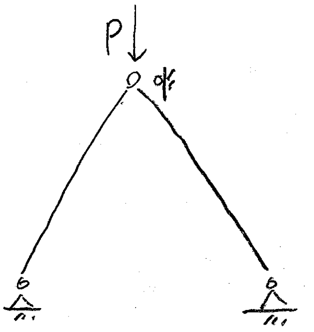
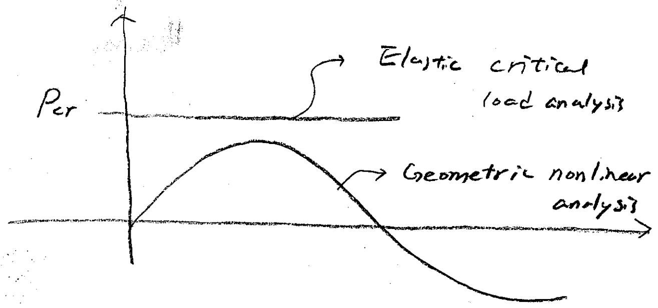
Chapter 9. Geometric nonlinear and Elastic Critical load analysis

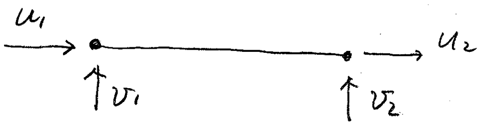
Application to complex multiple dof systems



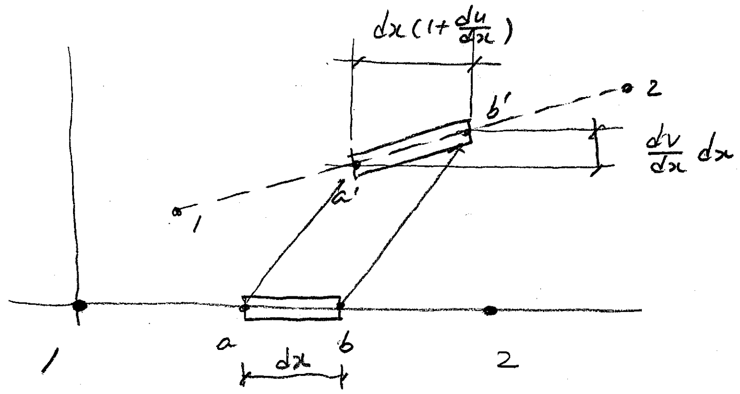


9.1 Geometric Stiffness matrices for Planar Elements

9.1.1 Axial Force member



4DOF



$$\varepsilon = \frac{\overline{a'b'} - \overline{ab}}{\overline{ab}}$$

$$\overline{a'b'} = dx\left[\left(1+\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right]^{1/2}$$

$$= dx\left[1 + 2\frac{du}{dx} + \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right]^{1/2}$$

$$\simeq dx\left[1 + \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2\right]$$

$$\varepsilon = \frac{\overline{a'b'} - \overline{ab}}{\overline{ab}} = \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2$$

Using the principle of virtual Displacement

$$\delta U = \int \delta \underline{\varepsilon}^T \underline{\sigma} \, dV$$

$$\delta \underset{\sim}{\varepsilon} = \delta\varepsilon = \left(\frac{d\delta u}{dx}\right) + \left(\frac{du}{dx}\right)\left(\frac{d\delta u}{dx}\right) + \left(\frac{dv}{dx}\right)\left(\frac{d\delta v}{dx}\right)$$

$$= \int \left(\frac{d\delta u}{dx}\right)^T EA \left(\frac{du}{dx}\right) dx + \int \sigma A \left[\frac{d\delta u}{dx}\frac{du}{dx} + \frac{d\delta v}{dx}\frac{dv}{dx}\right] dx$$

$\left\{ \sigma = \sigma_0 + \Delta\sigma \right.$ and $F = \sigma A$, neglecting small terms $\left(\frac{du}{dx}\right)^2, \left(\frac{dv}{dx}\right)^2$

$$\approx \int \left(\frac{d\delta u}{dx}\right)^T EA \left(\frac{du}{dx}\right) dx + F\left[\int \frac{d\delta u}{dx}\frac{du}{dx}\, dx + \int \frac{d\delta v}{dx}\frac{dv}{dx}\, dx\right]$$

$$\begin{cases} u = \underset{\sim}{f}\, \underset{\sim}{q}_u = \left[\underbrace{\left(1 - \frac{x}{L}\right)}_{f_1} \ \underbrace{\left(\frac{x}{L}\right)}_{f_2} \right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ v = \underset{\sim}{f}\, \underset{\sim}{q}_v = \left[\left(1 - \frac{x}{L}\right) \ \left(\frac{x}{L}\right) \right] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{cases}$$

$$= \delta \underset{\sim}{q}_u \underbrace{\left[EA \int \underset{\sim}{f}'^T \underset{\sim}{f}' dx\right]}_{K} \underset{\sim}{q}_u + \delta \underset{\sim}{q}_u \underbrace{\left[F \int \underset{\sim}{f}'^T \underset{\sim}{f}' dx\right]}_{G_u} \underset{\sim}{q}_u$$

$$+ \delta \underset{\sim}{q}_v \underbrace{\left[F \int \underset{\sim}{f}'^T \underset{\sim}{f}' dx\right]}_{G_v} \underset{\sim}{q}_v$$

$$\underset{\sim}{K} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{G} = \underset{\sim}{G}_u + \underset{\sim}{G}_v = \frac{F}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[\underset{\sim}{K}_e + \underset{\sim}{G}]\,\underset{\sim}{q} = \underset{\sim}{P}$$

$$\underset{\sim}{K} = \underset{\sim}{K}_e + \underset{\sim}{G}$$

$$[\underset{\sim}{K}_e + \underset{\sim}{G}]\,d\underset{\sim}{q} = d\underset{\sim}{P}$$ with updated geometry,

9.1.2 Combined Bending and Axial Force

$$\varepsilon = \varepsilon_{axial} + \varepsilon_{bending}$$

$$= \frac{du}{dx} + \frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right] - y\frac{d^2v}{dx^2}$$

$$= \frac{du}{dx} - y\frac{d^2v}{dx^2} + \frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right]$$

$$\begin{cases} u = \underset{\sim}{f}_u\,\underset{\sim}{q}_u = \left[\underbrace{\left(1-\frac{x}{L}\right)}_{f_{u1}} \quad \underbrace{\left(\frac{x}{L}\right)}_{f_{u2}}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ v = \underset{\sim}{f}_v\,\underset{\sim}{q}_v = \left[f_{v1} \quad f_{v2} \quad f_{v3} \quad f_{v4}\right]\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \end{cases}$$

$$\delta U = \int \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, dV$$

$$= \int \left(\frac{d\delta u}{dx}\right) EA \left(\frac{du}{dx}\right) dx + \int \left(\frac{d^2\delta v}{dx^2}\right) EI \left(\frac{d^2v}{dx^2}\right) dx$$

$$+ F\left[\int \frac{d\delta u}{dx}\frac{du}{dx}\,dx + \int \frac{d\delta v}{dx}\frac{dv}{dx}\,dx\right]$$

$$\left(\begin{aligned} &\delta\varepsilon = \frac{d\delta u}{dx} - y\frac{d^2\delta v}{dx^2} + \left[\left(\frac{du}{dx}\right)\left(\frac{d\delta u}{dx}\right) + \left(\frac{dv}{dx}\right)\left(\frac{d\delta v}{dx}\right)\right] \\ &\sigma = E\varepsilon \text{ or } \frac{F}{A} \\ &\int y \,\square\, dx = 0 \end{aligned}\right)$$

$u = \underset{\sim}{f_u}\, \underset{\sim}{q_u}$ and $v = \underset{\sim}{f_v}\, \underset{\sim}{q_v}$

$$\delta U = \delta \underset{\sim}{q_u} \underbrace{\left[EA \int \underset{\sim}{f_u}'^T \underset{\sim}{f_u}' \, dx \right]}_{\underset{\sim}{K_a}} \underset{\sim}{q_u} + \delta \underset{\sim}{q_v} \underbrace{\left[EI \int \underset{\sim}{f_v}''^T \underset{\sim}{f_v}'' \, dx \right]}_{\underset{\sim}{K_b}} \underset{\sim}{q_v}$$

$$+ \delta \underset{\sim}{q_u} \underbrace{\left[F \int \underset{\sim}{f_u}'^T \underset{\sim}{f_u}' \, dx \right]}_{\underset{\sim}{G_u}} \underset{\sim}{q_u} + \delta \underset{\sim}{q_v} \underbrace{\left[F \int \underset{\sim}{f_v}'^T \underset{\sim}{f_v}' \, dx \right]}_{\underset{\sim}{G_v}} \underset{\sim}{q_v}$$

$\underset{\sim}{K_a}$ = Elastic stiffness of axial force element

$\underset{\sim}{K_b}$ = " of flexure element

$\underset{\sim}{G_u}$ = Geometric stiffness for axial force action

$\underset{\sim}{G_v}$ = " for bending action

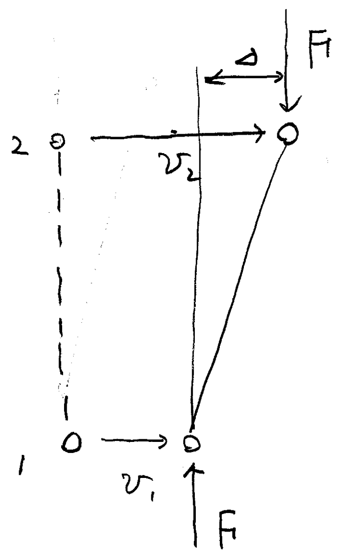
$$\underset{\sim}{G} = \underset{\sim}{G_u} + \underset{\sim}{G_v}$$

$$\underset{\sim}{G} = \frac{F}{L} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ & & \frac{2L^2}{15} & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ & & & 1 & 0 & 0 \\ & \text{Sym.} & & & \frac{6}{5} & -\frac{L}{10} \\ & & & & & \frac{2L^2}{15} \end{bmatrix}$$

