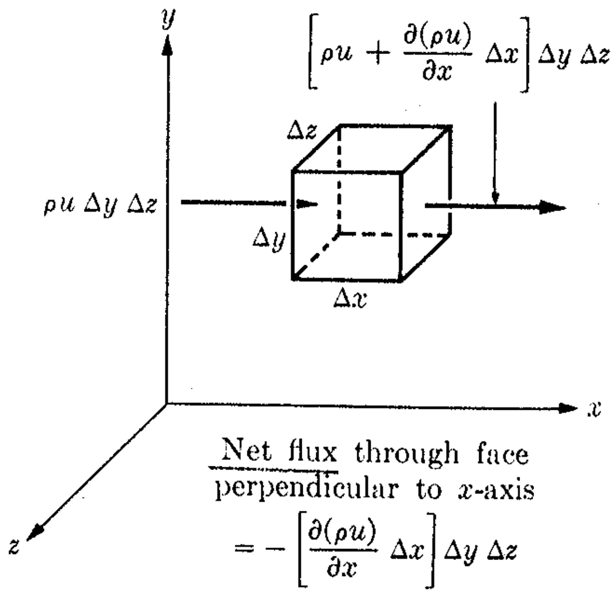
Chapter 6 Equations of Continuity and Motion

· Derivation of 3-D Eq.

$$\begin{cases} \text{conservation of mass} \rightarrow \text{Continuity Eq.} \\ \text{conservation of momentum} \rightarrow \text{Eq. of motion} \rightarrow \text{Navier-Strokes Eq.} \end{cases}$$

6.1 Continuity Equation



Consider differential (infinitesimal) control volume ($\Delta x \Delta y \Delta z$)

[Cf] Finite control volume – arbitrary CV $\rightarrow$ integral form equation

Apply principle of conservation of matter to the CV

$\rightarrow$ sum of net flux = time rate change of mass inside C.V.

1) mass flux per unit time

$$=\frac{mass}{time}=\rho\frac{vol}{time}=\rho Q=\rho u\Delta A$$

· net flux through face perpendicular to x -axis

= flux in –flux out

$$=\rho u\Delta y\Delta z-\left(\rho u+\frac{\partial(\rho u)}{\partial x}\Delta x\right)\Delta y\Delta z=-\frac{\partial(\rho u)}{\partial x}\Delta x\Delta y\Delta z$$

· net flux through face perpendicular to y -axis

$$=-\frac{\partial(\rho v)}{\partial y}\Delta x\Delta y\Delta z$$

· net flux through face perpendicular to z -axis

$$=-\frac{\partial(\rho w)}{\partial z}\Delta x\Delta y\Delta z \qquad \text{(A)}$$

2) time rate change of mass inside C.V.

$$=\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z) \qquad \text{(B)}$$

Thus, equating (A) and (B) gives

$$\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z)=-\frac{\partial(\rho u)}{\partial x}\Delta x\Delta y\Delta z-\frac{\partial(\rho v)}{\partial y}\Delta x\Delta y\Delta z-\frac{\partial(\rho w)}{\partial z}\Delta x\Delta y\Delta z$$

$$LHS = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = \rho \frac{\partial}{\partial t}(\Delta x \Delta y \Delta z) + \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

Since C.V. is fixed $\rightarrow$ $\dfrac{\partial(\Delta x \Delta y \Delta z)}{\partial t} = 0$

$$\therefore \quad LHS = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = 0$$

Cancelling terms makes

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} = 0} \qquad (6.1)$$

$\rightarrow$ Continuity Eq. for **c**ompressible fluid in unsteady flow (point form)

The 2nd term of Eq. (6.1) can be expressed as

$$\nabla \cdot (\rho \vec{q}) = \underbrace{\vec{q} \nabla \rho}_{\boxed{\text{I}}} + \underbrace{\rho \nabla \cdot \vec{q}}_{\boxed{\text{II}}}$$

(I): $\vec{q} \nabla \rho = (u\vec{i} + v\vec{j} + w\vec{k}) \left(\dfrac{\partial \rho}{\partial x}\vec{i} + \dfrac{\partial \rho}{\partial y}\vec{j} + \dfrac{\partial \rho}{\partial z}\vec{k} \right)$

gradient

$$= u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

divergence

(II): $\rho \nabla \cdot \vec{q} = \rho \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} \right)$

$$\therefore \nabla \cdot (\rho \vec{q}) = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \qquad (i)$$

Substituting (i) into Eq (6.1) yields

$$\underbrace{\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}}_{\boxed{\frac{d\rho}{dt}}} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\boxed{\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0} \qquad (6.2a)$$

$$\boxed{\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{q}) = 0} \qquad (6.2b)$$

[Re] Total derivative (total rate of density change)

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

1) For steady-state conditions

$$\rightarrow \quad \frac{\partial \rho}{\partial t} = 0$$

Then (6.1) becomes

$$\boxed{\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = \nabla \cdot (\rho \vec{q}) = 0} \qquad (6.3)$$

2) For incompressible fluid (whether or not flow is steady)

$$\rightarrow \quad \frac{d\rho}{dt} = 0$$

Then (6.2) becomes

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\nabla\cdot\vec{q}=0} \qquad (6.5)$$

[Re] Continuity equation derived using a finite CV method

Eq. (4.5a):

$$\int_{CV}\frac{\partial\rho}{\partial t}dV+\oint_{CS}\rho\vec{q}\cdot d\vec{A}=0 \qquad (4.5)$$

→ <u>volume-averaged (integrated) form</u>

● Gauss' theorem:

volume integral ↔ surface integral

– reduce dimensions by 1 (3D → 2D)

$$\int_{v}\left(\nabla\cdot\vec{X}\right)dV=\int_{A}\vec{X}\cdot d\vec{A}$$

Thus,

$$\oint_{CS}\rho\vec{q}\cdot d\vec{A}=\int_{CV}\nabla\cdot(\rho\vec{q})dV$$

Eq. (4.5) becomes

$$\int_{CV}\frac{\partial\rho}{\partial t}dV+\int_{CV}\nabla\cdot(\rho\vec{q})dV=\int_{CV}\left(\underline{\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{q})}\right)dV=0$$

Since integrands must be equal.

