# Chapter 2 Kinematics

→ nature of a flowing fluid without reference to the dynamics

## 2.1 The Velocity Field

velocity, acceleration ~ vector quantities

$$\vec{q} \qquad \vec{a}$$

Cartesian coordinates

$$x \qquad y \qquad z$$

$$u \qquad v \qquad w$$

$$a_x \qquad a_y \qquad a_z$$

### 2.1.1 Lagrangian approach

~ coordinates of moving particles are represented as function of time

~ follow a particular particle through the flow field → path line

At $t = t_0$ coordinates (position) of a particle $(a, b, c)$

At $t = t$ position of a particle $(x, y, z)$

$$x = f_1(\underline{a, b, c, t})$$

Independent variables

$$y = f_2(a, b, c, t)$$

$$z = f_3(a, b, c, t)$$

$$u = \frac{\partial x}{\partial t} \qquad a_x = \frac{\partial u}{\partial t} = \frac{\partial^2 x}{\partial t^2}$$

$$v = \frac{\partial y}{\partial t} \qquad a_y = \frac{\partial v}{\partial t} = \frac{\partial^2 y}{\partial t^2}$$

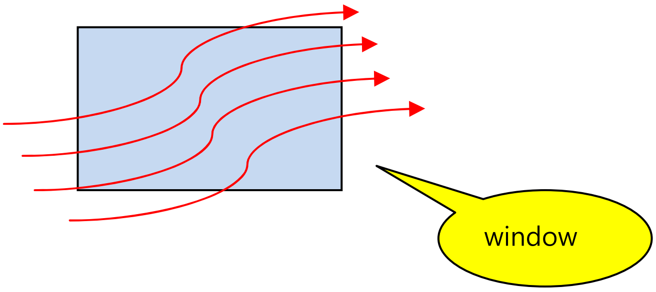
$$w = \frac{\partial z}{\partial t} \qquad a_z = \frac{\partial w}{\partial t} = \frac{\partial^2 z}{\partial t^2}$$

~ commonly used in the solid dynamics

~ convenient to identify a discrete particle, e.g. center of mass of spring - mass system

~ cumbersome when dealing with a fluid as a continuum of particles

→ Due to deformation of fluid, we are not usually concerned with the detailed history of an individual particle, but rather with <u>interrelation of flow properties at individual points</u> in the flow field.



### 2.1.2 Eulerian method

~ observer fixes attention at discrete points

~ notes flow characteristics in the vicinity of a fixed point as particles pass by

~ focused on the fluid which passes through a control volume that is fixed in space

~ familiar framework in which most fluid problems are solved

~ <u>instantaneous picture</u> of the velocities and accelerations of every particle

→ streamline

~ velocities at various points are given as <u>function of time</u>

$$\vec{q} = \vec{i}u + \vec{j}v + \vec{k}w$$

where $u = f_1\underline{(x, y, z, t)}$

$v = f_2(x, y, z, t)$

$w = f_3(x, y, z, t)$

Independent variables

$x, y, z, t$ = independent variables

$\vec{i}, \vec{j}, \vec{k}$ = unit vectors

### 2.1.3 Total Derivative

(1) Total change in velocity

= sum of partial derivatives of the four independent variables, *x, y, z, t*

$$x-\text{dir} \quad : \quad du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

total derivative: $$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{du}{dt}$$

$$= \underbrace{\frac{\partial u}{\partial t}}_{\text{local change due to unsteadiness}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{convective change due to translation}}$$

$$y-\text{dir} \quad : \quad \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$z-\text{dir} \quad : \quad \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

(2) Total rate of density change of <u>compressible</u> fluid

$$\rho = \rho(x,y,z,t)$$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{\partial \rho}{\partial t} + u_j\frac{\partial \rho}{\partial x_j}$$