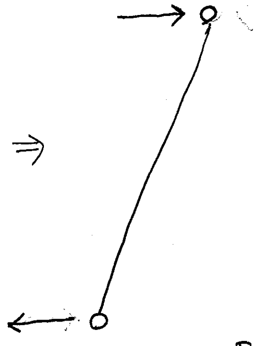
$F < 0 \Rightarrow \underset{\sim}{G}$ reduces stiffness $\underset{\sim}{K}$ $(= K_e + \underset{\sim}{G})$

$F > 0 \Rightarrow \underset{\sim}{G}$ increases

Meaning of Geometric Stiffness



$\Rightarrow$



$$P_{y2} = \frac{F\Delta}{L} = \frac{F}{L}(V_2 - V_1)$$

$$P_{y1} = -\frac{F\Delta}{L} = -\frac{F}{L}(V_2 - V_1)$$

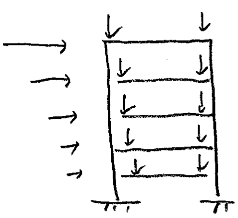
$$\begin{bmatrix} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \end{bmatrix} = \underbrace{\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\underset{\sim}{K}e}\begin{bmatrix} u_1 \\ V_1 \\ u_2 \\ V_2 \end{bmatrix} + \underbrace{\frac{F}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{\underset{\sim}{G}}\begin{bmatrix} u_1 \\ V_1 \\ u_2 \\ V_2 \end{bmatrix}$$

when $+F$ = tension.

$$\underset{\sim}{K} = \underset{\sim}{K}e + \underset{\sim}{G}$$

If the coordinates of nodes are continuously updated in the nonlinear analysis, we don't need to consider the geometric stiffness.

But, considering the geometric stiffness reduces computational effort to seek the final equilibrium position.



$\Rightarrow$ 주로 중력하중의 영향

$$[\underset{\sim}{K}_e + \underset{\sim}{G}]\, d\underset{\sim}{q} = d\underset{\sim}{P}$$

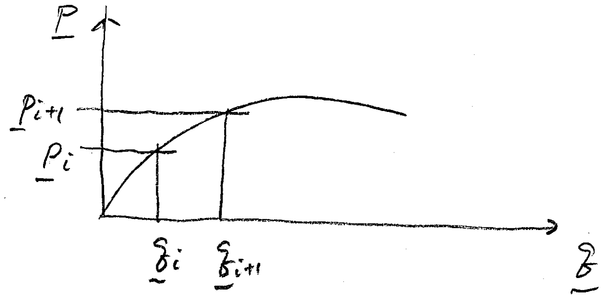
- geometric nonlinear analysis - step-by step analysis
- Analysis of critical load

1. Geometric nonlinear analysis

$$[\underset{\sim}{K}_e + \underset{\sim}{G}]\, d\underset{\sim}{q} = d\underset{\sim}{P} \quad \text{with updated geometry}$$

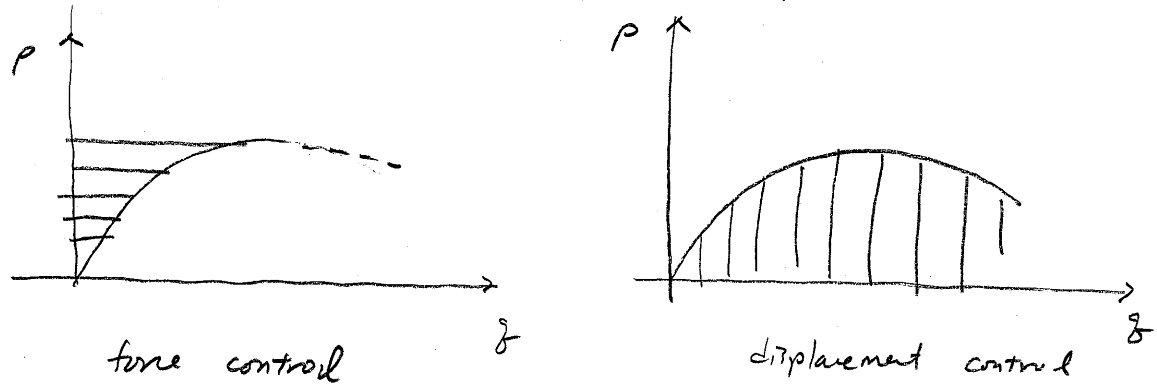
$\underset{\sim}{G}$ is the function of displacement or internal forces

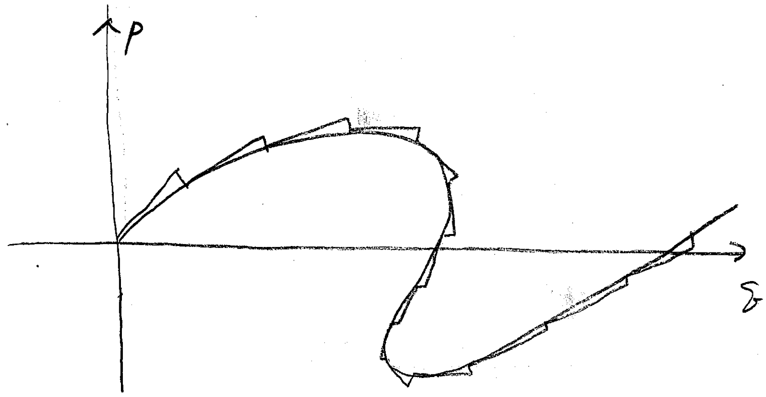
$\underset{\sim}{q}_{i+1}$ depends on $\underset{\sim}{q}_i$ or $\underset{\sim}{F}_i$



Iterative calculations are required to obtain a solution satisfying equilibrium and compatibility.

⇒ A Numerical Technique is required for the geometric nonlinear analysis.





arc-length method

2. Analysis of critical load (Bifurcation analysis, Eigenvalue problem)

$$[\underset{\sim}{K_e} + \underset{\sim}{G}]\ \underset{\sim}{q} = \underset{\sim}{P} \quad \text{(initial relationship)}$$

⇒ linearization Assumption : geometry does not significantly change until instability occur

$$\underset{\sim}{G}(P_{cr}) = \underset{\sim}{G}(\lambda P_i)$$

and

$$[\underset{\sim}{K_e} + \underset{\sim}{G}]\ d\underset{\sim}{q} = d\underset{\sim}{P}$$

Criterion for critical force

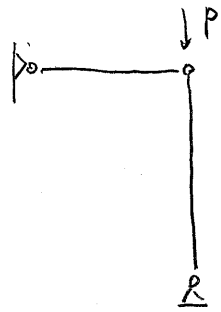
$$[\underset{\sim}{K_e} + \underset{\sim}{G}]\ d\underset{\sim}{q} = \underset{\sim}{0}$$

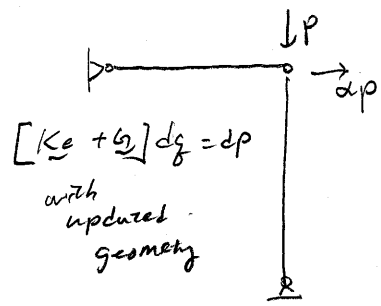
$[\underset{\sim}{K_e} + \underset{\sim}{G}]$ ⇒ only function of $\underset{\sim}{P}$ (not $\underset{\sim}{q}$)

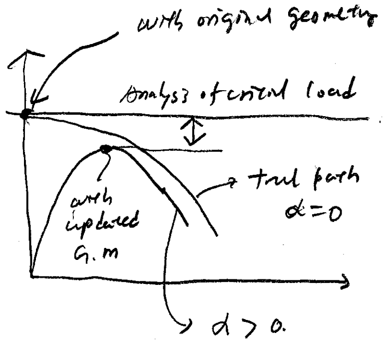
$\underset{\sim}{P}$는 geometry 변화의 함수가 아님

$d\underset{\sim}{q} = \underset{\sim}{0}$ ⇒ trivial solution

$|\underset{\sim}{K_e} + \underset{\sim}{G}| = 0$ ⇒ Eigenvalue problem







(force가 변위방향이어야 or) 변위를 일으키는 Force가 있을 경우에는 Analysis of critical Load의 결과와 정확한 해의 limit load는 차이가 클수 있다.

$$|\underset{\sim}{K}_G + \underset{\sim}{G}| = 0$$

$$\Rightarrow |\underset{\sim}{K}_G + \lambda \hat{\underset{\sim}{G}}(f_i)| = 0$$

solve λ (= eigenvalue)

$\underset{\sim}{\delta}$ (= eigenvector)

$\Rightarrow$ solution method for eigenvalue problem

1) method for finding the least λ (in absolute value)