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{q})=0$$

→ same as Eq. (6.1) → <u>point form</u>

[Cf] 1D Continuity equation in 1-D

$$\int \frac{\partial \rho}{\partial t} dA + \int \frac{\partial \rho u}{\partial x} dA = 0$$

$$\frac{\partial}{\partial t}\int \rho dA + \frac{\partial}{\partial x}\int \rho u dA = 0$$

For incompressible fluid flow

$$\rho \frac{\partial}{\partial t}\underbrace{\int dA}_{A} + \rho \frac{\partial}{\partial x}\underbrace{\int u dA}_{VA} = 0$$

where V = cross-sectional average velocity

$$\therefore \frac{\partial A}{\partial t} + \frac{\partial VA}{\partial x} = 0$$

Consider lateral inflow/outflow

$$\frac{\partial A}{\partial t} + \frac{\partial VA}{\partial x} = \int_{\sigma} q d\sigma$$

where q = flow through σ

For steady flow; $\frac{\partial A}{\partial t} = 0$

$$\therefore \quad \frac{\partial VA}{\partial x} = 0$$

$$VA = const. = Q$$

[Re] Continuity equation in polar (cylindrical) coordinates

u, r - radial

v, θ - azimuthal

w, z - axial

For compressible fluid of unsteady flow

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho u r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v)}{\partial \theta} + \frac{\partial(\rho w)}{\partial z} = 0}$$

For incompressible fluid

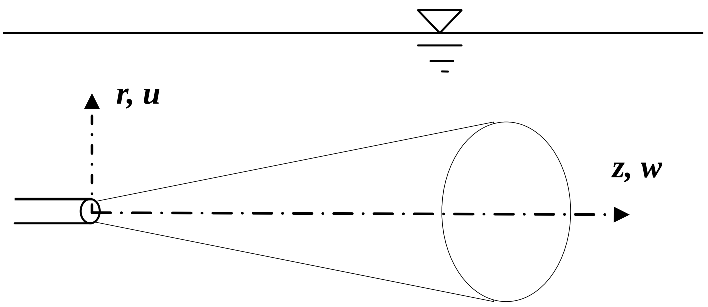
$$\frac{1}{r}\frac{\partial(ur)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

For incompressible fluid and flow of axial symmetry

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial \rho}{\partial r} = \frac{\partial \rho}{\partial \theta} = \frac{\partial \rho}{\partial z} = 0, \qquad \frac{\partial(\rho v)}{\partial \theta} = 0$$

$$\therefore \frac{1}{r}\frac{\partial(ur)}{\partial r} + \frac{\partial w}{\partial z} = 0 \quad \rightarrow \text{2-D boundary layer flow}$$

Example: submerged jet



[Re] Green's Theorem

1) Transformation of double integrals into line integrals

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C \left(F_1 dx + F_2 dy \right)$$

$$\iint_R \left(curl\ \vec{F} \right) \cdot \vec{k} dxdy = \oint_C \vec{F} \cdot d\vec{r}$$

$$\vec{F} = F_1 \vec{i} + F_2 \vec{j}$$

2) 1st form of Green's theorem

$$\iiint_T \left(f \nabla^2 g + grad\ f \cdot grad\ g \right) dV = \iint_S f \frac{\partial g}{\partial n} dA$$

3) 2nd form of Green's theorem

$$\iiint_T \left(f \nabla^2 g + g \nabla^2 f \right) dV = \iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial x} \right) dA$$

[Re] Divergence theorem of Gauss

→ transformation between volume integrals and surface integrals

$$\iiint_T div \vec{F}\ dV = \iint_S \vec{F} \cdot \vec{n}\ dA$$

Where n = outer unit normal vector of S

$$\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$$

$$\vec{n} = \cos\alpha \vec{i} + \cos\beta \vec{j} + \cos\gamma \vec{k}$$

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz$$

$$= \iint_S \left(F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma \right) dA$$

By the way

$$\iint_S \vec{F} \cdot \vec{n} dA = \iint_S \left(F_1 dydz + F_2 dzdx + F_3 dxdy \right)$$

$$\therefore \quad \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz$$

$$= \iint_S \left(F_1 dydz + F_2 dzdx + F_3 dxdy \right)$$

6.2 Stream Function in 2-D, Incompressible Flows

2-D incompressible continuity eq. is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (6.7)$$

Now, define stream function $\psi(x, y)$ as

$$u=-\frac{\partial \psi}{\partial y}$$

$$\boxed{\begin{aligned}\psi &= \int -u dy \\ \psi &= \int v dx\end{aligned}}$$

$$v=\frac{\partial \psi}{\partial x} \qquad (6.8)$$

Then LHS of Eq. (6.7) becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=-\cancel{\frac{\partial^2 \psi}{\partial x \partial y}}+\cancel{\frac{\partial^2 \psi}{\partial x \partial y}}=0$$

→ Thus, continuity equation is satisfied.

1) Apply stream function to the equation for a stream line in 2-D flow

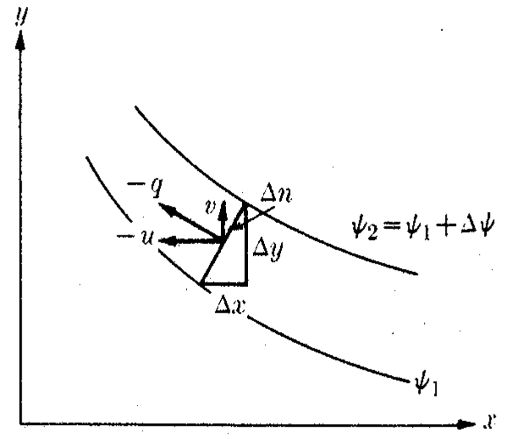
Eq. (2.10): $vdx-udy=0$ (6.11)

Substitute (6.8) into (6.11)

$$\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy=d\psi=0 \qquad (6.12)$$

ψ = constant (6.13)

→ The stream function is constant along a streamline.