For incompressible fluid, $\dfrac{d\rho}{dt} = 0$

For steady flow, $\dfrac{\partial \rho}{\partial t} = 0$

### 2.2 Steady versus Uniform motion

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + u_j\frac{\partial u}{\partial x_j}$$

$$a_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial v}{\partial t} + u_j\frac{\partial v}{\partial x_j}$$

$$a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + u_j\frac{\partial w}{\partial x_j}$$

Vector notation

$$\vec{a} = \vec{i}a_x + \vec{j}a_y + \vec{k}a_z$$

$$\vec{a} = \frac{d\vec{q}}{dt} = \underbrace{\frac{\partial \vec{q}}{\partial t}}_{\boxed{\text{local acceleration}}} + \underbrace{(\vec{q}\cdot\nabla)\,\vec{q}}_{\boxed{\text{convective acceleration}}}$$

i) steady motion: no changes with time at fixed point ⟷ unsteady motion

$$\frac{\partial \vec{q}}{\partial t} = 0 \quad \rightarrow \text{local acceleration} = 0$$

ii) uniform motion: no changes with space ⟷ non-uniform motion

$$(\vec{q}\cdot\nabla)\vec{q} = 0 \quad \rightarrow \text{convective acceleration} = 0$$

◈ Vector differential operators: $\nabla \rightarrow$ "del" or "nabla"

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Gradient: $\nabla f = grad\ f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

Divergence: $\nabla \cdot \vec{q} = div\, \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

[Re] Vector product

i) dot product → scalar

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\phi$$

$\phi$ = angle between the vectors

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1 \qquad (\cos 0^\circ = 1)$$

$$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{j} \cdot \vec{i} = \vec{k} \cdot \vec{j} = 0 \qquad (\because \cos 90^\circ = 0)$$

ii) cross product → vector

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\phi$$

Direction = perpendicular to the plane of $\vec{a}\ and\ \vec{b} \rightarrow$ right-hand rule

$$\vec{q}\cdot\nabla = (\vec{i}u + \vec{j}v + \vec{k}w)\cdot(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})$$

$$= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$(\vec{q}\cdot\nabla)\,\vec{q} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(\vec{i}u + \vec{j}v + \vec{k}w)$$

$$= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\vec{i}$$

$$+ \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\vec{j}$$

$$+ \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\vec{k}$$

$$\nabla^2 = \nabla\cdot\nabla = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2\phi = 0 \quad\rightarrow\quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \;\rightarrow\; \text{Laplace Eq.}$$

$$grad\ (u+v) = \nabla\,(u+v) = \nabla u + \nabla v$$

$$div\ (\vec{u}+\vec{v}) = \nabla\cdot(\vec{u}+\vec{v}) = \nabla\cdot\vec{u} + \nabla\cdot\vec{v}$$

$$grad\ (uv) = \nabla\,(uv) = v\nabla u + u\nabla v$$

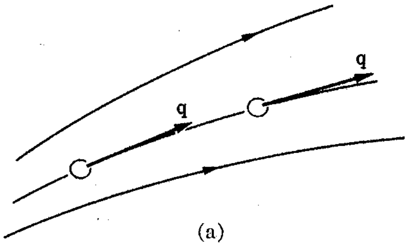
$$div\ (u\vec{v}) = \nabla\cdot(u\vec{v}) = \nabla u\cdot\vec{v} + u\nabla\cdot\vec{v}$$