Stodola method, Inverse Iteration method

2) method finding all λ's

Householder Tridiagonalization

QR - factorization

Stodola method

$$(\underset{\sim}{K}_e + \underset{\sim}{G})\, \underset{\sim}{v} = \underset{\sim}{0}$$

or $\underset{\sim}{K}_e \underset{\sim}{v} = -\underset{\sim}{G}\underset{\sim}{v}$ $\Rightarrow$ needs iterative calculations

choose an estimate $\underset{\sim}{v}^{(0)}$

and calculate an improved estimate :

$$* \quad \boxed{\underset{\sim}{K}}\, \underset{\sim}{v}^{(1)} = -\boxed{\hat{\underset{\sim}{G}}\, \underset{\sim}{v}^{(0)}}$$

$\hat{\underset{\sim}{G}}$: geometric stiffness with $\underset{\sim}{F}_i$ (initial force vector)

$\underset{\sim}{K}$ ↓ already processed by Gauss Elemination

$\hat{\underset{\sim}{G}}\, \underset{\sim}{v}^{(0)}$ ↳ element by element and assemble

Compare $\underset{\sim}{v}^{(1)}$ to $\underset{\sim}{v}^{(0)}$

If the two vectors are parallel to within some specified tolerance, terminate the iterations and use $\underset{\sim}{v}^{(1)}$ to compute λ

If the vectors are not parallel, $\underset{\sim}{v}^{(0)} \leftarrow \underset{\sim}{v}^{(1)}$ and go to (*)

If $\underset{\sim}{v}^{(0)} \,/\!/\, \underset{\sim}{v}^{(1)}$

$$\underset{\sim}{K}\underset{\sim}{v}^{(1)} = -\hat{\underset{\sim}{G}}\underset{\sim}{v}^{(0)} \qquad \underset{\sim}{v}^{(0)} = \lambda \cdot \underset{\sim}{v}^{(1)}$$

$$= -\lambda \hat{\underset{\sim}{G}} \underset{\sim}{v}^{(1)}$$

$$\underset{\sim}{v}^{(1)T}\underset{\sim}{K}\underset{\sim}{v}^{(1)} = -\lambda\, \underset{\sim}{v}^{(1)T}\hat{\underset{\sim}{G}}\underset{\sim}{v}^{(1)}$$

$$\lambda = -\frac{\underset{\sim}{v}^{(1)T}\underset{\sim}{K}\underset{\sim}{v}^{(1)}}{\underset{\sim}{v}^{(1)T}\hat{\underset{\sim}{G}}\underset{\sim}{v}^{(1)}} = \frac{\underset{\sim}{v}^{(1)T}\hat{\underset{\sim}{G}}\underset{\sim}{v}^{(0)}}{\underset{\sim}{v}^{(1)T}\hat{\underset{\sim}{G}}\underset{\sim}{v}^{(1)}} \quad (\underset{\sim}{K}\underset{\sim}{v}^{(1)} = -\hat{\underset{\sim}{G}}\underset{\sim}{v}^{(0)})$$

How to test $\underset{\sim}{v}^{(1)} \parallel \underset{\sim}{v}^{(0)}$

$$\underset{\sim}{v}^{(0)} \cdot \underset{\sim}{v}^{(1)} = \|\underset{\sim}{v}^{(0)}\| \, \|\underset{\sim}{v}^{(1)}\| \cos\theta$$

$$\cos\theta = \frac{\underset{\sim}{v}^{(0)} \cdot \underset{\sim}{v}^{(1)}}{\|\underset{\sim}{v}^{(0)}\| \, \|\underset{\sim}{v}^{(1)}\|}$$

If $\underset{\sim}{v}^{(0)} \parallel \underset{\sim}{v}^{(1)}$, $\theta = 0 \Rightarrow \cos\theta = 1$ $\quad$ θ : angle between $\underset{\sim}{v}^{(0)}$ and $\underset{\sim}{v}^{(1)}$

Compare $|\cos\theta|$ and 1

How does the Stodola method work?

$$\underset{\sim}{K}\,\underset{\sim}{v} = +\lambda\,\underset{\sim}{G}\,\underset{\sim}{v} \quad \text{or} \quad (\underset{\sim}{K} - \lambda\underset{\sim}{G})\,\underset{\sim}{v} = \underset{\sim}{0}$$

$$|\underset{\sim}{K} - \lambda\underset{\sim}{G}| = 0$$

the characteristic equation is nth order.

$$\lambda^n + P_{n-1}\lambda^{n-1} + \cdots + P_0 = 0$$

The solutions $\lambda_1, \lambda_2, \cdots, \lambda_n \Rightarrow$ Eigenvalues

If $\underset{\sim}{A}$ is positive definite ($\underset{\sim}{A} = \underset{\sim}{A}^T$, and $\underset{\sim}{x}^T\underset{\sim}{A}\,\underset{\sim}{x} > 0$ for all $\underset{\sim}{x} \neq 0$)

Eigenvalues are real number:

$$\begin{array}{cccc} \lambda_1, & \lambda_2, & \cdots, & \lambda_n \\ \downarrow & \downarrow & & \downarrow \\ \underset{\sim}{v}_1, & \underset{\sim}{v}_2, & \cdots, & \underset{\sim}{v}_n \end{array} \Rightarrow \text{Eigen vectors}$$

Orthogonality of eigen vectors

Consider any two of the eigenvectors $\underset{\sim}{v}_r$ and $\underset{\sim}{v}_s$ $(r \neq s)$

$$\underset{\sim}{K}\,\underset{\sim}{v}_r = -\lambda_r \underset{\sim}{G}\,\underset{\sim}{v}_r \qquad \underset{\sim}{K}\,\underset{\sim}{v}_s = -\lambda_s \underset{\sim}{G}\,\underset{\sim}{v}_s$$

$$\underset{\sim}{v}_s^T \underset{\sim}{K}\,\underset{\sim}{v}_r = -\lambda_r \underset{\sim}{v}_s^T \underset{\sim}{G}\,\underset{\sim}{v}_r \qquad \underset{\sim}{v}_r^T \underset{\sim}{K}\,\underset{\sim}{v}_s = -\lambda_s \underset{\sim}{v}_r^T \underset{\sim}{G}\,\underset{\sim}{v}_s \quad - (1)$$

$\underset{\sim}{K}$ and $\underset{\sim}{G}$ are symmetric

$$(\underset{\sim}{v}_s^T \underset{\sim}{K}\,\underset{\sim}{v}_r)^T = \underset{\sim}{v}_r^T \underset{\sim}{K}\,\underset{\sim}{v}_s = -\lambda_r \underset{\sim}{v}_r^T \underset{\sim}{G}\,\underset{\sim}{v}_s \quad - (2)$$

$$(1)-(2) \qquad 0 = (\lambda_r - \lambda_s)\, \underset{\sim}{v}_r^T \underset{\sim}{G}\,\underset{\sim}{v}_s = 0$$

$$\lambda_r \neq \lambda_s \Rightarrow \underset{\sim}{v}_r^T \underset{\sim}{G}\,\underset{\sim}{v}_s = 0 \Rightarrow \underset{\sim}{v}_s^T \underset{\sim}{G}\,\underset{\sim}{v}_r = 0$$

$$\underset{\sim}{v}_s^T \underset{\sim}{K}\,\underset{\sim}{v}_r = 0 \Rightarrow \underset{\sim}{v}_r^T \underset{\sim}{K}\,\underset{\sim}{v}_s = 0$$

$\Rightarrow$ orthogonality of eigenvectors

In the Stodola method,

$$\underset{\sim}{K}\,\underset{\sim}{v}^{(1)} = -\underset{\sim}{G}\,\underset{\sim}{v}^{(0)}$$

$$\underset{\sim}{v}^{(0)} = \alpha_1 \underset{\sim}{v}_1 + \alpha_2 \underset{\sim}{v}_2 + \cdots + \alpha_n \underset{\sim}{v}_n$$

$\underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_n \Rightarrow$ Eigenvectors (independent)

$$\underset{\sim}{v}^{(1)} = \beta_1 \underset{\sim}{v}_1 + \beta_2 \underset{\sim}{v}_2 + \cdots + \beta_n \underset{\sim}{v}_n$$

What is the relation between β_r and α_r ?

$$\underset{\sim}{K}\, \underset{\sim}{v}^{(1)} = -\underset{\sim}{G}\, \underset{\sim}{v}^{(0)}$$

$$\underset{\sim}{v}_r^T \underset{\sim}{K}\, \underset{\sim}{v}^{(1)} = -\underset{\sim}{v}_r^T \underset{\sim}{G}\, \underset{\sim}{v}^{(0)}$$

$$\left\{\begin{aligned} \underset{\sim}{v}_r^T \underset{\sim}{K}\, \underset{\sim}{v}^{(1)} &= \underset{\sim}{v}_r^T \underset{\sim}{K}\, (\beta_1 \underset{\sim}{v}_1 + \cdots + \beta_r \underset{\sim}{v}_r + \cdots + \beta_n \underset{\sim}{v}_n) \\ &= \beta_r \underset{\sim}{v}_r^T \underset{\sim}{K}\, \underset{\sim}{v}_r \qquad (\because\ \underset{\sim}{v}_r^T \underset{\sim}{K}\, \underset{\sim}{v}_s = 0) \\ \underset{\sim}{v}_r^T \underset{\sim}{G}\, \underset{\sim}{v}^{(0)} &= \underset{\sim}{v}_r^T \underset{\sim}{G}\, (\alpha_1 \underset{\sim}{v}_1 + \cdots + \alpha_r \underset{\sim}{v}_r + \cdots + \beta_n \underset{\sim}{v}_n) \\ &= \alpha_r \underset{\sim}{v}_r^T \underset{\sim}{G}\, \underset{\sim}{v}_r \end{aligned}\right.$$

