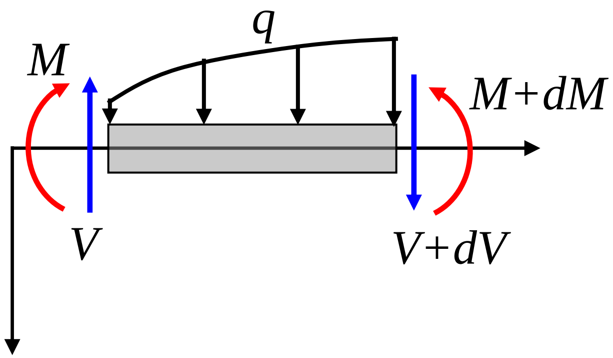
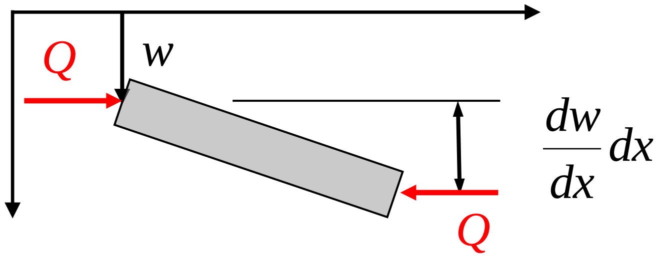
Chapter 1

Beam Equations

1.1 Governing Equation for a Beam with Axial Force





- **Equilibrium for vertical force**

$$(V+dV)-V+(q-kw-m\frac{d^2w}{dt^2})dx=0 \rightarrow \frac{dV}{dx}-kw=-q+m\frac{d^2w}{dt^2}$$

- **Equilibrium for moment**

$$(M+dM)-M-Vdx+(q-kw-m\frac{d^2w}{dt^2})dx\frac{dx}{2}-Q\frac{dw}{dx}dx=0 \rightarrow \frac{dM}{dx}-Q\frac{dw}{dx}=V$$

- **Elimination of shear force**

$$\frac{d^2M}{dx^2}-Q\frac{d^2w}{dx^2}-kw=-q+m\frac{d^2w}{dt^2}$$

- **Strain-displacement relation**

$$\varepsilon = -\frac{d^2 w}{dx^2} y + \frac{du}{dx}$$

- **Stress-strain relation (Hooke law)**

$$\sigma = E\varepsilon = -E\frac{d^2 w}{dx^2} y + E\frac{du}{dx}$$

- **Definition of Moment**

$$M = \int_A \sigma y dA = \int_A E\varepsilon y dA = -\int_A \left(E\frac{d^2 w}{dx^2} y^2 - E\frac{du}{dx} y\right) dA = -EI\frac{d^2 w}{dx^2}$$

- **Beam Equation with Axial Force**

$$EI\frac{d^4 w}{dx^4} + Q\frac{d^2 w}{dx^2} + kw = q - m\frac{d^2 w}{dt^2}$$

1.2 General Solution

- **Characteristic Equation for** $Q > 0$

$$w = e^{\lambda x}$$

$$e^{\lambda x}(\lambda^4 + \beta^2\lambda^2) = 0 \rightarrow \lambda = \pm\beta i,\ 0 \qquad \text{where} \quad \beta^2 = \frac{Q}{EI}$$

- **Homogeneous solution**

$$w = Ae^{\beta ix} + Be^{-\beta ix} + Cx + D$$

- **Homogeneous solution**

$$\begin{aligned} w &= A(\cos\beta x + i\sin\beta x) + B(\cos\beta x - i\sin\beta x) + Cx + D \\ &= (A + B)\cos\beta x + i(A - B)\sin\beta x + Cx + D \\ &= A\cos\beta x + B\sin\beta x + Cx + D \end{aligned}$$

