11

Work and Energy Methods

11.1 Introduction. Several work and energy methods have already been presented and used for computing deflections and analyzing statically indeterminate structures. The method of virtual work was developed in Arts. 8.3 to 8.7, and Castigliano's first and second theorems were derived in Arts. 8.16 and 8.15, respectively.

Several more energy relationships will be needed for the development of the methods of systematic analysis to be discussed in Part II. Specifically, the **principle of stationary total potential energy** and the **principle of stationary total complementary potential energy** will be utilized in these subsequent discussions. Before developing these two new principles, it will be helpful to review some basic definitions. Then some of the previous developments regarding work and energy will also be reviewed, after which these concepts will be expanded to obtain the additional principles needed in the following chapters.

11.2 Some Basic Definitions and Concepts. Perhaps it should only be necessary at this point to suggest that the reader review his elementary dynamics, concentrating particularly on the basic concepts and definitions introduced during the elementary applications of Newton's laws of motion. However, those ideas that are particularly pertinent to the present developments in this chapter will be restated here for ready reference.

A **force** may be defined as any action that tends to change the state of motion (or rest) of the body to which it is applied. When the point of application of an active force moves, then the force is said to do **work** $\mathscr{W}$ equal to the product of the force and the lineal displacement of its point of application in the direction of the force.[1]

[1] In this case, the words *force* and *displacement* are used in a generalized sense which may also be interpreted to mean *couple* and *rotational displacement*, respectively.

If a force is *very* gradually applied to a body, there will be essentially only static deformation and displacement of the body, with no essential accelerations or changes of velocity involved. In such cases, due to the deformation of the body, the work done by the force is stored up in the body by a particular form of potential energy referred to as **strain energy**. If, on the other hand, a force is *not* gradually applied, accelerations and changes of velocity, as well as deformation of the body, will be produced and the work done by the force may be converted into both strain energy and a change in the kinetic energy of the body.

Most often we associate **potential energy** $\mathscr{V}$ with the capability (or potential) of a weight to do work. We measure such potential by selecting arbitrarily some convenient datum plane such as shown in Fig. 11.1*a* and then computing $\mathscr{V}_o$, the initial potential energy (or the initial potential) of the weight, to be the product of D_o, the initial distance of the weight from the datum, and W_1, the force of gravity acting on the weight.

Actually, of course, any active force P, whether it acts vertically or horizontally or in any direction, has the potential for doing work, and therefore it has potential energy. As in the case of the weight, any force P has an initial potential energy $\mathscr{V}_o$ equal to PD_o, where D_o is the distance to a convenient datum measured in the sense of, and along the line of action of, the force.

If the weight W_1 in Fig. 11.1*b* were pushed off of its supporting ledge, it would fall freely. As it fell, the weight would steadily acquire more and more kinetic energy, the amount of such energy at any instant being exactly equal to the potential energy W_1D it had lost between that instant and the start of its fall. On the other hand, consider the situation shown in Fig. 11.1*c*, where the weight is very gradually transferred from the ledge to the supporting spring. It can be imagined that this gradual tranfer is accomplished with the assistance of a friendly genie who at the start of the transfer process bears the entire load of the weight. But as the transfer proceeds, the spring gradually deflects and absorbs part of the load of the weight. Finally, the spring has deflected enough for the resisting force of the spring to carry the entire load and the genie can be discharged for the time being. Depending on whether the force-deflection characteristics of the spring are linear or nonlinear, the load-transfer process can be depicted by Fig. 11.1*d* or *e*, respectively.

During the gradual assumption of the weight W_1 by the spring, the weight has *lost* potential energy by the amount of $-W_1\Delta_1$. Note that this amount is numerically equal to the area of the rectangle *Oabc* in either Fig. 11.1*d* or *e*. On the other hand, the area *Obc* under force-deflection curve *Ob* in either of these figures represents the strain energy stored in the spring. During the gradual loading of the spring, the resisting force R of the spring at any displacement is equal and opposite to the net load; i.e., the difference between the weight and the portion of the weight being supported by the genie. The curve *Ob* therefore could also represent the net load-deflection curve; and for such an interpretation the area *Obc* would be equal to the work done by the net load during the gradual loading of the spring.

The area *Oab* above the curve *Ob* represents that part of the potential energy lost by the weight that is *not* transformed into strain energy stored in the spring.