2) Apply stream function to the law of conservation of mass

$$-qdn = -udy + vdx \tag{6.14}$$

Substitute (6.8) into (6.14)

$$-qdn = \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi \tag{6.15}$$

→ Change in ψ $(d\psi)$ between adjacent streamlines is equal to the volume rate of flow per unit width.

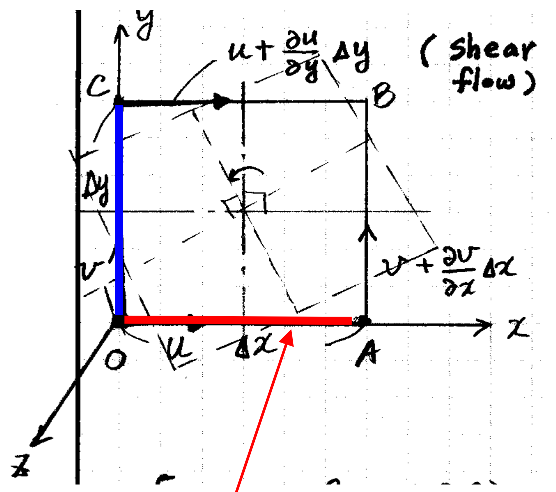
3) Stream function in cylindrical coordinates

$$v_r = -\frac{\partial \psi}{r\partial \theta} \quad \text{radial}$$

$$v_\theta = \frac{\partial \psi}{\partial r} \quad \text{azimuthal}$$

6.3 Rotational and Irrotational Motion

6.3.1 Rotation and vorticity



Assume the rate of rotation of fluid element Δx *and* Δy about z-axis is positive when it rotates counterclockwise.

- time rate of rotation of Δx-face about z-axis

$$= \frac{1}{\Delta t}\frac{\left[\left\{v+\left(\frac{\partial v}{\partial x}\right)\Delta x\right\}-v\right]\Delta t}{\Delta x} = \frac{\partial v}{\partial x}$$

- time rate of rotation of Δy-face about z-axis

$$= -\frac{1}{\Delta t}\frac{\left[u+\left(\frac{\partial u}{\partial y}\Delta y\right)-u\right]\Delta t}{\Delta y} = -\frac{\partial u}{\partial y}$$

net rate of rotation = average of sum of rotation of Δx-and Δy-face

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

Doing the same way for x-, and y-axis

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad (6.16a)$$

1) Rotation

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

$$= \frac{1}{2}\left(\nabla \times \vec{q}\right) = \frac{1}{2} curl\ \vec{q} \qquad (6.16b)$$

Magnitude:

$$|\vec{\omega}| = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$

a) Ideal fluid $\rightarrow$ irrotational flow

$$\nabla \times \vec{q} = 0$$

$$\omega_x = \omega_y = \omega_z = 0$$

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \qquad (6.17)$$

b) Viscous fluid $\rightarrow$ rotational flow

$$\nabla \times \vec{q} \neq 0$$

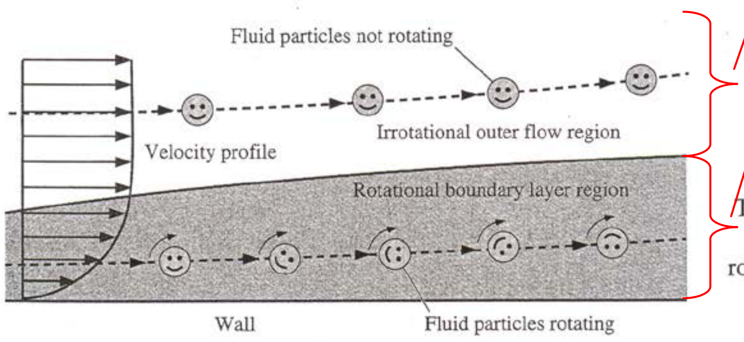


FIGURE 4–24
The difference between rotational and irrotational flow: fluid elements in a rotational region of the flow rotate, but those in an irrotational region of the flow do not.

2) Vorticity

$$\vec{\zeta} = curl\ \vec{q} = \nabla \times \vec{q} = 2\vec{\omega}$$

[Re] Rotation in cylindrical coordinates

$$\omega_r = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)$$

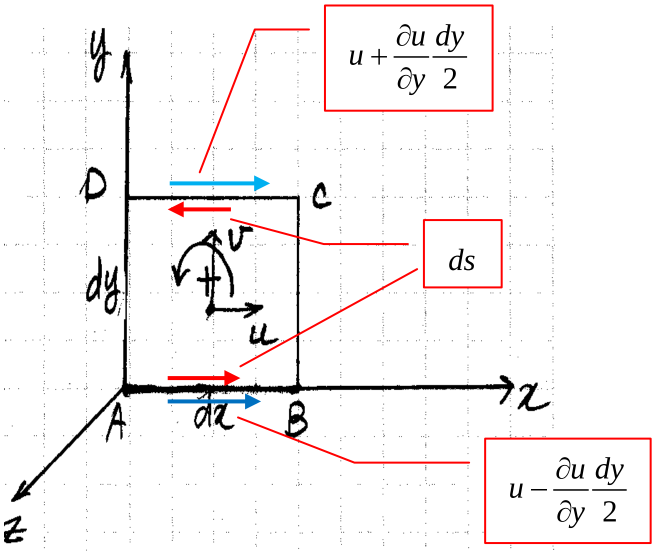
$$\omega_\theta = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} - \frac{\partial vz}{\partial r}\right)$$

$$\omega_z = \frac{1}{2}\left(-\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\right)$$