- **Characteristic Equation for** $Q < 0$

$$w = e^{\lambda x}$$

$$e^{\lambda x}(\lambda^4 - \beta^2\lambda^2) = 0 \rightarrow \lambda = \pm\beta,\ 0 \qquad \text{where} \quad \beta^2 = \left|\frac{Q}{EI}\right|$$

- **Homogeneous solution for** $Q<0$

$$w = Ae^{\beta x} + Be^{-\beta x} + Cx + D$$
$$= (A+B)\frac{e^{\beta x}+e^{-\beta x}}{2} + (A-B)\frac{e^{\beta x}-e^{-\beta x}}{2} + Cx + D$$
$$= A\cosh\beta x + B\sinh\beta x + Cx + D$$

- **General Solution**

$$w = w_h + w_p = A\cos\beta x + B\sin\beta x + Cx + D + w_p$$

$$EI\frac{d^4(w_h+w_p)}{dx^4} + Q\frac{d^2(w_h+w_p)}{dx^2} = EI\frac{d^4 w_h}{dx^4} + Q\frac{d^2 w_h}{dx^2} + EI\frac{d^4 w_p}{dx^4} + Q\frac{d^2 w_p}{dx^2} = EI\frac{d^4 w_p}{dx^4} + Q\frac{d^2 w_p}{dx^2} = q$$

- **Four Boundary Conditions for Simple Beams**

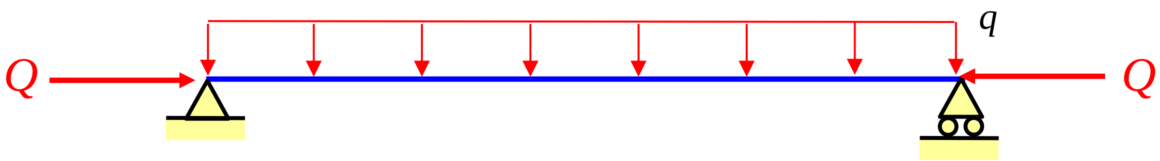
$$w(0) = A + D + w_p(0) = 0$$
$$M(0) = -EIw''(0) = -EI(-A\beta^2 + w_p''(0)) = 0$$
$$w(L) = A\cos\beta L + B\sin\beta L + CL + D + w_p(L) = 0$$
$$M(L) = -EIw''(L) = -EI(-A\beta^2\cos\beta L - B\beta^2\sin\beta L + w_p''(L)) = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -\beta^2 & 0 & 0 & 0 \\ \cos\beta L & \sin\beta L & L & 1 \\ -\beta^2\cos\beta L & -\beta^2\sin\beta L & 0 & 0 \end{bmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} + \begin{pmatrix} w_p(0) \\ w_p''(0) \\ w_p(L) \\ w_p''(L) \end{pmatrix} = 0 \rightarrow \mathbf{KX} + \mathbf{F} = 0$$

- The homogenous solution is for the boundary conditions, while the particular solution is for the equilibrium.

1.3 Analysis of Beam-Columns and Buckling Load

- **Beam Column - Simple Beam Subject to a Uniformly Distributed Load**



- Governing Equation

$$EI\frac{d^4w}{dx^4}+Q\frac{d^2w}{dx^2}=q$$

- Particular Solution

$$w_p(x)=\frac{q}{2Q}x^2$$

- General Solution

$$w(x)=A\cos\beta x+B\sin\beta x+Cx+D+\frac{q}{2Q}x^2$$

$$w''(x)=-A\beta^2\cos\beta x-B\beta^2\sin\beta x+\frac{q}{Q}$$

- Boundary Condition

$$w(0) = A + D = 0 \quad , \quad M(0) = -EIw''(0) = -EI(-A\beta^2 + \frac{q}{Q}) = 0 \rightarrow A = \frac{q}{Q\beta^2}, D = -\frac{q}{Q\beta^2}$$

$$w(L) = B\sin\beta L + CL + \frac{q}{Q\beta^2}(\cos\beta L - 1) + \frac{q}{2Q}L^2 = 0$$