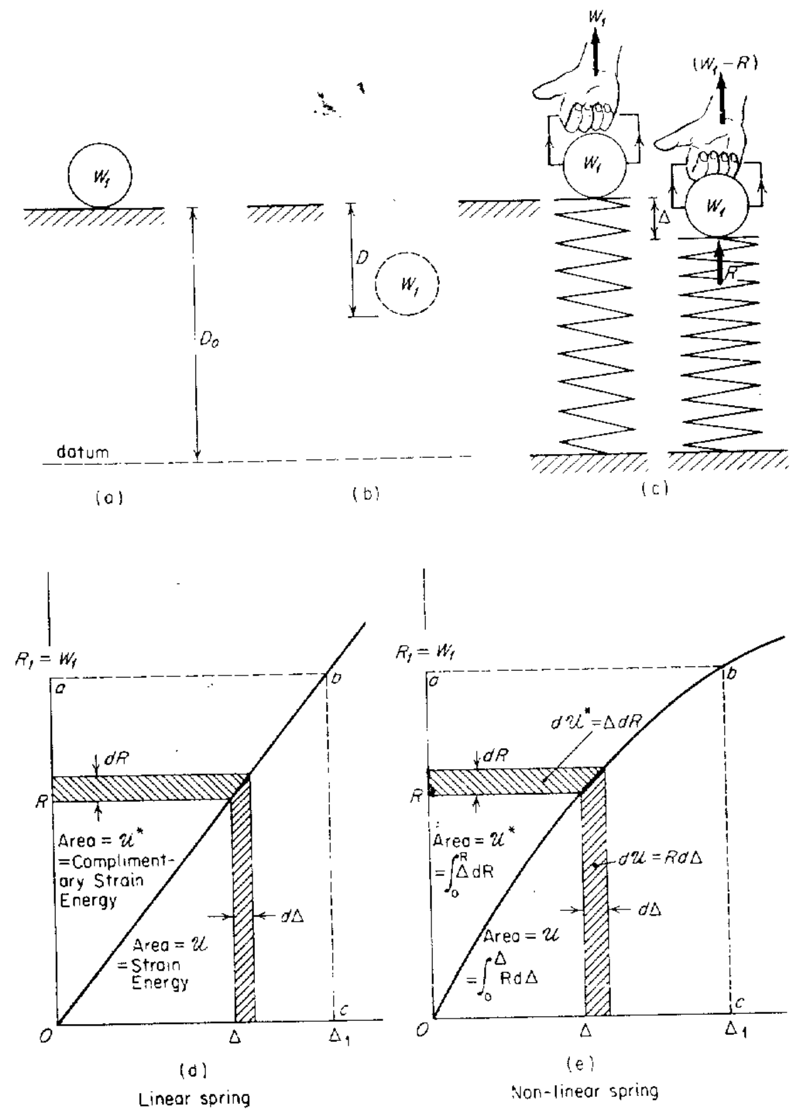

Figure 11.1 Work and energy relationships

In this case, the area Oab represents the work done by the genie in partially holding back the weight so that the net load on the spring gradually increases from zero to the final value W_1.

In the parlance of structural mechanics, the work done by the genie and therefore lost to the system is called the **complementary work** done by the net load. Likewise, interpreting Fig. 11.1d and e as they are actually drawn to represent the

force-deflection behavior of the spring, the areas Obc above the curves are said to represent the **complementary strain energy** of the spring.

11.3 Principle of Stationary Total Potential Energy. There is a general principle of rigid-body mechanics that can be represented by Fig. 11.2, in which identical balls are shown resting on three different types of surfaces representing the cases of stable, neutral, and unstable equilibrium, respectively. In each case, a small horizontal displacement of the ball would not be accompanied by any essential vertical displacement. Therefore, for such small horizontal displacements of the balls from their equilibrium positions shown, there is no change in the potential energies associated with the weights W_1. Reasoning from situations like that represented in Fig. 11.2a, it can be stated that the potential energy of a system has a stationary value when the system is in equilibrium and this value is a minimum when the equilibrium is stable.