6.3.2 Circulation

Γ = line integral of the tangential velocity component about any closed contour S

$$\Gamma = \oint \vec{q} \cdot d\vec{s} \tag{6.19}$$



– take line integral from A to B, C, D, A ~ infinitesimal CV

$$d\Gamma \cong \left[u - \frac{\partial u}{\partial y}\frac{dy}{2}\right]dx + \left[v + \frac{\partial v}{\partial x}\frac{dx}{2}\right]dy - \left[u + \frac{\partial u}{\partial y}\frac{dy}{2}\right]dx - \left[v - \frac{\partial v}{\partial x}\frac{dx}{2}\right]dy$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy$$

$$d\Gamma \cong \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy$$

$$\Gamma = \iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dA = \iint_A 2\omega_z dA = \iint_A \left(\nabla \times \vec{q}\right)_z dA \tag{6.20}$$

For irrotational flow,

circulation $\Gamma = 0$ (if there is no singularity vorticity source).

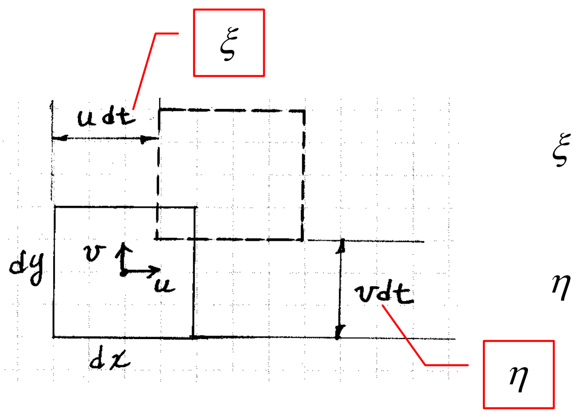
[Re] Fluid motion and deformation of fluid element

$$\text{Motion}\begin{cases}\text{translation}\\ \text{rotation}\end{cases}$$

$$\text{Deformation}\begin{cases}\text{linear deformation}\\ \text{angular deformation}\end{cases}$$

(1) Motion

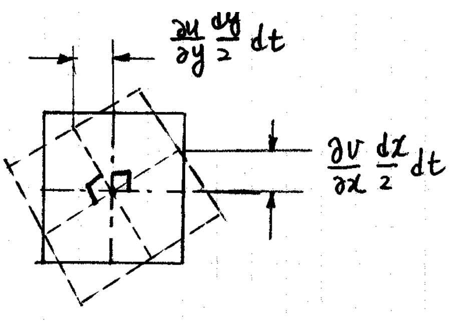
1) Translation: ξ, η



$$\xi = u\ dt, \quad u = \frac{d\xi}{dt}$$

$$\eta = v\ dt, \quad v = \frac{d\eta}{dt}$$

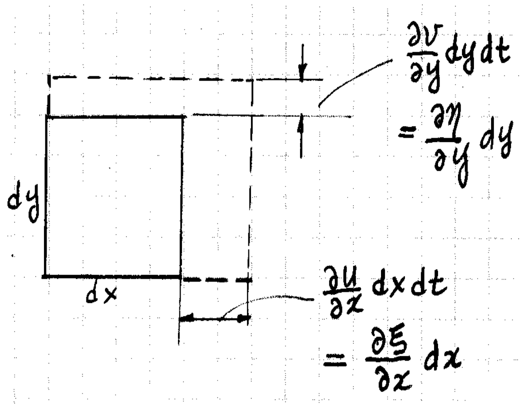
2) Rotation ← Shear flow



$$\omega_x = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

(2) Deformation

1) Linear deformation – normal strain



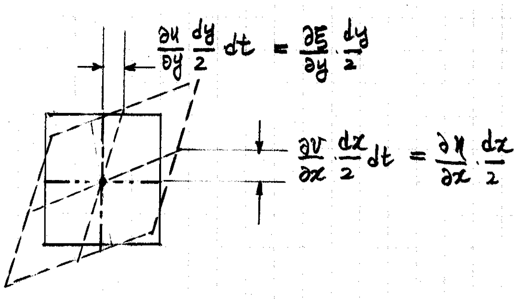
$$\varepsilon_x = \frac{\partial \xi}{\partial x}$$

$$\varepsilon_y = \frac{\partial \eta}{\partial y}$$

i) For compressible fluid, changes in temperature or pressure cause change in volume.

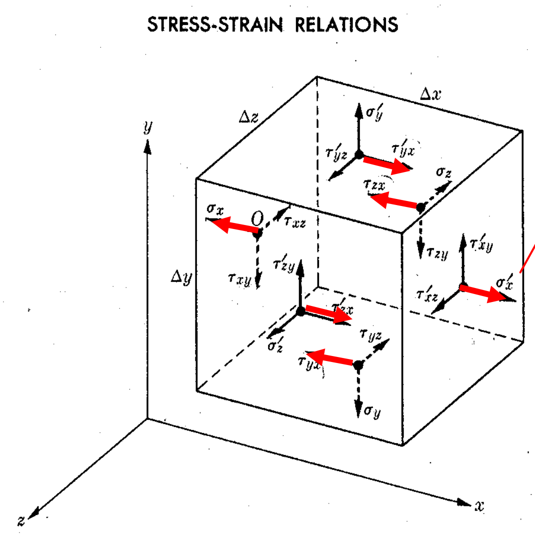
ii) For incompressible fluid, if length in 2-D increases, then length in another 1-D decreases in order to make total volume unchanged.

