phase, between them a quasi-bottom exists where there is no vertical flow. If we denote its vertical distance from the SWL as $|z_0|$ so that

$$\frac{\partial \phi_1}{\partial z} = 0 \qquad \text{on} \quad z = -|z_0|$$

the dispersion relationship becomes

$$\sigma^2 = gk \tanh k|z_0|$$

Waves over rigid, porous bottom



At the bottom, we need the kinematic and dynamic matching conditions:

$$-\frac{\partial \phi_1}{\partial z} = w_2 \qquad \text{on} \quad z = -h$$

$$p_1 = p_2 \qquad \text{on} \quad z = -h$$

In the porous medium, we use the equation for unsteady Darcy's flow:

$$\frac{1}{n}\frac{\partial \vec{u}_2}{\partial t} = -\frac{\nu}{K}\vec{u}_2 - \frac{1}{\rho}\nabla p_2$$

where n = porosity and K = permeability constant. Using $\nu = \mu / \rho$, the above equation can be written as

$$\frac{K}{\nu n}\frac{\partial \vec{u}_2}{\partial t} = -\vec{u}_2 - \frac{K}{\mu}\nabla p_2$$

Since $\vec{u}_2 \propto e^{i(kx - \sigma t)}$, the above equation becomes

$$-\frac{i}{n}\frac{\sigma K}{\nu}\vec{u}_2 = -\vec{u}_2 - \frac{K}{\mu}\nabla p_2$$

$K \sim 10^{-9} \sim 10^{-12}$ m^2 and $\nu \sim 10^{-6}$ m^2/s. Therefore, $(\sigma K/\nu) \sim 10^{-3} \sim 10^{-6}$. Thus, the preceding equation can be approximated to

$$\vec{u}_2 = -\frac{K}{\mu}\nabla p_2$$

Using the continuity equation, $\nabla \cdot \vec{u}_2 = 0$, we get

$$\nabla^2 p_2 = 0$$

which is the governing equation in the porous medium.

As with the waves over a mud bottom, assume the solutions as

$$\phi_1 = \left(A\cosh k(h+z) + B\sinh k(h+z)\right)e^{i(kx-\sigma t)}$$

and

$$p_2 = De^{k(h+z)}e^{i(kx-\sigma t)}$$

which give non-zero vertical velocity at the bottom and exponential decay of pore pressure as $h + z \to -\infty$.

As done for viscous mud bottom, LKFSBC and LDFSBC give

$$A = \frac{ia_0 \cosh kh}{\sigma k}(gk - \sigma^2 \tanh kh)$$

$$B = \frac{ia_0 \cosh kh}{\sigma k}(\sigma^2 - gk \tanh kh)$$

Dynamic matching condition at the bottom:

$$\rho \frac{\partial \phi_1}{\partial t} = p_2 \qquad \text{on} \quad z = -h$$

or

$$-i\sigma\rho A = D$$
$$A = \frac{D}{-i\rho\sigma}$$

Kinematic matching condition at the bottom:

$$-\frac{\partial \phi_1}{\partial z} = -\frac{K}{\mu}\frac{\partial p_2}{\partial z} \qquad \text{on} \quad z = -h$$

or

$$-Bk = -\frac{K}{\mu} kD$$
$$B = \frac{K}{\mu} D$$

Now

$$\frac{A}{B} = \frac{1}{-i\rho\sigma\left(\frac{K}{\mu}\right)} = \frac{i}{\frac{K\sigma}{\nu}} = \frac{gk - \sigma^2 \tanh kh}{\sigma^2 - gk \tanh kh}$$

which gives

$$i(\sigma^2 - gk \tanh kh) = \frac{K\sigma}{\nu}(gk - \sigma^2 \tanh kh)$$

The LHS and RHS are imaginary and real, respectively. Therefore we get

$$\sigma^2 = gk \tanh kh \quad \text{and} \quad \sigma^2 = gk \coth kh$$

which are in conflict. To satisfy the dispersion relationship, the wave number should be complex:

$$k = k_r + ik_i$$

The real part is related to the wave length, while the imaginary part determines the spatial damping:

$$\eta = a_0 e^{i(kx-\sigma t)} = a_0 e^{-k_i x} e^{i(k_r x - \sigma t)}$$

In intermediate depth, where we can assume that $K\sigma/\nu << 1$ and $k_i h << 1$ (or no significant damping), Reid and Kajiura (1957) obtained

$$\sigma^2 \cong gk_r \tanh k_r h$$

$$k_i \cong \frac{2k_r (K\sigma/\nu)}{2k_r h + \sinh 2k_r h}$$

which is shown in Figure 9.6 of textbook.

In shallow water where $|kh| < \pi/10$, k_i and k_r are given by Eqs. (9.95) and (9.96), respectively.