$$\beta_r \underset{\sim}{v}_r^T \underset{\sim}{K}\, \underset{\sim}{v}_r = -\alpha_r \underset{\sim}{v}_r^T \underset{\sim}{G}\, \underset{\sim}{v}_r$$

$$\beta_r = -\alpha_r \frac{\underset{\sim}{v}_r^T \underset{\sim}{G}\, \underset{\sim}{v}_r}{\underset{\sim}{v}_r^T \underset{\sim}{K}\, \underset{\sim}{v}_r}$$

$$\underset{\sim}{K}\, \underset{\sim}{v}_r = -\lambda_r \underset{\sim}{G}\, \underset{\sim}{v}_r \qquad \lambda_r = -\frac{\underset{\sim}{v}_r^T \underset{\sim}{K}\, \underset{\sim}{v}_r}{\underset{\sim}{v}_r^T \underset{\sim}{G}\, \underset{\sim}{v}_r}$$

$$\therefore \beta_r = \frac{1}{\lambda_r} \alpha_r \qquad r = 1, \cdots, n$$

$$\begin{aligned} \underset{\sim}{v}^{(1)} &= \beta_1 \underset{\sim}{v}_1 + \beta_2 \underset{\sim}{v}_2 + \cdots + \beta_n \underset{\sim}{v}_n \\ &= \frac{\alpha_1}{\lambda_1} \underset{\sim}{v}_1 + \frac{\alpha_2}{\lambda_2} \underset{\sim}{v}_2 + \cdots + \frac{\alpha_n}{\lambda_n} \underset{\sim}{v}_n \end{aligned}$$

$$\underset{\sim}{v}^{(k)} = \frac{1}{\lambda_1^k} \alpha_1 \underset{\sim}{v}_1 + \frac{1}{\lambda_2^k} \alpha_2 \underset{\sim}{v}_2 + \cdots + \frac{1}{\lambda_n^k} \alpha_n \underset{\sim}{v}_n$$

assuming $|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_n|$

$$\underset{\sim}{v}^{(k)} = \frac{1}{\lambda_1^k}\left[\alpha_1 \underset{\sim}{v}_1 + \left(\frac{\lambda_1}{\lambda_2}\right)^k \alpha_2 \underset{\sim}{v}_2 + \cdots + \left(\frac{\lambda_1}{\lambda_n}\right)^k \alpha_n \underset{\sim}{v}_n \right]$$

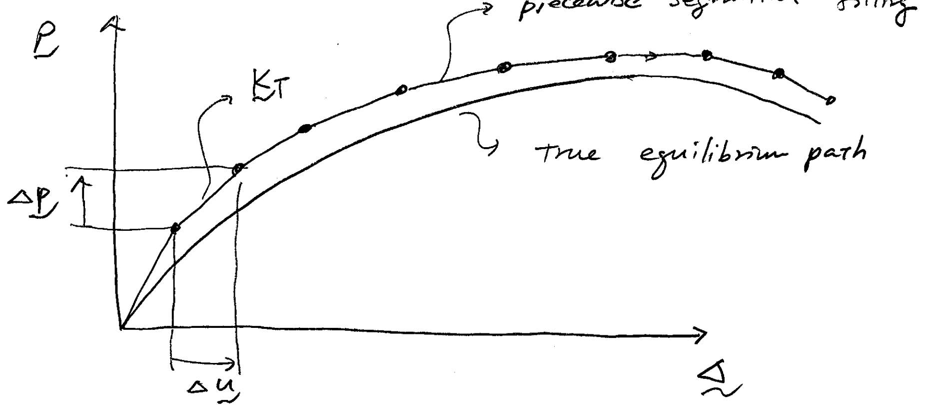
$$k \to \infty, \quad \left(\frac{\lambda_1}{\lambda_2}\right)^k = \cdots = \left(\frac{\lambda_1}{\lambda_n}\right)^k = 0$$

$$\underset{\sim}{v}^{(k)} = \frac{1}{\lambda_1^k} \alpha_1 \underset{\sim}{v}_1$$

$$\underset{\sim}{v}^{(k-1)} = \frac{1}{\lambda_1^{k-1}} \alpha_1 \underset{\sim}{v}_1 = \lambda_1 \underset{\sim}{v}^{(k)}$$

Solution of Nonlinear Equilibrium Equations

Incremental Analysis



At each loading step, tangent stiffness is used.

The tangen stiffness is calculated based on the total displacement.

The force equilibium between external force and internal force is assumed to be satisfied at each loading step.

No iterative calcutions is required.

This method is convenient in numerical calculations, But cannot be used for structures showing very complicated behavior.

Very small seep sizes for loadings are required to obtain solutions close to the actual behavior.

$$\underset{\sim}{K}_T \, \Delta \underset{\sim}{u} = \Delta \underset{\sim}{P} \quad \text{solve } \Delta \underset{\sim}{u}$$

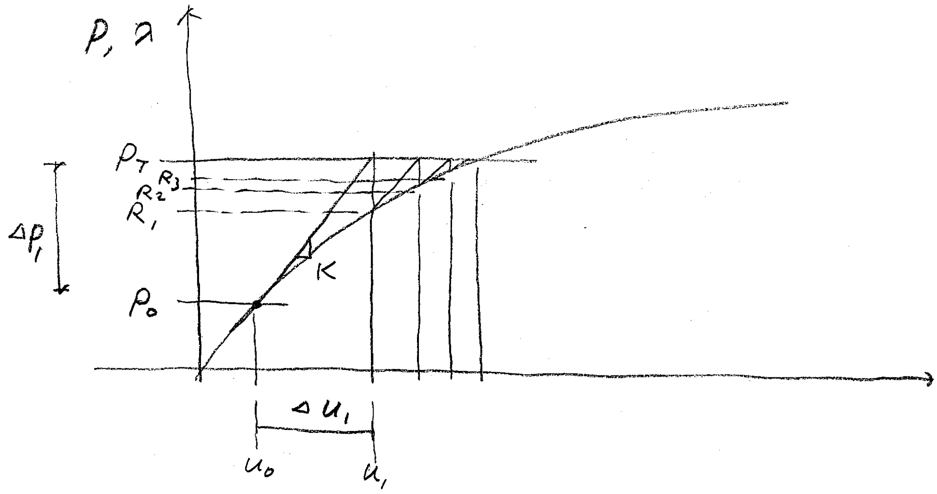
(force control과 displacement control 모두 사용이 가능함

$$\underset{\sim}{u} = \underset{\sim}{u} + \Delta \underset{\sim}{u}$$

$$\underset{\sim}{u} \rightarrow \text{calculate } \underset{\sim}{K}_T$$

($\underset{\sim}{K}_T$ 를 정확히 산출해야 함)

Force control Method



$$\underset{\sim}{K}\,\Delta\underset{\sim}{u} = \Delta\underset{\sim}{P} + \underset{\sim}{R}$$

$$\underset{\sim}{K}\,\Delta\underset{\sim}{u}_1 = \Delta\underset{\sim}{P}_1$$

$$K\,\Delta\underset{\sim}{u}_2 = \Delta\underset{\sim}{P}_2$$

$$\vdots$$

$$\Delta\underset{\sim}{u} = \Sigma\Delta u_i$$

$\underset{\sim}{K}\,\Delta\underset{\sim}{u}_1 = \Delta\underset{\sim}{P}_1$ solve $\Delta\underset{\sim}{u}_1$

$\Delta\underset{\sim}{u}_1 \rightarrow$ internal force $\underset{\sim}{Q}_1$ $(\underline{u} = \underline{u} + \Delta\underline{u})$

Compare $\underset{\sim}{P}_T$ $(= \underset{\sim}{P}_0 + \Delta P_1)$ and $\underset{\sim}{Q}_1$ (by assembling the member forces)

$\left[\text{check tolerance} \leq \dfrac{\|(\underset{\sim}{P}_T - \underset{\sim}{Q}_1)\|}{\|\underset{\sim}{P}_T\|} \right] \xrightarrow[\text{yes}]{}$ terminate to the next load step

↓ NO

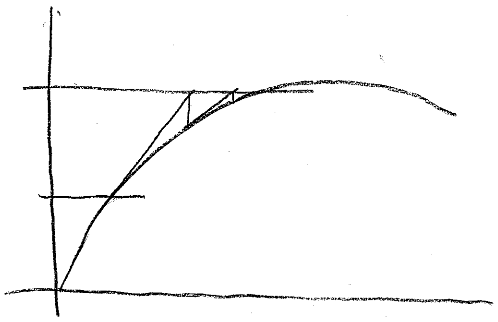
$$\Delta\underset{\sim}{P}_2 = \underset{\sim}{P}_T - \underset{\sim}{Q}_1 = \underset{\sim}{R}_1$$

$\underset{\sim}{K}\,\Delta\underset{\sim}{u}_2 = \Delta\underset{\sim}{P}_2$ solve $\Delta\underset{\sim}{u}_2$