$$M(L) = -EIw''(L) = -EI(-A\beta^2\cos\beta L - B\beta^2\sin\beta L + \frac{q}{Q}) = 0$$

$$-\frac{q}{Q}\cos\beta L - B\beta^2\sin\beta L + \frac{q}{Q} = 0 \rightarrow B\beta^2\sin\beta L = -\frac{q}{Q}(\cos\beta L - 1)$$

$$B\beta^2 2\sin\frac{\beta L}{2}\cos\frac{\beta L}{2} = -\frac{q}{Q}(1 - 2\sin^2\frac{\beta L}{2} - 1) \rightarrow B = \frac{1}{\beta^2}\frac{q}{Q}\tan\frac{\beta L}{2}$$

$$C = -\frac{q}{2Q}L$$

- Final solution

$$w(x) = \frac{q}{\beta^2 Q}(\cos\beta x - 1) + \frac{q}{\beta^2 Q}\tan\frac{\beta L}{2}\sin\beta x - \frac{q}{2Q}Lx + \frac{q}{2Q}x^2$$

$$M(x) = -EIw''(x) = EI(\frac{q}{Q}\cos\beta x + \frac{q}{Q}\tan\frac{\beta L}{2}\sin\beta x - \frac{q}{Q})$$

- Deflection at the Mid-span

$$w\left(\frac{L}{2}\right) = \frac{q}{\beta^2 Q}\left(\cos\frac{\beta L}{2} - 1\right) + \frac{1}{\beta^2}\frac{q}{Q}\tan\frac{\beta L}{2}\sin\frac{\beta L}{2} - \frac{q}{8Q}L^2$$

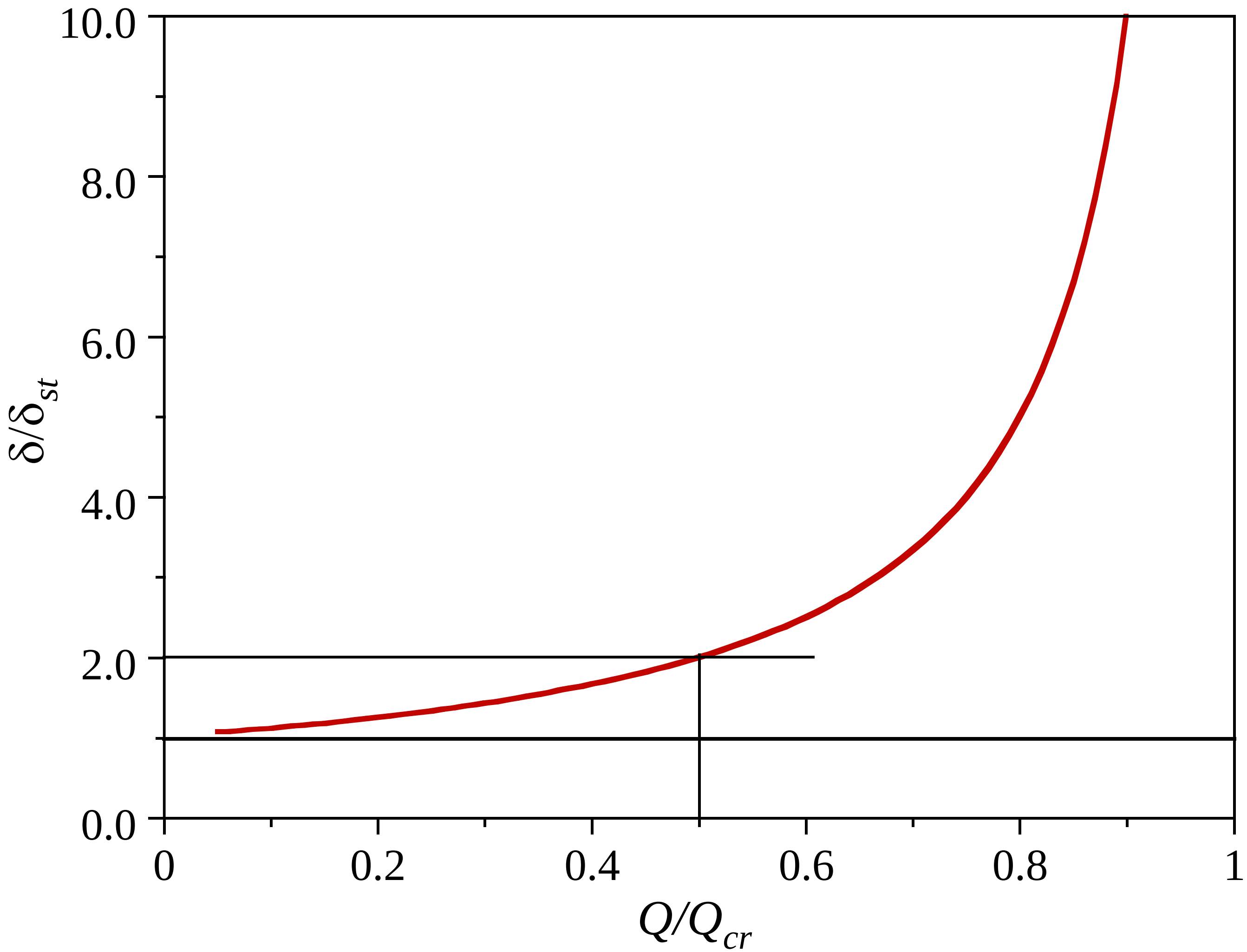
$$= \frac{q}{\beta^2 Q}\left(\cos\frac{\beta L}{2} + \tan\frac{\beta L}{2}\sin\frac{\beta L}{2} - 1\right) - \frac{q}{8Q}L^2$$