2) Angular deformation– shear strain



$$\gamma_{xy} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial x}$$

6.4 Equations of Motion



$$\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x$$

• Apply Newton's 2nd law of motion

$$\sum \vec{F} = m\vec{a} \qquad \text{(A)}$$

$$\Delta F_x = \Delta m a_x$$

• External forces = surface force + body force

· Surface force:

~ normal force + tangential force

· Body forces:

~ due to gravitational or electromagnetic fields, no contact

~ act at the centroid of the element → centroidal force

Consider only gravitational force

$$\vec{g} = \vec{i}g_x + \vec{j}g_y + \vec{k}g_z$$

LHS of (A):

$$\Delta F_x = (\rho \Delta x \Delta y \Delta z) g_x$$ body force (B)

$$-\sigma_x \Delta y \Delta z + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \Delta x \right) \Delta y \Delta z$$ normal force

$$-\tau_{yx} \Delta x \Delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \Delta y \right) \Delta x \Delta z$$ tangential force

$$-\tau_{zx} \Delta x \Delta y + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \Delta z \right) \Delta x \Delta y$$ tangential force

Divide (B) by volume of element

$$\frac{\Delta F_x}{\Delta x \Delta y \Delta z} = \rho g_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \quad \text{(C)}$$

RHS of (A):

$$\frac{\Delta m a_x}{\Delta x \Delta y \Delta z} = \rho a_x \quad \text{(D)}$$

Combine (C) and (D)

$$\boxed{\begin{aligned}
&\rho g_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho a_x \\
&\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho a_y \\
&\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \rho a_z
\end{aligned}} \quad (6.21)$$

6.4.1 Navier-Stokes equations

– Eq (6.21) ~ general equation of motion

– For Newtonian fluids (with single viscosity coeff.), use stress-strain relation given in (5.29) and (5.30)

→ Navier-Stokes equations

Eq. (5.29):

$$\sigma_x = \underbrace{-p}_{\text{pressure}} + \underbrace{2\mu\frac{\partial u}{\partial x} - \left(\frac{2}{3}\right)\mu(\nabla\cdot\vec{q})}_{\text{normal stress due to fluid deformation and viscosity}}$$

$$\sigma_y = -p + 2\mu\frac{\partial v}{\partial y} - \left(\frac{2}{3}\right)\mu(\nabla\cdot\vec{q})$$

$$\sigma_z = -p + 2\mu\frac{\partial w}{\partial z} - \left(\frac{2}{3}\right)\mu(\nabla\cdot\vec{q})$$

Eq. (5.30):

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Substitute Eqs. (5.29) & (5.30) into (6.21)

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu(\nabla\cdot\vec{q})\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] = \rho a_x$$

Assume constant viscosity (neglect effect of pressure and temperature on viscosity variation)

$$\rho g_x - \frac{\partial p}{\partial x} + \mu \frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla \cdot \vec{q}) \right] + \mu \frac{\partial}{\partial y}\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] + \mu \frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right] = \rho a_x$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}$$

Expand and simplify

$$L.H.S = \rho g_x - \frac{\partial p}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \right) + \mu \left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z} \right)$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] + \frac{1}{3}\mu \left(\underline{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}} \right)$$

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)}$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] + \frac{1}{3}\mu \frac{\partial}{\partial x}(\nabla \cdot \vec{q})$$

$$\rho g_x - \frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] + \frac{1}{3}\mu \frac{\partial}{\partial x}(\nabla \cdot \vec{q}) = \rho a_x$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] + \frac{1}{3}\mu \frac{\partial}{\partial y}(\nabla \cdot \vec{q}) = \rho a_y$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] + \frac{1}{3}\mu \frac{\partial}{\partial z}(\nabla \cdot \vec{q}) = \rho a_z \qquad (6.24)$$

→ Navier-Stokes equation for compressible fluids with constant viscosity

♦ Vector form

$$\boxed{\rho\vec{g} - \nabla p + \mu\nabla^2\vec{q} + \frac{\mu}{3}\nabla(\nabla\cdot\vec{q}) = \rho\frac{\partial\vec{q}}{\partial t} + \rho(\vec{q}\cdot\nabla)\vec{q}}$$

where $\vec{a} = \dfrac{d\vec{q}}{dt} = \dfrac{\partial\vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}$ --- Eq. (2.5)

1) For inviscid (ideal) fluid flow, $(\mu = 0)$ → viscous forces are neglected.

$$\rho\vec{g} - \nabla p = \rho\frac{\partial\vec{q}}{\partial t} + \rho(\vec{q}\cdot\nabla)\vec{q}$$

→ Euler equations for ideal fluid

2) For incompressible fluids, $\nabla\cdot\vec{q} = 0$ (Continuity Eq.)

$$\boxed{\rho\vec{g} - \nabla p + \mu\nabla^2\vec{q} = \rho\frac{\partial\vec{q}}{\partial t} + \rho(\vec{q}\cdot\nabla)\vec{q}} \qquad (6.25)$$

Define acceleration due to gravity as

$$\left.\begin{aligned} g_x &= -g\frac{\partial h}{\partial x} \\ g_y &= -g\frac{\partial h}{\partial y} \\ g_z &= -g\frac{\partial h}{\partial z} \end{aligned}\right\} \quad \vec{g} = -g\nabla h$$

where h = vertical direction measured positive upward

For Cartesian axes oriented so that h and z coincide

$$g_x = g_y = 0 \quad , \quad \frac{\partial h}{\partial z} = 1$$

$$g_z = -g$$

→ minus sign indicates that acceleration due to gravity is in the negative h direction

Then, N-S equation for incompressible fluids and isothermal flows are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -g\frac{\partial h}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -g\frac{\partial h}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right]$$

$$\frac{\partial w}{\partial t} + \underline{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}} = -g\frac{\partial h}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} + \underline{\frac{\mu}{\rho}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]} \qquad (6.28)$$

Local acceleration

Convective acceleration

Body force per mass

Pressure force per mass

Viscosity force per mass

Eq. (6.28): unknowns - u, v, w, p

→ We need one more equation to obtain a solution when the boundary conditions are specified.