$$div\ grad\ u = \nabla\cdot\nabla u = \nabla^2 u$$

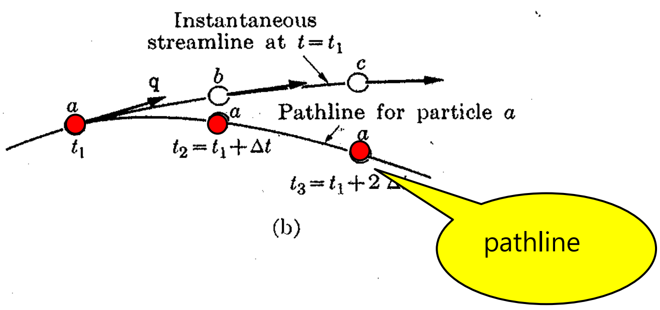
## 2.3 Streamlines vs Path line

### 2.3.1 Flow lines

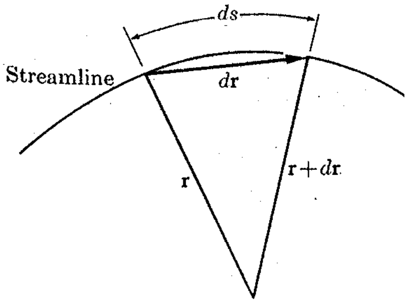
streamline, path line, streak line



(a)



(b)



(c)

$\mathbf{dr} = \mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz$

In limit $d\mathbf{r}/ds = \mathbf{n} =$ unit tangent vector

$d\mathbf{r} = \mathbf{n}\, ds =$ element of length along streamline

(1) streamline

= imaginary line connecting a series of points in space at a given instant in such a manner that all particles falling on the line at that instant have velocities whose vectors are tangent to the line

= instantaneous curves which are everywhere tangent to the velocity vector

= a line that is (at a given instant) tangent to the velocity at every point on it

* stream tube = small imaginary tube bounded by streamlines

* stream filament = if cross section of stream tube is infinitesimally small

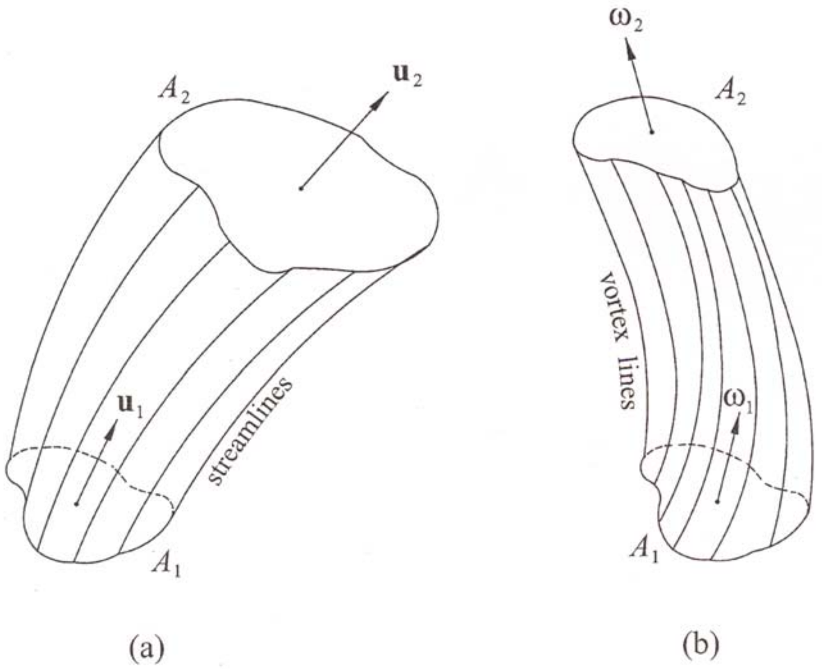


**FIGURE 2.2** (a) Stream tube and (b) vortex tube subtended by a contour of area $A_1$ in a flow field.

(2) path line

= trajectory of a particle of fixed identity as time passes

(3) streak line

= a line connecting all the particles that have passed successfully through a particular given point (injection point)

= current location of all particles which have passed through a fixed point in space

[Ex] dye stream in water, smoke filament in air

* For steady flow, streamline = path line = streak line

◈ How can we photo 3 lines?