$$= \frac{q}{\beta^2 Q}\left(\frac{1}{\cos\frac{\beta L}{2}} - 1 - \frac{L^2\beta^2}{8}\right)$$

$$= \frac{5qL^4}{384EI}\frac{384}{5\pi^4}\frac{\pi^4 EI^2}{Q^2 L^4}\left(\frac{1}{\cos\frac{\pi}{2}\sqrt{L^2 Q/\pi^2 EI}} - 1 - \frac{\pi^2}{8}\frac{QL^2}{\pi^2 EI}\right)$$

$$= \delta_{st}\,0.7884\left(\frac{Q_{cr}}{Q}\right)^2\left(\frac{1}{\cos\frac{\pi}{2}\sqrt{Q/Q_{cr}}} - 1 - \frac{\pi^2}{8}\frac{Q}{Q_{cr}}\right)$$

$$= \delta_{st}\frac{0.7884}{t^2}\left(\frac{1}{\cos\frac{\pi}{2}\sqrt{t}} - 1 - \frac{\pi^2}{8}t\right) \qquad \left(t = \frac{Q}{Q_{cr}}\right)$$


10.0
8.0
6.0
4.0
2.0
0.0
δ/δst
0
0.2
0.4
0.6
0.8
1
Q/Qcr

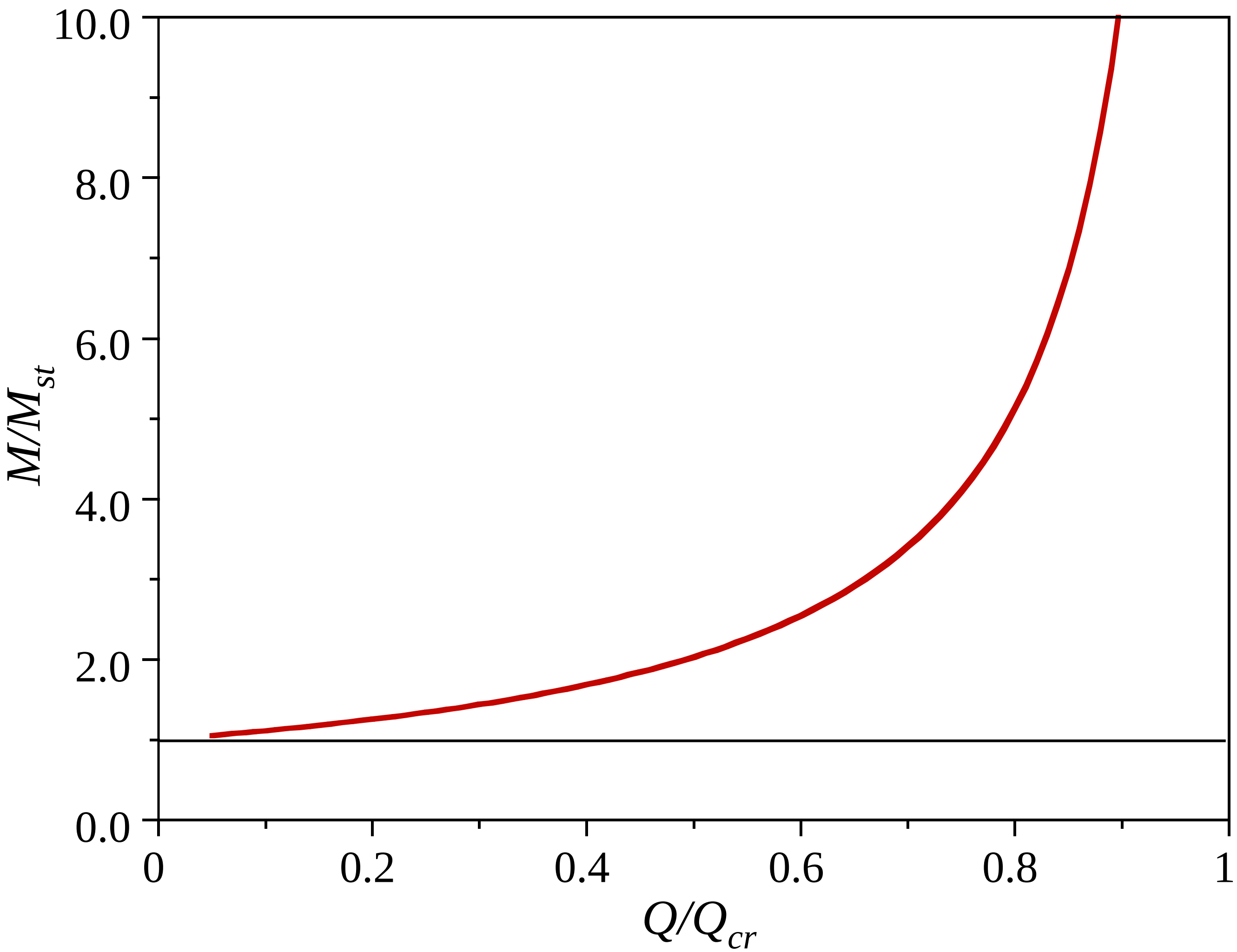
- Moment at the Mid-span

$$M(\frac{L}{2}) = EI\frac{q}{Q}(\cos\beta\frac{L}{2} + \tan\frac{\beta L}{2}\sin\frac{\beta L}{2} - 1)$$

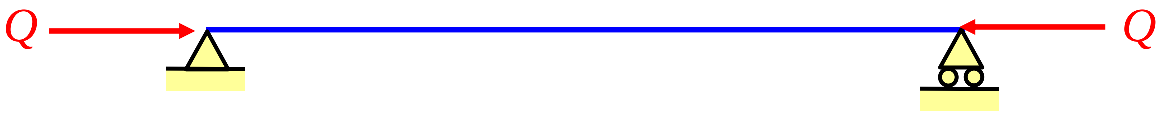
$$= \frac{qL^2}{8}\frac{8}{\pi^2}\frac{\pi^2 EI}{QL^2}(\cos\beta\frac{L}{2} + \tan\frac{\beta L}{2}\sin\frac{\beta L}{2} - 1)$$