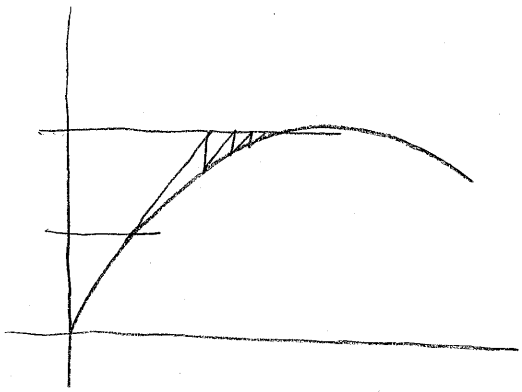
⇣

$$\underset{\sim}{K}\,\Delta\underset{\sim}{u} = \underset{\sim}{P}$$

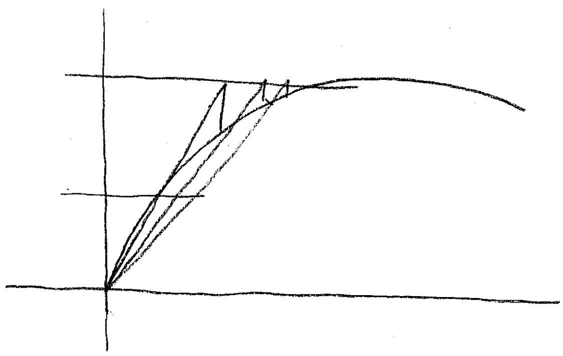
$$\begin{bmatrix} & & \\ & \underset{\sim}{K} & \\ & & \end{bmatrix} \underbrace{\begin{bmatrix} \Delta u_1 \\ \vdots \\ \Delta u_n \end{bmatrix}}_{\text{unknowns}} = \underbrace{\begin{bmatrix} \Delta P_1 \\ \Delta P_2 \\ \vdots \\ \Delta P_n \end{bmatrix}}_{\text{knowns}} = \underbrace{\lambda \begin{bmatrix} P_{p1} \\ \vdots \\ P_{pn} \end{bmatrix}}$$



tangent stiffness



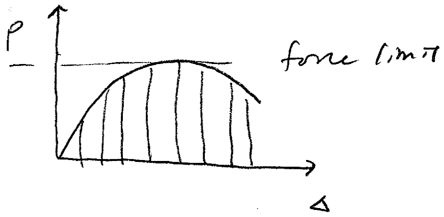
initial stiffness



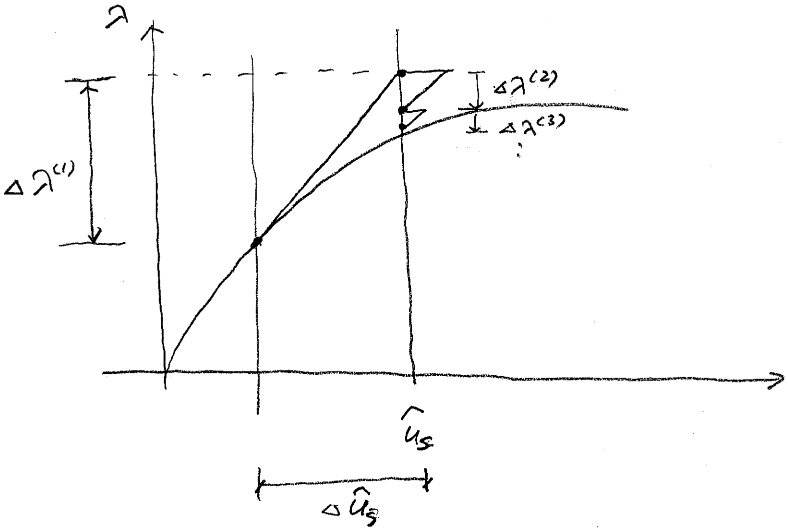
secant stiffness

iterative calculation을 위하여 다양한 stiffness를 사용할 수 있다.

Displacement Control Method



u_s : controlled dof



$$\underset{\sim}{K}\,\Delta \underset{\sim}{u} = \underset{\sim}{P}$$

unknowns

$$\begin{bmatrix} & & \\ & \underset{\sim}{K} & \\ & & \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \vdots \\ \Delta u_s \\ \vdots \\ \Delta u_n \end{bmatrix} = \lambda \begin{bmatrix} P_{p1} \\ \vdots \\ P_{pn} \end{bmatrix}$$

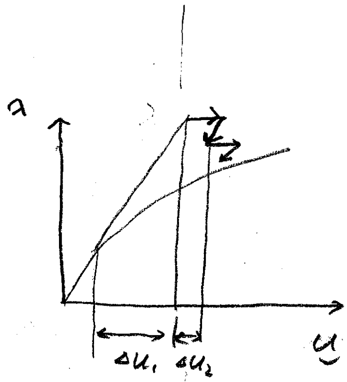
unknowns knowns

$$\underset{\sim}{K}\,\Delta \underset{\sim}{u}^{I} = \underset{\sim}{P}_p$$

$$\underset{\sim}{K}\,\Delta \underset{\sim}{u} = \underset{\sim}{P} + \underset{\sim}{R}$$

$$\underset{\sim}{K}\,\Delta \underset{\sim}{u}^{II} = \underset{\sim}{R}$$

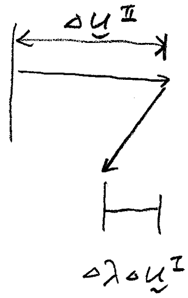
$$\Delta \underset{\sim}{u} = \Delta\lambda\,\Delta \underset{\sim}{u}^{I} + \Delta \underset{\sim}{u}^{II}$$

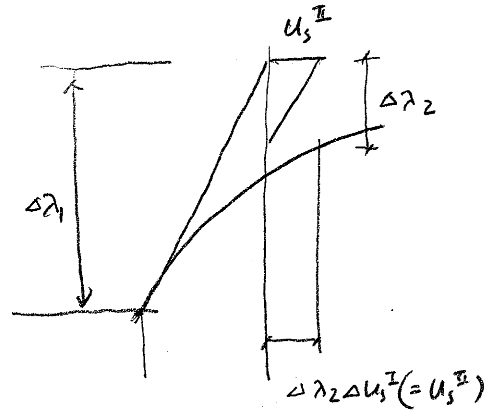


condition :

$$\Delta u_s = \Delta\lambda\,\Delta u_s^{I} + \Delta u_s^{II} = \hat{u}_s$$

$$\Delta\lambda = \frac{\hat{u}_s - \Delta u_s^{II}}{\Delta u_s^{I}}$$





$\mathbf{K}\, \Delta \mathbf{u}^{I} = \mathbf{P}_p$ solve $\Delta \mathbf{u}^{I}$

$$\Delta\lambda_1 = \frac{\hat{u}_s}{\Delta u_s^{I}}$$

$\Delta \mathbf{u}_1 = \Delta\lambda_1 \Delta \mathbf{u}^{I}$ $\quad \mathbf{u}_1 = \mathbf{u}_0 + \Delta \mathbf{u}_1$

$\mathbf{u}_1 \rightarrow \boldsymbol{\varepsilon}_1 \rightarrow$ internal force $\mathbf{Q}_1$

$\mathbf{R}_1 = \lambda_1 \mathbf{P}_p - \mathbf{Q}_1$

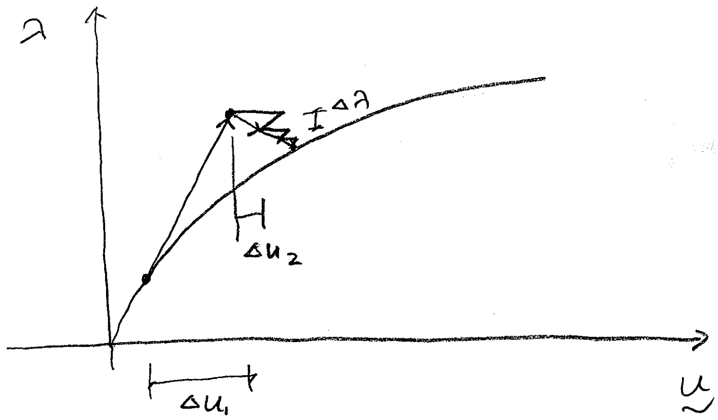
check $\mathbf{R}_1 \approx \mathbf{0}$ $\xrightarrow{\text{yes}}$ stop to the next step

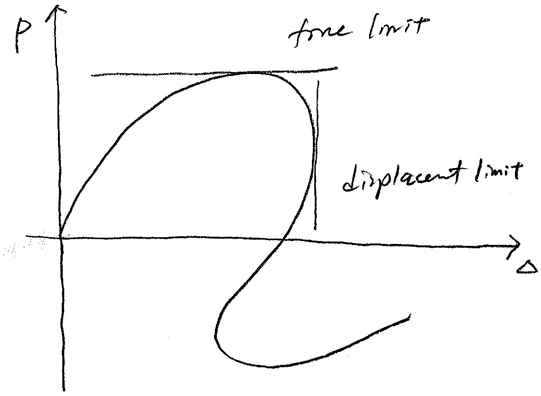
solve $\mathbf{K}\, \Delta \mathbf{u}_2^{II} = \mathbf{R}_1$

$$\Delta\lambda_2 = \frac{-u_s^{II}}{\Delta u_s^{I}} \qquad (\Delta u_{2s} = 0)$$

$\Delta \mathbf{u}_2 = \Delta\lambda_2 \Delta \mathbf{u}^{I} + \Delta \mathbf{u}_2^{II}$ $\quad \mathbf{u}_2 = \mathbf{u}_1 + \Delta \mathbf{u}_2$