(1) streamline: spread bunch of reflectors on the flow field, then take a instant shot

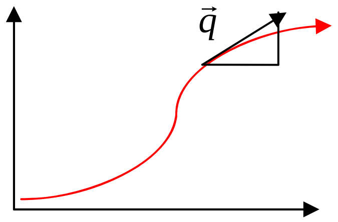
(2) path line: put only one particle on the flow field, then take long-time exposure

(3) streak line: take a instant shot of dye injecting from one slot of the dye tanks

### 2.3.2 Differential equations for flow lines

(1) Streamline

By virtue of definition of a streamline (velocity vector $\vec{q}$ is tangent to the streamline), it's slope in the $xy$-plane, $\frac{dy}{dx}$, must be equal to that of the velocity, $\frac{v}{u}$.



$$\frac{dy}{dx} = \frac{v}{u}$$

By similarly treating the projections on the *xz* plane and on the *yz* plane

$$\frac{dz}{dx} = \frac{w}{u}; \quad \frac{dz}{dy} = \frac{w}{v}$$

$$\rightarrow \qquad \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

→ Integration of the differential equation for streamline yields equation of streamline.

For 2-D Cartesian coordinates

$$\frac{dx}{u} = \frac{dy}{v} \rightarrow \frac{dy}{dx} = \frac{v}{u} \rightarrow v\,dx - u dy = 0$$

◈ Vector form of equation of streamline

$$\vec{q} \times d\vec{r} = 0$$

$$\vec{q} = \vec{i}u + \vec{j}v + \vec{k}w$$

$$d\vec{r} = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz$$

$= \vec{n}\,ds =$ element of length along streamline

$$\vec{q} \times d\vec{r} = \vec{i}vdz + \vec{j}wdx + \vec{k}udy - \vec{i}wdy - \vec{j}udz - \vec{k}vdx$$

$$= \vec{i}\,(vdz - wdy) + \vec{j}\,(wdx - udz) + \vec{k}\,(udy - vdx) = 0$$

(2) Path line

Since the particle is moving with the fluid at its local velocity

$$\frac{dx}{dt} = u\,; \quad \frac{dy}{dt} = v\,; \quad \frac{dz}{dt} = w$$

[App] Vector Products

(1) dot product → scalar

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

(2) cross product → vector

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left( a_2 b_3 - a_3 b_2 \right) \vec{i} + \left( a_3 b_1 - a_1 b_3 \right) \vec{j} + \left( a_1 b_2 - a_2 b_1 \right) \vec{k}$$

$$curl\ \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \vec{i} + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \vec{j} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{k}$$

[Ex] Vorticity: $\xi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$   3–D flow, $\xi = \nabla \times \vec{V}$

For irrotational flow, $\xi = 0$ ; $\nabla \times \vec{V} = 0$

$$\bullet\, curl(u\vec{v}) = \nabla \times (u\vec{v}) = \nabla u \times \vec{v} + u \nabla \times \vec{v}$$

$$\bullet\, curl\, grad\, u = \nabla \times \nabla u = 0$$

$$\bullet\, divcurl\ \vec{u} = \nabla \cdot \left( \nabla \times \vec{u} \right) = 0$$

## Homework Assignment 1

Due: 1 week from today

1. The velocity of an inviscid, incompressible fluid as it steadily approaches the stagnation point at the leading edge of a sphere of radius $R$ is

$$u = u_s \left( 1 + \frac{R^3}{x^3} \right)$$

What is the fluid acceleration at **(a)** $x = -3R$, **(b)** $x = -2R$, and **(c)** $x = -R$?

**(d)** When and $u_s$=2 m/s and $R$ = 3 cm, what is the magnitude of the acceleration at $x = -2R$?

2. The velocity field in a flow system is given by

$$\vec{q} = 5\vec{i} + (x + y^2)\vec{j} + 3xy\vec{k}$$

What is the fluid acceleration **(a)** at (1, 2, 3) and **(b)** at (-1, -2, -3)?

3. A nozzle is shaped such that the axial-flow velocity increases linearly from 2 to 18 m/s in a distance of 1.20 m. What is the convective acceleration **(a)** at the inlet and **(b)** at the exit of the nozzle?