$$= M_{st}\frac{8}{\pi^2}\frac{Q_{cr}}{Q}(\frac{1}{\cos\frac{\pi}{2}\sqrt{Q/Q_{cr}}} - 1)$$

$$= M_{st}\,0.811\frac{1}{t}(\frac{1}{\cos\frac{\pi}{2}\sqrt{t}} - 1)$$


10.0
8.0
6.0
4.0
2.0
0.0
M/M_{st}
0
0.2
0.4
0.6
0.8
1
Q/Q_{cr}

- **Buckling Analysis of Simple Beam**



- Boundary Condition

$$w(0) = A + D = 0 \ , \quad w''(0) = -A\beta^2 = 0 \rightarrow A = 0$$

$$w(L) = B\sin\beta L + CL = 0 \ , \quad w''(L) = -B\beta^2 \sin\beta L = 0 \rightarrow B = C = 0$$

- Characteristic Equation

$$A = B = C = D = 0 \rightarrow w = 0 \ \ \textbf{(???) or} \ \ \beta L = n\pi \rightarrow Q = \frac{n^2\pi^2 EI}{L^2} \ , \quad n = 1,2,3\cdots$$

$$w = B\sin\beta x = B\sin\frac{n\pi}{L}x$$

- **Buckling Analysis of Fixed-Fixed Beam**

- Boundary Condition

$$w(0) = A + D = 0 \ , \quad w'(0) = \beta B + C = 0$$

$$w(L) = A\cos\beta L + B\sin\beta L + CL + D = 0 \ , \quad w'(L) = -A\beta\sin\beta L + B\beta\cos\beta L + C = 0$$

or

- Characteristic Equation

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \beta & 1 & 0 \\ \cos\beta L & \sin\beta L & L & 1 \\ -\beta\sin\beta L & \beta\cos\beta L & 1 & 0 \end{bmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow Det\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \beta & 1 & 0 \\ \cos\beta L & \sin\beta L & L & 1 \\ -\beta\sin\beta L & \beta\cos\beta L & 1 & 0 \end{bmatrix} \right) = 0$$

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & \beta & 1 & 0 \\ \cos\beta L & \sin\beta L & L & 1 \\ -\beta\sin\beta L & \beta\cos\beta L & 1 & 0 \end{vmatrix} = \begin{vmatrix} \beta & 1 & 0 \\ \sin\beta L & L & 1 \\ \beta\cos\beta L & 1 & 0 \end{vmatrix} - \begin{vmatrix} 0 & \beta & 1 \\ \cos\beta L & \sin\beta L & L \\ -\beta\sin\beta L & \beta\cos\beta L & 1 \end{vmatrix}$$

$$-\beta-(-\beta\cos\beta L)-(-\beta(\cos\beta L+\beta L\sin\beta L)+\beta)=\beta(2\cos\beta L-2+\beta L\sin\beta L)=0$$

$$2\cos\beta L-2+\beta L\sin\beta L=2(\cos\beta L-1)+\beta L\sin\beta L=-4\sin^2\frac{\beta L}{2}+2\beta L\sin\frac{\beta L}{2}\cos\frac{\beta L}{2}=0$$

$$\sin\frac{\beta L}{2}\left(\frac{\beta L}{2}\cos\frac{\beta L}{2}-\sin\frac{\beta L}{2}\right)=0\rightarrow\sin\frac{\beta L}{2}=0\ \text{ or }\ \frac{\beta L}{2}\cos\frac{\beta L}{2}-\sin\frac{\beta L}{2}=0$$