These considerations suggest the validity of the principle of stationary total potential energy for deformable structural systems. For such a system, the total potential energy of the system consists in part of the potential of the active applied loads $\mathcal{V}$, the remainder consisting of the strain energy stored in the structure $\mathcal{U}$; both these parts can be expressed mathematically in terms of the relevant independent displacements of the system. If the system is in stable equilibrium under the loads, the total potential energy of the system must be at a minimum. Therefore, for small displacements of the system from the equilibrium position, there must be no significant changes in the total potential energy $\mathcal{U} + \mathcal{V}$, or

$$\delta_D(\mathcal{U} + \mathcal{V}) = 0 \tag{11.1}$$

where the subscript D has been added to the variation symbol δ to emphasize that only deformations and displacements are to be varied. When the following minimizing conditions are expressed mathematically, n equations are obtained, from which the n displacements Δ can be computed, thereby defining the deflected position of the system in its equilibrium position under the active applied loads:

$$\frac{\partial}{\partial \Delta_1}(\mathcal{U} + \mathcal{V}) = 0, \quad \frac{\partial}{\partial \Delta_2}(\mathcal{U} + \mathcal{V}) = 0, \quad \ldots, \frac{\partial}{\partial \Delta_n}(\mathcal{U} + \mathcal{V}) = 0 \qquad (11.2)^1$$

In the setup of these equations, the strain energy of the system can be evaluated from expressions such as those developed in Art. 8.15. The potential of any load P_j can be expressed as $\mathcal{V}_{oj} - P_j \Delta_j$, and such quantities may be summed for all loads of the system to obtain the potential of the entire load system:

$$\mathcal{V} = \sum^{n} (\mathcal{V}_{oj} - P_j \Delta_j) \tag{11.3}$$

in which $\mathcal{V}_{oj}$ is a constant that represents the potential (with respect to some convenient datum) of the load P_j in its original position on the unloaded, and hence undeformed, system. With respect to the process of applying Eq. (11.2), note that

[1] Basically these equations represent force-equilibrium conditions in the directions of the displacements Δ.

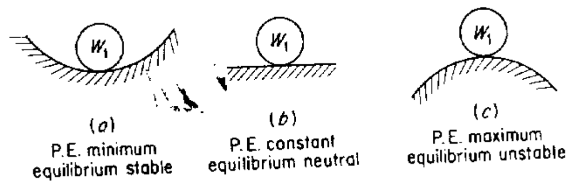

Figure 11.2 Stationary Potential Energy

the value of $\mathscr{V}_{oj}$ need not in fact be computed for each load since these values of $\mathscr{V}_{oj}$, being constants, contribute nothing when partially differentiated in turn with respect to each of the deflections, Δ.

Although we shall not do so formally in this presentation, the principle of total potential energy we have been considering can be rigorously derived by an application of the theorem of virtual deformations, discussed in Art. 11.4. The total potential energy principle has been expressed mathemetically by Eqs. (11.1) and (11.2), and may be stated in words as follows:

Principle of stationary total potential energy:

Of all geometrically compatible deformation states of a structural system that also satisfy the deflection boundary conditions, those that also satisfy the force equilibrium requirements give stationary and minimum values to the total potential energy.

It should be noted that this principle is valid theoretically for all structural systems, linear or otherwise, and that even large deformations causing significant changes in geometry can, in principle, be handled. Admittedly, however, cases involving large deflections and/or nonlinear behavior result in very cumbersome solutions which may be impractical to complete.

It is also particularly important to recognize that the principle of stationary total potential energy is not based in any way on the concept of the conservation of energy. If it were, the principle would have to account not only for the various forms of potential energy possessed by the system but also for the kinetic energy involved and any losses or diversions of energy that might occur. The principle of stationary total potential energy is simply a very useful relationship regarding all the potential energy involved in the system. The various terms of the relationship can often be evaluated without much difficulty, and the solution of the resulting equations leads to useful results.

11.4 Virtual-Work Theorems. Several virtual-work relationships have already been discussed. Bernoulli's principle of virtual work for rigid bodies was developed in Art. 8.3. The principle of virtual work for deformable bodies was developed in Art. 8.4 and converted into a convenient form for use in deflection computations in Art. 8.5. This latter principle of virtual work is restated here for convenient reference.

Principle of virtual work for deformable bodies:

If a deformable body is in equilibrium under a virtual Q-force system and remains in equilibrium while the body is subjected to a small and compatible deformation, the external virtual work done by the external Q-forces acting on the body is equal to the internal virtual work of deformation done by the internal Q stresses.