Arc-length method



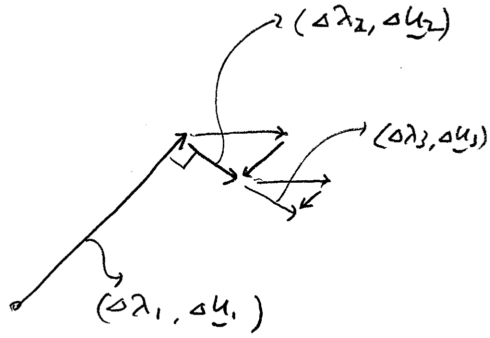


$$\underset{\sim}{K}\,\Delta\underset{\sim}{u} = \underset{\sim}{P} + \underset{\sim}{R}$$

$$\underset{\sim}{K}\,\Delta\underset{\sim}{u}^{\mathrm{I}} = \underset{\sim}{P}_p$$

$$\underset{\sim}{K}\,\Delta\underset{\sim}{u}^{\mathrm{II}} = \underset{\sim}{R}$$

$$\Delta\underset{\sim}{u} = \Delta\lambda\,\Delta\underset{\sim}{u}^{\mathrm{I}} + \Delta\underset{\sim}{u}^{\mathrm{II}}$$



condition

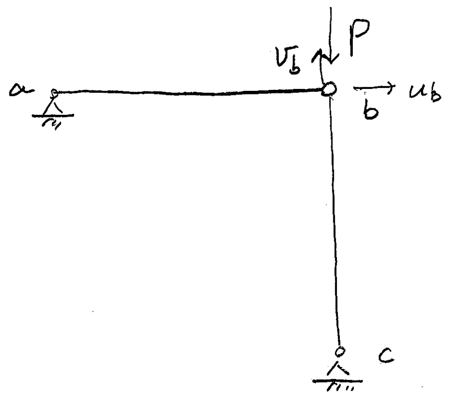
$$\overrightarrow{(\Delta\lambda_1, \Delta\underset{\sim}{u}_1)} \perp \overrightarrow{(\Delta\lambda_i, \Delta\underset{\sim}{u}_i)}$$

$$\Delta\lambda_1 \cdot \Delta\lambda_i + \Delta\underset{\sim}{u}_1 \cdot \Delta\underset{\sim}{u}_i = 0 \qquad i = 2, 3, \ldots$$

$$\Delta\lambda_1 \cdot \Delta\lambda_i + \Delta\underset{\sim}{u}_1 \cdot (\Delta\lambda_i \Delta\underset{\sim}{u}^{\mathrm{I}} + \Delta\underset{\sim}{u}_i^{\mathrm{II}}) = 0$$

$$\Delta\lambda_i = \frac{-\Delta\underset{\sim}{u}_1 \cdot \Delta\underset{\sim}{u}_i^{\mathrm{II}}}{\Delta\lambda_1 + \Delta\underset{\sim}{u}_1 \cdot \Delta\underset{\sim}{u}^{\mathrm{I}}}$$

Example 9.1



$$\underset{\sim}{K}_e = \underset{\sim}{K}_{e(ab)} + \underset{\sim}{K}_{e\,(bc)}$$

$$= 200 \begin{bmatrix} \frac{2}{4000} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{5\times10^3}{4000} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 0 \\ 0 & 2.5\times10^3 \end{bmatrix}$$

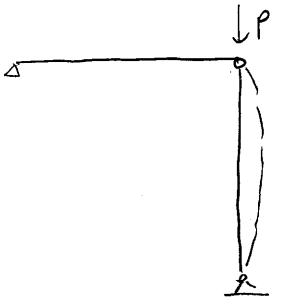
$$\underset{\sim}{G} = \underset{\sim}{G}_{(bc)} = \frac{-P}{4000}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{K} = \underset{\sim}{K}_e + \underset{\sim}{G} = \begin{bmatrix} \left(\frac{1}{10} - \frac{P}{4000}\right) & 0 \\ 0 & \left(\frac{2.5\times10^3}{10} - \frac{P}{4000}\right) \end{bmatrix}$$

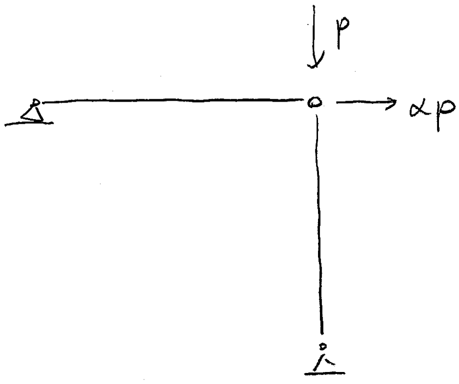
$$|\underset{\sim}{K}| = 0 \Rightarrow \left(\frac{1}{10} - \frac{P}{4000}\right) \times \left(\frac{2.5\times10^3}{10} - \frac{P}{4000}\right) = 0$$

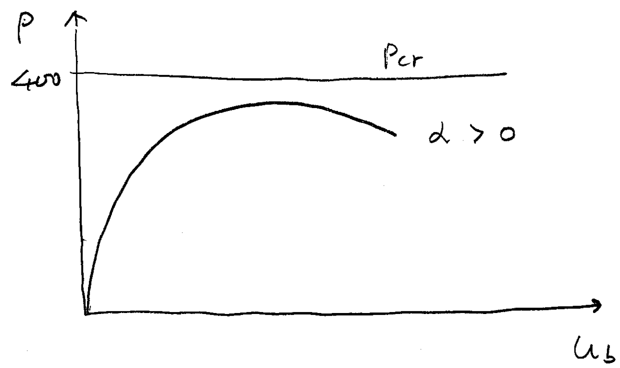
$$\underbrace{P = 400}_{\uparrow\ \text{solution}} \quad \text{or} \quad 400 \times 2.5\times10^3$$

But we need to consider ~~the~~ non-sway buckling mode



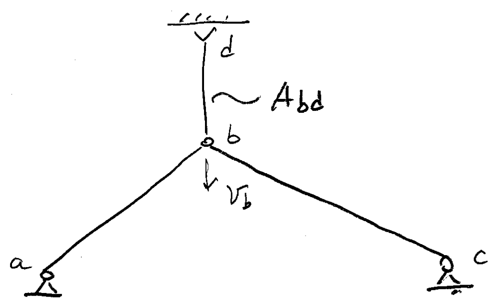
$$P_{cr} = \frac{\pi^2 EI}{L^2}$$





P_{cr} overestimates the true limit force resulting from step-by-step nonlinear analysis

Example 9.2

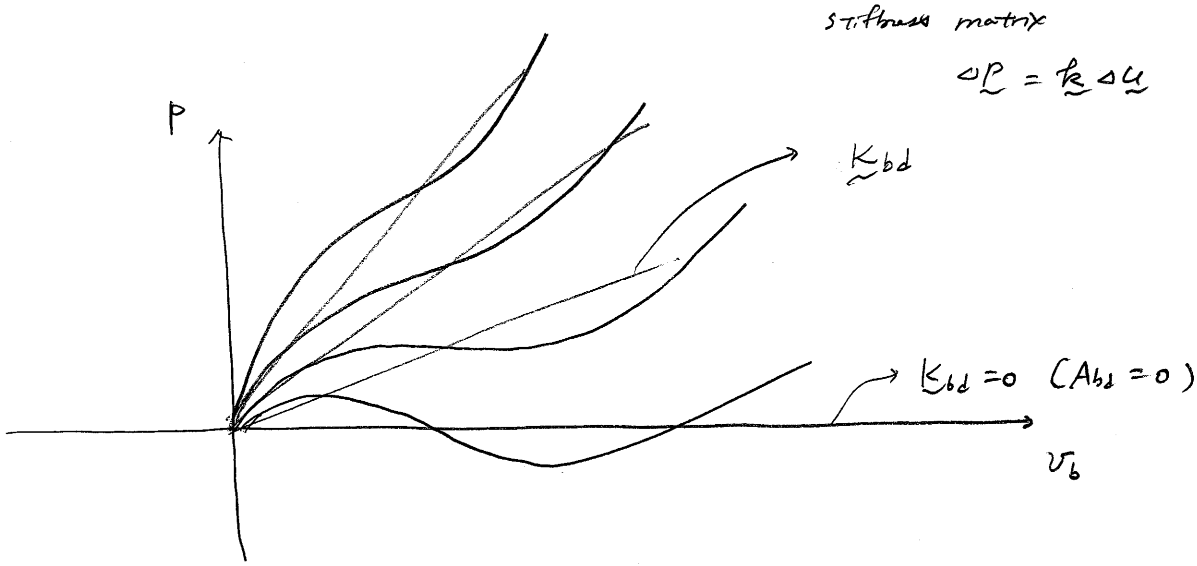


$$\underset{\sim}{k} = \underset{\sim}{T}^T \underset{\sim}{k}' \underset{\sim}{T}$$

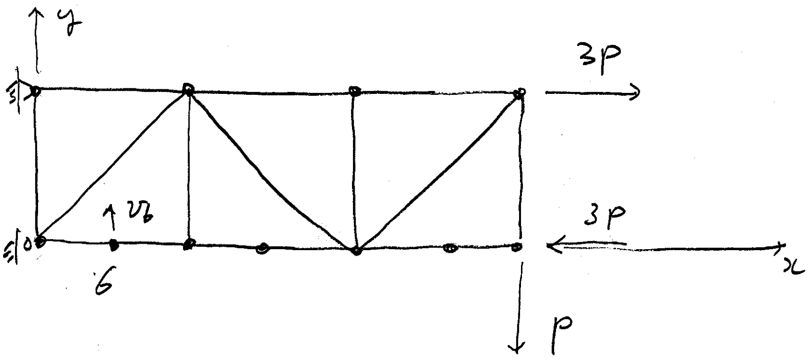
$$\underset{\sim}{k} = \underset{\sim}{k}_e + \underset{\sim}{k}_G$$