- Eigenvalues

$$w(0)=A+D=0\ ,\quad w'(0)=\beta B+C=0$$

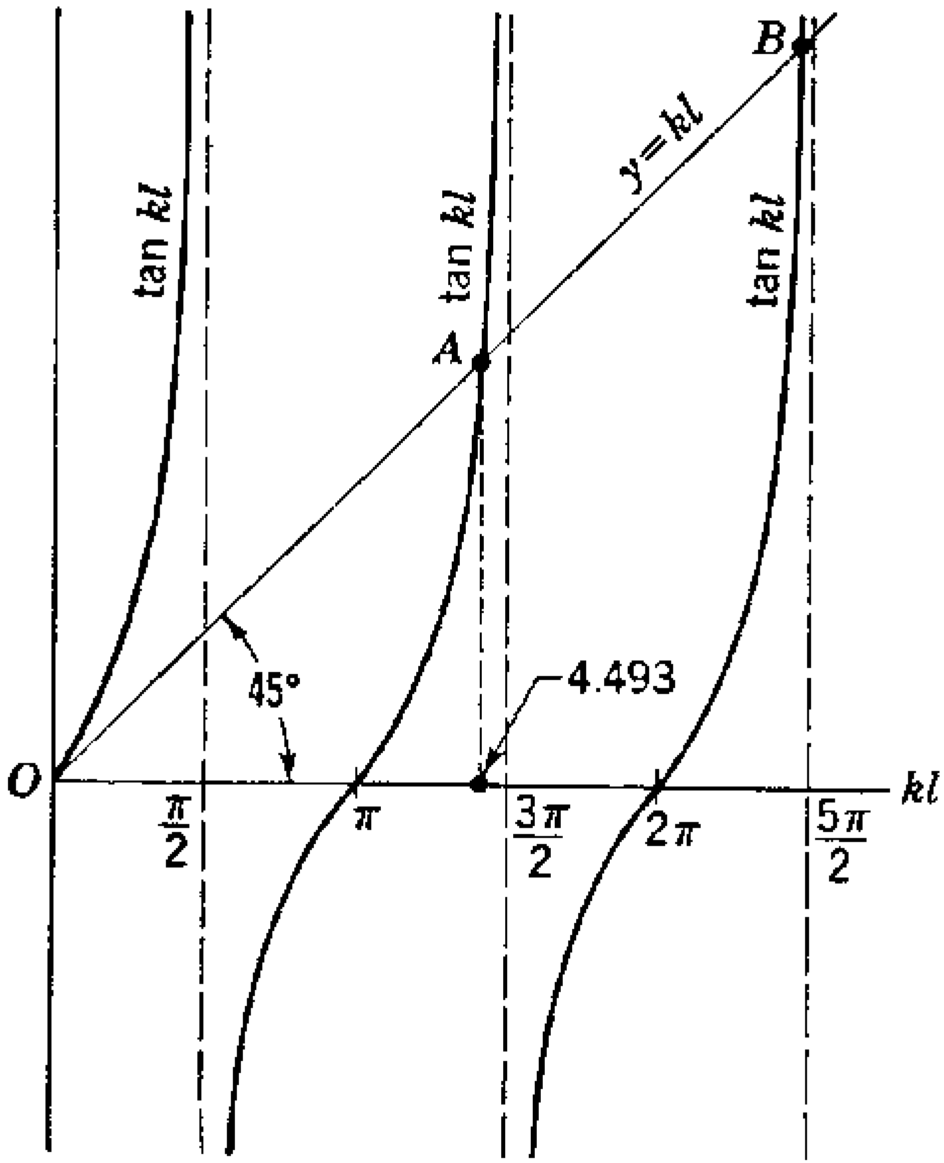
$$w(L)=A\cos\beta L+B\sin\beta L+CL+D=0\ ,\quad w'(L)=-A\beta\sin\beta L+B\beta\cos\beta L+C=0$$

Symmetric modes: $\sin\frac{\beta L}{2}=0\ \rightarrow\frac{\beta L}{2}=n\pi\rightarrow Q=\frac{4n^2\pi^2 EI}{L^2},\quad n=1,2,3\cdots$

$$w(0)=A+D=0\ ,\quad w'(0)=\beta B+C=0,\ w(L)=A+CL+D=0\ ,\quad w'(L)=\beta B+C=0$$

$$A+D=0\rightarrow A=-D\rightarrow w=A(\cos\frac{2n\pi}{L}x-1)\ \text{ for }\ A\neq 0$$