In order to visualize this statement mathematically, it is convenient to think of this principle in the form applicable to frameworks,

$$\overbrace{\sum Q\delta = \sum F_Q}^{\text{virtual-force system}}\,\Delta L \qquad \text{(11.4)}$$

deformations caused by actual forces

Two important corollaries can be reasoned from this principle of virtual work. The first may be stated as follows:

Theorem of virtual forces for deformable bodies:

If, during any statically possible virtual variation from the equilibrium state of the virtual Q-force system, the external virtual work done by the virtual variation of the external Q forces is equal to the internal virtual work done by the virtual variation of the internal Q stresses, then the unchanging displacements and strains of the deformation system to which the deformable body is being subjected are internally compatible and externally consistent with the support constraints imposed on the body.

Mathematically this theorem may be expressed as follows:

$$\overbrace{\sum \delta_F(Q)\,\delta = \sum \delta_F(F_Q)}^{\text{variation of virtual-force system}}\,\Delta L \qquad \text{(11.5)}$$

compatible deformation

where the subscript F has been added to the variation symbol δ to emphasize that only the external Q forces and the internal Q stresses are to be varied. The second corollary may be stated as follows:

Theorem of virtual deformations for deformable bodies:

If, during any consistent virtual variation of a compatible deformation system to which a deformable body is subjected, the external virtual work done by the external Q forces during the virtual variations of the deformations is equal to the internal virtual work done by the internal Q-stresses during the variation, then the external Q forces, which have remained constant during the variation of the deformations, are in a state of equilibrium with the internal Q stresses.

Mathematically this second theorem may be expressed as follows:

virtual-force system in equilibrium

$$\sum Q\,\delta_D(\delta) = \sum F_Q\,\delta_D(\Delta L) \tag{11.6}$$

variation of deformation system

where the subscript D has been added to the variation symbol $\boldsymbol{\delta}$ to emphasize that only deformations and displacements are to be varied.

It should be noted that the principle of virtual work and its two corollaries are valid for any type of deformation no matter what its cause and for linear or nonlinear material. The only limitation is that the deformations should not be so large that the equilibrium conditions of the structure need to be altered to include the effects of these large displacements.

11.5 Principle of Stationary Total Complementary Potential Energy. Like potential energy, the complementary potential energy of a system can often be readily evaluated. Also, as in the case of potential energy, there is a useful relationship involving the complementary potential energy of a structural system. This complementary energy relationship is also *not* based on the concept of the conservation of energy. The relationship may be stated as follows:

Principle of stationary total complementary potential energy:

Of all force states satisfying the equilibrium equations and the force boundary conditions, those that also satisfy the geometric compatibility requirements give stationary and minimum values to the total complementary potential energy.

Mathematically this relationship involving π^*, the total complementary potential energy, which is equal to the sum of the complementary strain energy $\mathscr{U}^*$ and the complementary potential energy of the active applied loads $\mathscr{V}^*$, may be written

$$\delta_F(\pi^*) = \delta_F(\mathscr{U}^* + \mathscr{V}^*) = 0 \tag{11.7}$$

where the subscript F has been added to the variation symbol $\boldsymbol{\delta}$ to emphasize that only the external forces and internal stresses are to be varied. Basically this relationship leads to statements of the geometric compatibility conditions.

The complementary strain energy $\mathscr{U}^*$ can be evaluated in a manner similar to the evaluation of the strain energy $\mathscr{U}$. Whereas $\bar{\mathscr{U}}$, the density, i.e., the strain energy per unit volume, of the strain energy $\mathscr{U}$, is computed from the expression

$$\bar{\mathscr{U}} = \int \sigma\, d\varepsilon \tag{11.8}$$

the density $\bar{\mathscr{U}}^*$ of the complementary strain energy $\mathscr{U}^*$ is computed from the reciprocal relationship

$$\bar{\mathscr{U}}^* = \int \varepsilon\, d\sigma \tag{11.9}$$

Comparisons of these expressions with similar quantities for the spring shown on Figs. 11.1*d* and *e* reveals that the computation of $\bar{\mathscr{U}}$ is associated so to speak with the area *under* the curve and the computation of $\bar{\mathscr{U}}^*$ is associated with the area *above* the curve. Applying Eq. (11.9) to a particular structural member would lead to the value of its complementary strain energy.