→ Eq. of continuity for incompressible fluid

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

♦ Boundary conditions

1) kinematic BC: velocity normal to any rigid boundary (wall) equal the boundary velocity (velocity = 0 for stationary boundary)

2) physical BC: no slip condition (continuum stick to a rigid boundary)

→ tangential velocity relative to the wall vanish at the wall surface

♦ General solutions for Navier-Stocks equations are not available because of the nonlinear, 2nd-order nature of the partial differential equations.

→ Only particular solutions may be obtained by simplifications.

→ Numerical solutions are usually sought.

♦ Navier-Stocks equations in cylindrical coordinates for constant density and viscosity

r - component:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta{}^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$

$$=\rho g_r-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right\}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]$$

θ - component:

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}-\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$=\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right\}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]$$

z - component:

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$

$$=\rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]$$

Continuity eq. for incompressible fluid

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)+\frac{\partial}{\partial z}(v_z)=0$$

Normal & shear stresses for constant density and viscosity

$$\sigma_r=-p+2\mu\frac{\partial v_r}{\partial r}$$

$$\sigma_\theta=-p+2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)$$

$$\sigma_z=-p+2\mu\frac{\partial v_z}{\partial z}$$

$$\tau_{r\theta}=\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z}=\mu\left[\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]$$

$$\tau_{zr}=\mu\left[\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right]$$

6.5 Examples of Laminar Motion

- N-S equations are important in viscous flow problems.

♦ Laminar motion

~ orderly state of flow in which macroscopic fluid particles move in layers

~ viscosity effect is dominant

♦ Laminar flow through a tube (pipe) of constant diameter

~ instantaneous velocity at any point is always unidirectional (along the axis of the tube)

~ no-slip condition @ boundary wall

~ apply concept of the Newtonian viscosity

~ velocity gradient gives rise to viscous force within the fluid

~ low Re

[Re] Reynolds number = inertial force / viscous force = destabilizing force / stabilizing force

♦ Viscous force

~ dissipative

~ have a stabilizing or damping effect on the motion

~ use Reynolds number

[Cf] Turbulent flow

~ unstable flow

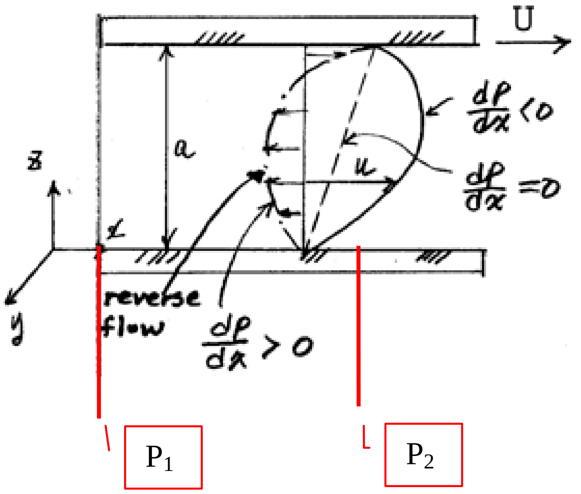
~ instantaneous velocity is no longer unidirectional

~ destabilizing force > stabilizing force

~ high Re

6.5.1 Laminar flow between parallel plates

Consider the two-dimensional, steady, laminar flow between parallel plates in which either of two surfaces is moving at constant velocity and there is also an external pressure gradient.



♦ Assumptions:

2-D flow $\rightarrow$ $v = 0$; $\dfrac{\partial(\ )}{\partial y} = 0$

steady flow $\rightarrow$ $\dfrac{\partial(\ )}{\partial t} = 0$

parallel flow $\rightarrow$ $w = 0$; $\dfrac{\partial w}{\partial(\ )} = 0$

z-axis coincides with h $\rightarrow$ $\dfrac{\partial h}{\partial x} = \dfrac{\partial h}{\partial y} = 0$; $\dfrac{\partial h}{\partial z} = 1$

♦ External pressure gradient

$p_1 > p_2$

i) $\dfrac{\partial p}{\partial x} < 0$ $\rightarrow$ pressure gradient assists the viscously induced motion to overcome the shear force at the lower surface

ii) $\dfrac{\partial p}{\partial x} > 0$ $\rightarrow$ pressure gradient resists the motion which is induced by the motion of the upper surface

$p_1 < p_2$

Continuity eq. for two-dimensional, parallel flow:

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial w}{\partial z}} = 0$$

$$\rightarrow \begin{cases} \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u = f(z) \text{ only} \end{cases}$$

Steady flow

Continuity eq. for incompressible fluid

N-S Eq.:

2D flow

$$x-dir.:\ \cancel{\frac{\partial u}{\partial t}} + u\cancel{\frac{\partial u}{\partial x}} + v\cancel{\frac{\partial u}{\partial y}} + w\cancel{\frac{\partial u}{\partial z}}$$

parallel flow

$$= -g\cancel{\frac{\partial h}{\partial x}} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left[\cancel{\frac{\partial^2 u}{\partial x^2}} + \cancel{\frac{\partial^2 u}{\partial y^2}} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\therefore\quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial z^2}\right) \qquad (6.31a)$$

$$z-dir.:\ \cancel{\frac{\partial w}{\partial t}} + u\cancel{\frac{\partial w}{\partial x}} + v\cancel{\frac{\partial w}{\partial y}} + w\cancel{\frac{\partial w}{\partial z}}$$

$$= -g\frac{\partial h}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left[\cancel{\frac{\partial^2 w}{\partial x^2}} + \cancel{\frac{\partial^2 w}{\partial y^2}} + \cancel{\frac{\partial^2 w}{\partial z^2}}\right]$$

$$\therefore\quad 0 = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (6.31b)$$

(6.31b): $\dfrac{\partial p}{\partial z} = -\rho g = -\gamma$

$$\therefore \quad p = -\gamma z + f(x) \tag{6.32}$$

→ hydrostatic pressure distribution normal to flow

→ For any orientation of z-axis. in case of a parallel flow, pressure is distributed hydrostatically in a direction normal to the flow.