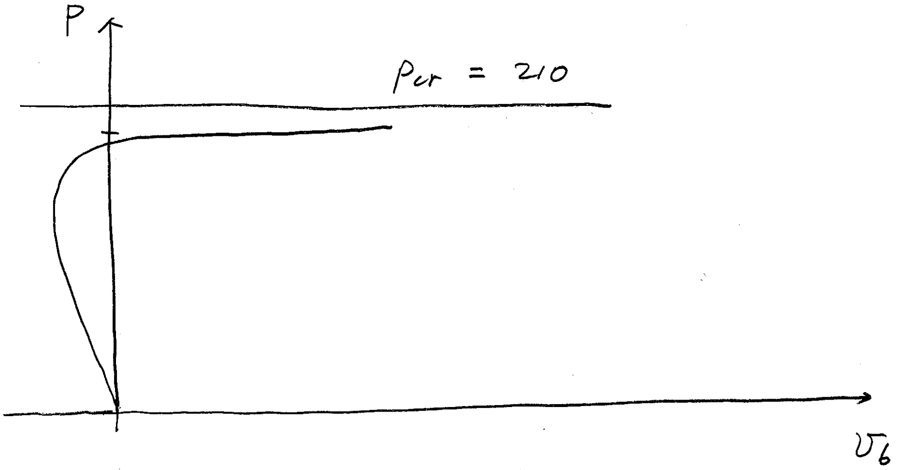
$\underset{\sim}{k}$ with updated geometry can be used as the tangent stiffness matrix

$$\Delta \underset{\sim}{P} = \underset{\sim}{k}\, \Delta \underset{\sim}{u}$$



Example 9.7



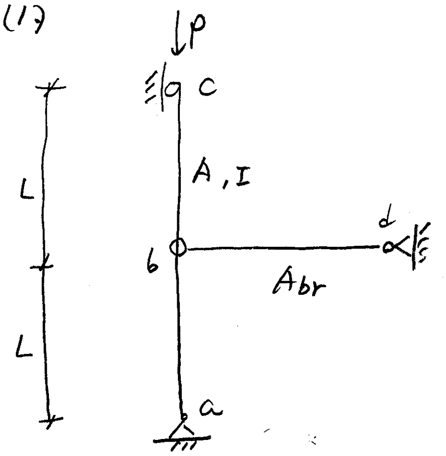


the Buckling mode results from linearized stability analysis is different from the deflected shape results from the step-by-step nonlinear analysis.

The result indicates that under small deflection, the buckling mode due to compression force 3p is dominant in the behavior, on the other hand, under large deflection, the sway mode due to downward load p is dominant.

Example 9.8

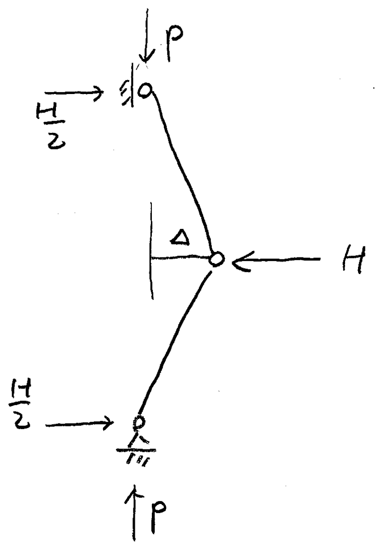
(1)

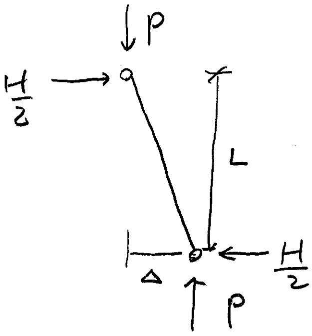


$A = 7.08 \text{ in}^2$

$I = 18.2 \text{ in}^2$

$L = 13' = (13)(12) \text{ in.}$

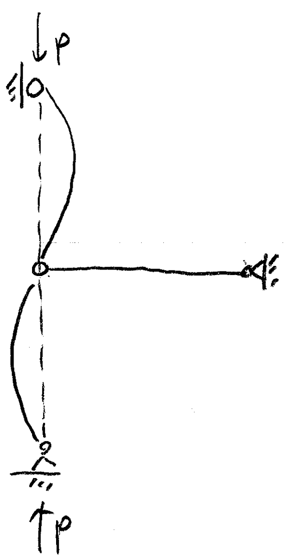




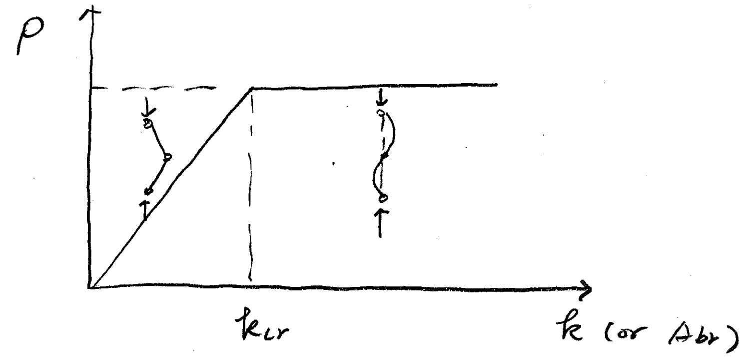
$P \cdot \Delta = \frac{H}{2} L$

$H = k \cdot \Delta$

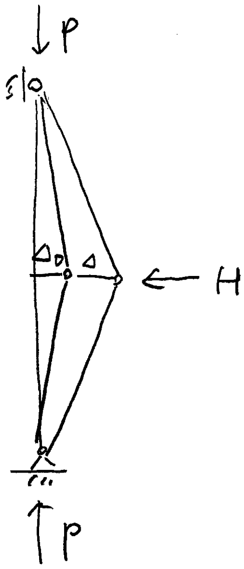
$\therefore P = \frac{kL}{2}$

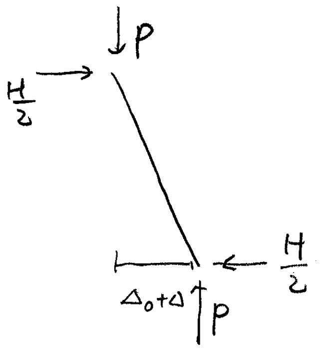


$P_{cr} = \frac{\pi^2 EI}{L^2}$ $\qquad \frac{k_{cr} L}{2} = \frac{\pi^2 EI}{L^2}$ $\qquad k_{cr} = \frac{2\pi^2 EI}{L^3}$



(2)





$A_{br} = 1.27 \text{ in}^2$

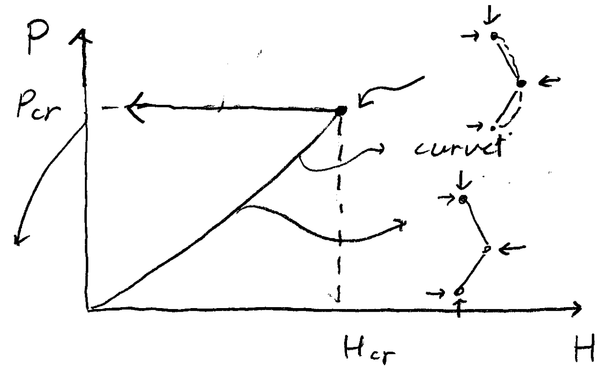
$k = \frac{A_{br}E}{L}$

$$P(\Delta_0+\Delta) = \frac{H}{2}\cdot L$$

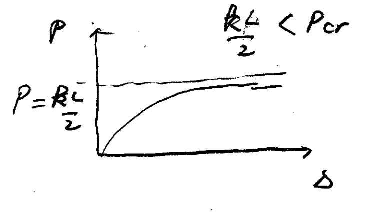
$$H = k\Delta$$

$$P\left(\Delta_0+\frac{H}{k}\right) = \frac{HL}{2}$$

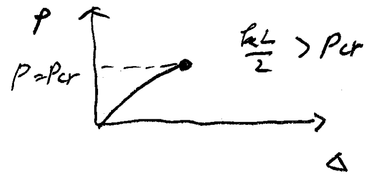
$$P = \frac{HL}{2\left(\Delta_0+\frac{H}{k}\right)}$$



$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

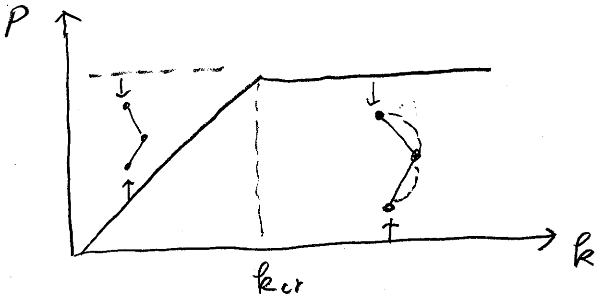


$$P = \frac{kL}{2\left(1+\frac{\Delta_0}{\Delta}\right)}$$

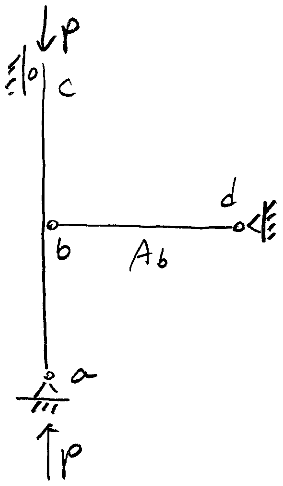


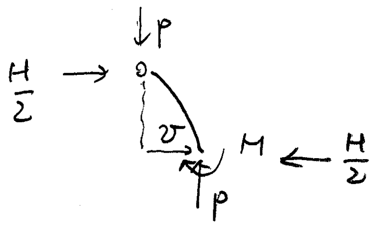
P is the function of Δ as well as k.