In somewhat similar fashion, the value of the complementary potential energy $\mathscr{V}^*$ of an active load P can be determined. The work $\mathscr{W}$ done by such a load moving through a displacement Δ in the sense of its line of action is equal to $\int P\,d\Delta$. On the other hand, the complementary work $\mathscr{W}$ done by such a load, as the load varies is equal to $\int \Delta dP$. However, from the definition of the potential energy (or potential) of a load, it is apparent that the load *loses* potential energy in an amount equal to the work it does as it is displaced. Therefore, the variation of the potential energy $\delta_D\mathscr{V}$ as the displacement varies is equal to the *negative* of the variation of the work $\delta_D\mathscr{W}$, or

$$\delta_D\mathscr{V} = -\delta_D\mathscr{W} = -P\,d\Delta \tag{11.10}$$

But in an analogous manner, if the loads instead of the displacements are varied, the variation of the complementary potential energy $\delta_F\mathscr{V}^*$ is equal to the negative of the variation of the complementary work $\delta_F\mathscr{W}^*$, or

$$\delta_F\mathscr{V}^* = -\delta_F\mathscr{W}^* = -\Delta dP \tag{11.11}$$

Equation (11.7), the mathematical expression of the principle of stationary total complementary potential energy, follows directly from an application of the theorem of virtual forces. Referring to Eq. (11.5) and the statement of this theorem, if the Q-force system mentioned there is considered to be the actual P-force system causing deformation of the structure, then the external work done is equal to the variation of the complementary work $\delta_F\mathscr{W}^*$ and the internal virtual work is equal to the variation of the complementary strain energy $\delta_F\mathscr{U}^*$, or

$$\delta_F\mathscr{W}^* = \delta_F\mathscr{U}^*$$

From which,

$$\delta_F(\mathscr{U}^* - \mathscr{W}^*) = 0$$

Substituting in this from Eq. (11.11) leads to

$$\delta_F(\mathscr{U}^* + \mathscr{V}^*) = 0$$

which confirms the relationship expressed in Eq. (11.7).

11.6 Complementary-Strain-Energy Theorem. In 1889, ten years after Castigliano published his well-known first and second theorems, Engesser published a paper on statically indeterminate structures in which he stated what we shall call the complementary-strain-energy-theorem.

Complementary-strain-energy theorem:

In any structure the material of which is linearly or nonlinearly elastic, the first partial derivative of the complementary strain energy $\mathscr{U}^*$ with respect to any particular force P_n is equal to the displacement δ_n of the point of application of that force in the direction of its line of action.

Mathematically, this theorem may be expressed as follows:

$$\frac{\partial \mathscr{U}^*}{\partial P_n} = \delta_n \tag{11.12}$$

This theorem is closely related to Castigliano's second theorem, which, of course, involves ordinary strain energy and is restricted to linearly elastic materials.

11.7 References

For additional information on Engesser's theorem refer to Matheson's book. For additional discussion of work and energy methods in general, Matheson and the other four following references are recommended:

1. Matheson, J. A. L.: "Hyperstatic Structures," 2 nd ed. vol. I, Butterworth's Scientific Publications, London, 1971.
2. Argyris, J. H., and S. Kelsey: "Energy Theorems and Structural Analysis," Butterworth's Scientific Publications, London, 1960.
3. Przemieniecki, J. S.: "Theory of Matrix Structural Analysis," McGraw-Hill Book Company, New York, 1968.
4. Crandall, S. H.: "Engineering Analysis," McGraw-Hill Book Company, 1956.
5. Oden, J. T.: "Mechanics of Elastic Structures," McGraw-Hill Book Company, New York, 1967.

Part II

Introduction to Systematic Structural Analysis

One objective of Part II is to familiarize the reader with the inputs and outputs of automated structural-analysis procedures. The other is to explain how the descriptive data of a linear structural-analysis problem can be systematically transformed into the quantities describing the structural response.

Chapter 12 is designed to accomplish the first objective. Article 12.1 is a review of the basics of matrix algebra. Articles 12.2 to 12.6 deal with the quantitative descriptions of the geometry, the material, the deflection boundary conditions, and the force boundary conditions of structures. Then follows a discussion of the quantities describing the structural response and the methods of transforming the basic data into the description of structural response.

Chapters 13 to 16 deal with the second objective. In Chap. 13, the transformation of the description of a vector (such as position, deflection, and force vectors) from one cartesian coordinate system to another is first established. Then the relationships between kinematically equivalent deflections and between statically equivalent forces are developed.

Chapter 14 develops flexibility and stiffness relationships between the vertex forces and the vertex deflections of a structural element with two or more vertices. The relationships between the descriptions of element flexibility and element stiffness relations in more than one cartesian coordinate system are also established.

In Chap. 15, the systematic analysis by the displacement method is detailed. Economical generation of the matrices involved is discussed, and methods of solving the equations for the unknown deflection components are given.