Anti-symmetric modes: $\frac{\beta L}{2}\cos\frac{\beta L}{2}-\sin\frac{\beta L}{2}=0\rightarrow\frac{\beta L}{2}=\tan\frac{\beta L}{2}\rightarrow Q=\frac{8.18\pi^2 EI}{L^2}$



tan kl
tan kl
tan kl
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A
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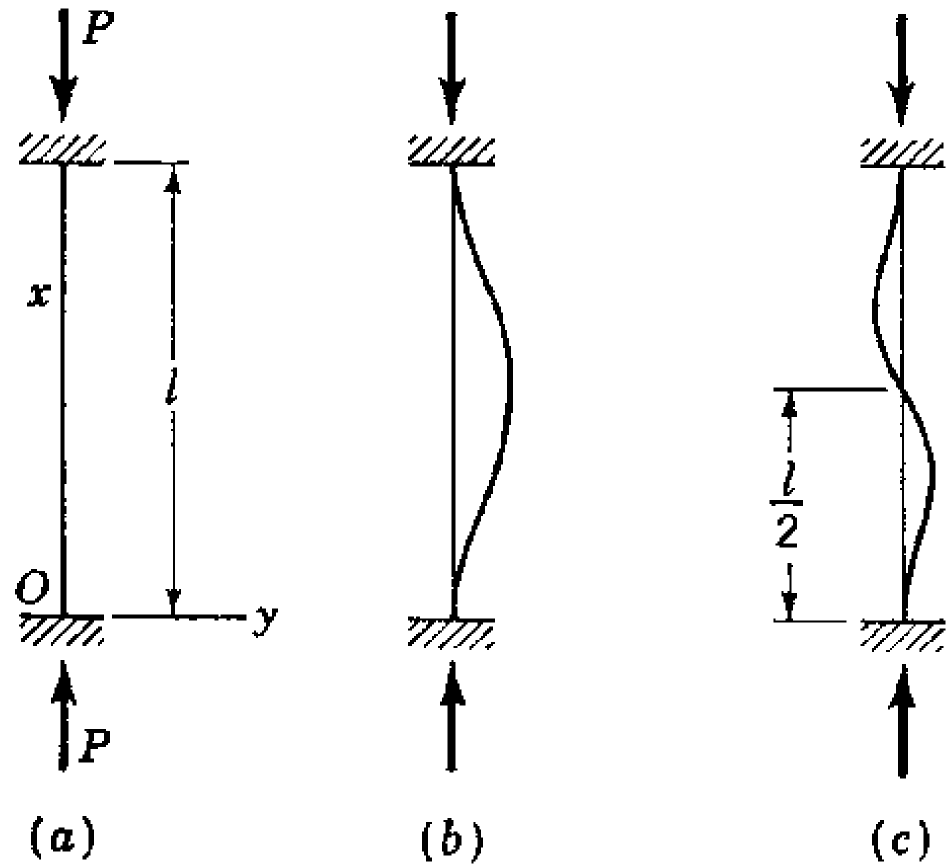
π
3π/2
2π
5π/2



P
x
l
O
y
P
l/2
(a)
(b)
(c)

- **Buckling Analysis of Cantilever Beam**

- Boundary Condition

$$w(0) = A + D = 0$$

$$w'(0) = \beta B + C = 0$$

$$M(L) = -EIw''(L) = -EI(-A\beta^2 \cos\beta L - B\beta^2 \sin\beta L) = 0$$

$$V(L) = -EI\frac{d^3w}{dx^3} - Q\frac{dw}{dx} = 0$$

- Buckling load

$$Q = \frac{n^2\pi^2 EI}{4L^2}, \quad n = 1,2,3\cdots$$