$\Delta \to \infty \quad P \to \frac{kL}{2}$



(B.)





$$M - Pv + \frac{H}{2}x = 0$$

$$EIv'' + Pv = \frac{H}{2}x$$

$$v_p = \frac{H}{2P}x$$

$$v_h = A\sin kx + B\cos kx$$

$$v = A\sin Kx + B\cos Kx + \frac{H}{2P}x$$

$$K^2 = \frac{P}{EI}$$

$$H = v(x=L)\cdot k$$

$$k = \frac{A_b E}{L_b}$$

$x = 0 \qquad B = 0$

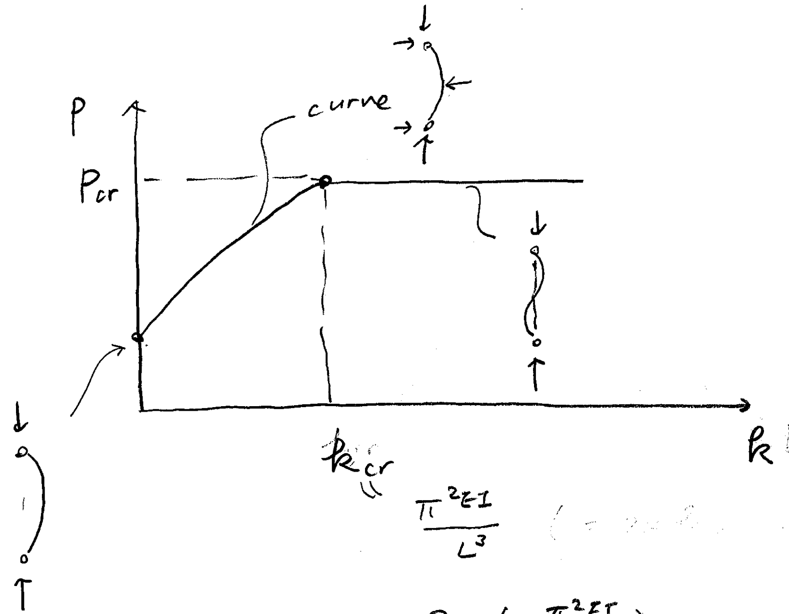
$x = 2L \qquad v = A\sin K(2L) + \frac{H(2L)}{2P} = 0$

$$A = \frac{-HL}{P\sin 2KL}$$

$$v = \left(\frac{-HL}{P\sin 2KL}\right)\sin Kx + \frac{H}{2P}x$$

$$v(x=L) = \frac{-HL}{P\sin 2KL}\sin KL + \frac{HL}{2P} = \frac{H}{k}$$

$$P = \frac{kL}{2}\left(1 - \frac{1}{\cos KL}\right)$$



when $P = P_{cr} \left(= \frac{\pi^2 EI}{L^2}\right)$,

$K = \frac{\pi}{L}$ $\quad \cos KL = -1$

$P_{cr} = \frac{kL}{2}$ (2) $\quad k_{cr} = \frac{\pi^2 EI}{L^3}$ ($= 2 \times k_{cr}$ in case (1))

the resistance by the column itself sould be considered.

text book is incorrect

when initial offset of $2L/500$ is considered,

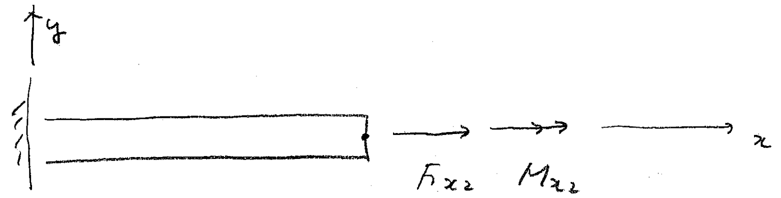
step by step analysis is required because, initially

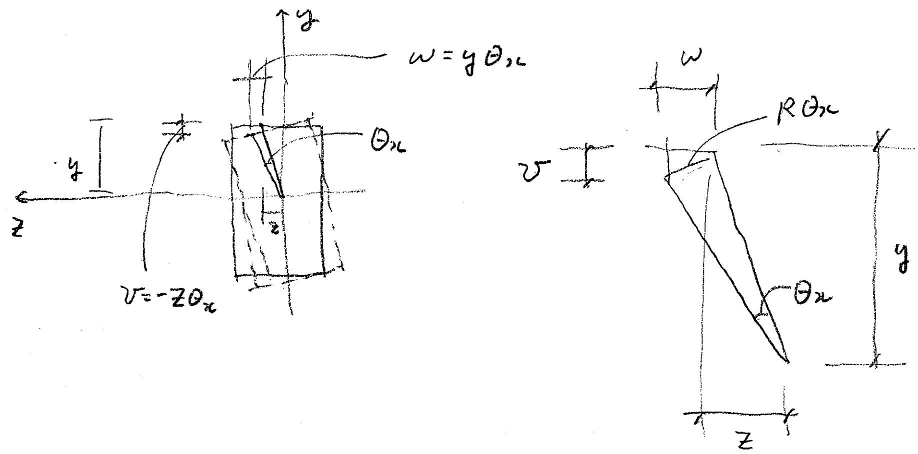
P-Δ effect increased the member moments.

But, the critial load is not expected to be significantly different from that obtained from the critial load analysis neglecting the initial offset.

9.2 Combined Torsion and Axial force

. only St. Venant torque is considered.





$w = R\theta_x \cdot \frac{y}{R}$

$= y\theta_x$

$v = -z\theta_x$, $w = y\theta_x$ $\qquad \overline{ab}$ = length in longitudinal axis

$\frac{dv}{dx} = -z\frac{d\theta_x}{dx}$, $\quad \frac{dw}{dx} = y\frac{d\theta_x}{dx}$ $\qquad \overline{a'b'} = dx\left[\left(1+\frac{du}{dx}\right)^2+\left(\frac{dv}{dx}\right)^2+\left(\frac{dw}{dx}\right)^2\right]^{\frac{1}{2}}$

$$\varepsilon = \frac{\overline{a'b'} - \overline{ab}}{\overline{ab}} \simeq \frac{du}{dx} + \frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right]$$

ε_g with respect to θ_x

$$\varepsilon_g = \frac{1}{2}\left[z^2\left(\frac{d\theta_x}{dx}\right)^2 + y^2\left(\frac{d\theta_x}{dx}\right)^2\right] = \frac{1}{2}[z^2+y^2]\left(\frac{d\theta_x}{dx}\right)^2$$

$\Rightarrow$ warping strain which in the presence of an axial stress is a source of internal virtual work

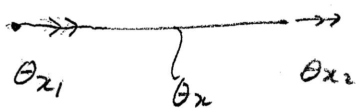
$$\delta U_{ig} = \int \sigma_x \, \delta\varepsilon_g \, dV$$

$$= \frac{1}{2}\int_V \sigma_x \, \delta\left(\frac{d\theta_x}{dx}\right)^2 [z^2+y^2] \, dA\, dx$$

$\sigma_x = F_x / A \qquad I_p = \int (y^2 + z^2)\, dA$

$$\delta W_{1g} = \frac{1}{2} \frac{F_x I_p}{A} \int_0^L \delta \left(\frac{d\theta}{dx}\right)^2 dx$$

$$= \frac{F_x I_p}{A} \int_0^L \left(\frac{d\delta\theta}{dx}\right)\left(\frac{d\theta}{dx}\right) dx$$

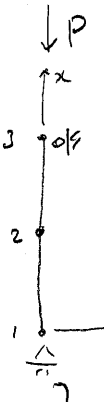


$$\theta_x = \left[\left(1 - \frac{x}{L}\right) \quad \frac{x}{L} \right] \begin{bmatrix} \theta_{x1} \\ \theta_{x2} \end{bmatrix}$$

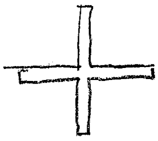
$$= \frac{F_x I_p}{A} \, \delta \underset{\sim}{q}^T \int_0^L \underset{\sim}{N}'^T \underset{\sim}{N}' dx \, \underset{\sim}{q} \qquad \underset{\sim}{q} = \begin{bmatrix} \theta_{x1} \\ \theta_{x2} \end{bmatrix}$$

$$\underset{\sim}{K}_g = \frac{F_x I_p}{AL} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Example 9.9



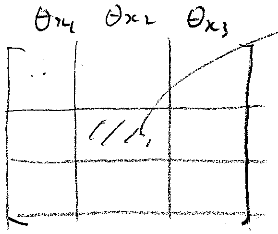
Torsionally fixed end



no warping torsional resistance

only pure torsion is considered

$$\underset{\sim}{K} = \underset{\sim}{K}_e + \underset{\sim}{K}_g$$



$$\left[\frac{GJ}{\left(\frac{L}{2}\right)} + \frac{(-P) I_p}{A \left(\frac{L}{2}\right)} \right] \times 2$$

$$\left[\frac{GJ}{L} + \frac{(-P) I_p}{AL} \right] 4\, \theta_{x2} = 0$$

$$P = \frac{GJA}{I_p}$$ Torsional buckling Load