(6.31a): $\dfrac{\partial p}{\partial x} \rightarrow \dfrac{dp}{dx}$ ~ independent of z

$$\therefore \quad \dfrac{dp}{dx} = \mu \dfrac{\partial^2 u}{\partial z^2} \tag{A}$$

Pressure drop

Energy loss due to viscosity

Integrate (A) twice w.r.t. z

$$\iint \frac{dp}{dx} dzdz = \iint \mu \frac{\partial^2 u}{\partial z^2} dzdz$$

$$\int \frac{dp}{dx} z\, dz = \int \mu \frac{\partial u}{\partial z} dz + \int C_1\, dz$$

$$\frac{dp}{dx}\frac{z^2}{2} = \mu u + C_1 z + C_2 \tag{6.33}$$

Use the boundary conditions,

i) $z = 0, \quad u = 0 \;\rightarrow\; \dfrac{dp}{dx} \times 0 = \mu(0) + C_2 \qquad \therefore\; C_2 = 0$

ii) $z = h, \quad u = U \;\rightarrow\; \dfrac{dp}{dx}\dfrac{a^2}{2} = \mu U + C_1 a$

$$\therefore \; C_1 = \frac{1}{a}\left(\frac{dp}{dx}\frac{a^2}{2} - \mu U\right)$$

∴ (6.33) becomes

$$\frac{dp}{dx}\frac{z^2}{2} = \mu u + \frac{1}{a}\left(\frac{dp}{dx}\frac{a^2}{2} - \mu U\right) z$$

$$\therefore \quad \mu u = \frac{z}{a}\mu U - \frac{dp}{dx}\left(\frac{az}{2} - \frac{z^2}{2}\right)$$

$$\boxed{u(z) = u = \frac{U}{a} z - \frac{a}{2\mu}\frac{dp}{dx}\left(1 - \frac{z}{a}\right) z} \qquad (6.34)$$

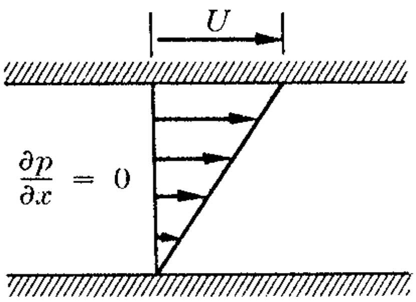
Velocity driven

Pressure driven

i) If $\frac{dp}{dx} = 0 \rightarrow$ Couette flow (plane Couette flow)

$$u = \frac{U}{a} z$$

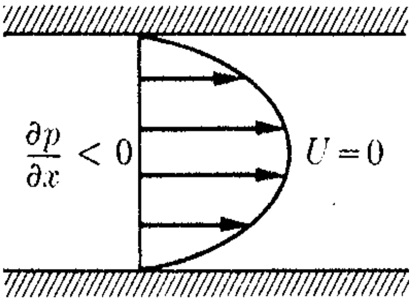
→ driving mechanism = U (velocity)



ii) If $U = 0 \rightarrow$ 2-D Poiseuille flow (plane Poiseuille flow)

$$u = \frac{1}{2\mu}\frac{dp}{dx}(z - a) z \quad \sim \text{parabolic}$$

→ driving mechanism = external pressure gradient, $\dfrac{dp}{dx}$



$u_{\max}$ @ $z = \dfrac{a}{2}$

$$u_{\max} = -\frac{a^2}{8\mu}\frac{dp}{dx}$$

V = average velocity

$$= \frac{Q}{A} = \frac{2}{3}u_{\max} = -\frac{a^2}{12\mu}\frac{dp}{dx}$$

[Re] detail

$$Q = \int_0^a u\,dz = \int_0^a \frac{1}{2\mu}\frac{dp}{dx}\left(z^2 - az\right)dz = -\frac{1}{12\mu}\frac{dp}{dx}a^3$$

$$A = a \times 1 \qquad \therefore\ V = \frac{Q}{A} = -\frac{a^2}{12\mu}\frac{dp}{dx} = \frac{2}{3}u_{\max}$$

6.5.2 Laminar flow in a circular tube of constant diameter

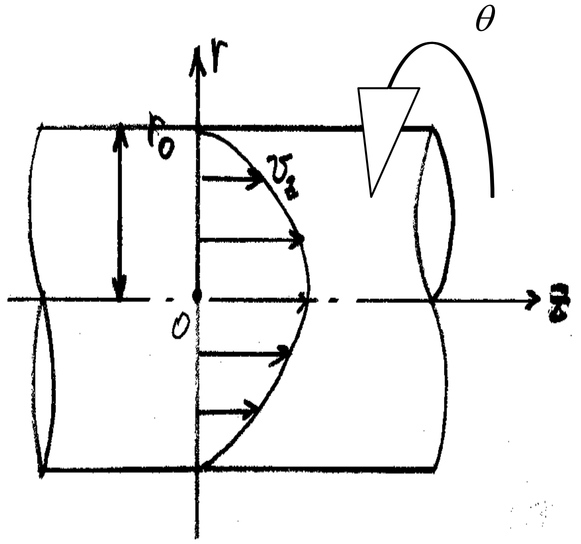
→ Hagen-Poiseuille flow

→ Poiseuille flow: steady laminar flow due to pressure drop along a tube

$$\frac{\partial p}{\partial x} < 0$$

Assumptions:

– use cylindrical coordinates



parallel flow → $v_r = 0$, $v_\theta = 0$

$v_z \neq 0$

Continuity eq. → $\dfrac{\partial v_z}{\partial z} = 0$

$\dfrac{\partial v_z}{\partial r} \neq 0$

paraboloid → $\dfrac{\partial v_z}{\partial \theta} = 0$

steady flow → $\dfrac{\partial v_z}{\partial t} = 0$

Eq. (6.29c) becomes

$$0 = \underline{-\frac{\partial p}{\partial z}} + \rho g_z + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r} \right) \qquad \text{(A)}$$

By the way,

$$-\frac{\partial p}{\partial z} + \rho g_z = -\frac{\partial}{\partial z}(p + \gamma h) = -\frac{d}{dz}(p + \gamma h)$$

$$\left[\rho g_z = -\rho g \frac{\partial h}{\partial z} \right]$$

independent of r

r – comp. Eq. →

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r$$

$$\to \frac{\partial}{\partial r}(p + \gamma r) = 0$$

Then (A) becomes

$$\frac{d}{dz}(p+\gamma h)=\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

$$\frac{1}{\mu}\frac{d}{dz}(p+\gamma h)r=\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \qquad \text{(B)}$$

Integrate (B) twice w.r.t. r

$$\frac{1}{\mu}\frac{d}{dz}(p+\gamma h)\frac{r^2}{2}=r\frac{\partial v_z}{\partial r}+C_1 \qquad \text{(C)}$$

$$\frac{1}{2\mu}\frac{d}{dz}(p+\gamma h)r=\frac{\partial v_z}{\partial r}+\frac{C_1}{r}$$

$$\frac{1}{2\mu}\frac{d}{dz}(p+\gamma h)\frac{r^2}{2}=v_z+C_1\ln r+C_2 \qquad \text{(D)}$$

Using BCs

$$r=0\,,\ v_z=v_{z_{\max}} \quad \rightarrow \quad \text{(C)}: \quad C_1=0$$

$$r=r_0\,,\ v_z=0 \quad \rightarrow \quad \text{(D)}: \quad C_2=\frac{1}{2\mu}\frac{d}{dz}(p+\gamma h)\frac{r_0^{\,2}}{2} \qquad \text{(D1)}$$

Then, substitute (D1) into (D) to obtain v_z

$$\therefore \quad v_z=\frac{1}{4\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]\left(r_0^{\,2}-r^2\right)$$

$$v_z=-\frac{d}{dz}(p+\gamma h)\frac{r_0^{\,2}}{4\mu}\left[1-\left(\frac{r}{r_0}\right)^2\right] \qquad (6.39)$$

→ equation of a paraboloid of revolution

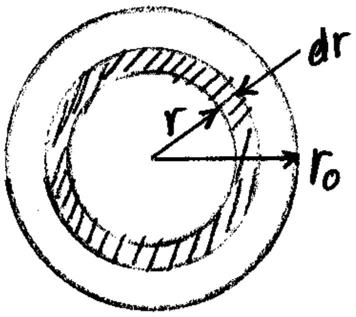
piezometric pressure

(1) maximum velocity, $v_{z_{\max}}$

$$v_{z_{\max}} \quad @ \quad r = 0$$

$$v_{z_{\max}} = -\frac{d}{dz}(p+\gamma h)\frac{r_0^{\ 2}}{4\mu}$$

(2) mean velocity, V_z



$$dQ = v_z dA$$

$$= \frac{1}{4\mu}\left[-\frac{d}{dz}(p+\gamma h)\right](r_0^{\ 2}-r^2)2\pi r dr$$

$$Q = \int_0^{r_0}\frac{1}{4\mu}\left[-\frac{d}{dz}(p+\gamma h)\right](r_0^{\ 2}-r^2)2\pi r dr$$

$$= \frac{\pi}{2\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]\left[r_0^{\ 2}\frac{r^2}{2}-\frac{r^4}{4}\right]_r^{r_0} = \frac{\pi r_0^{\ 4}}{8\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]$$

$$\therefore \quad V_z = \frac{Q}{A} = \frac{Q}{\pi r_0^{\ 2}} = \frac{r_0^{\ 2}}{8\mu}\left[-\frac{d}{dz}(p+\gamma h)\right] = \frac{v_{z_{\max}}}{2} \qquad \text{(E)}$$

[Cf] For 2 - D Poiseuille flow $V = \frac{2}{3}u_{\max}$

(3) Head loss per unit length of pipe

Total head = piezometric head + velocity head

Here, velocity head is constant.

Thus, total head = piezometric head

$$\frac{h_f}{L} \equiv \frac{1}{\gamma}\left[-\frac{d}{dz}(p+\gamma h)\right] = \frac{8\mu V_z}{\gamma r_0^{\ 2}} = \frac{32\mu V_z}{\gamma D^2} \tag{6.42}$$

(E)

where $D = 2r_0 =$ diameter

[Re] Consider Darcy-Weisbach Eq.

$$\frac{h_f}{L} = f\,\frac{1}{D}\frac{V_z^{\ 2}}{2g} \tag{F}$$

$h_f =$ head loss due to friction

$f =$ friction factor

Combine (6.42) and (F)

$$\frac{32\mu V_z}{\gamma D^2} = f\,\frac{1}{D}\frac{V_z^{\ 2}}{2g}$$

$$f = \frac{64}{V_z}\frac{\nu}{D} = \frac{64}{V_z D/\nu} = \frac{64}{\mathrm{Re}}$$ → For laminar flow

(4) Shear stress

$$\tau_{zr} = \mu\left(\cancel{\frac{\partial v_r}{\partial z}} + \frac{\partial v_z}{\partial r}\right) = \mu\frac{\partial v_z}{\partial r} \tag{G}$$

Differentiate (6.39) w.r.t. r

$$\frac{\partial v_z}{\partial r} = \frac{d}{dz}\left(p+\gamma h\right)\frac{1}{2\mu}r \tag{H}$$

Linear profile

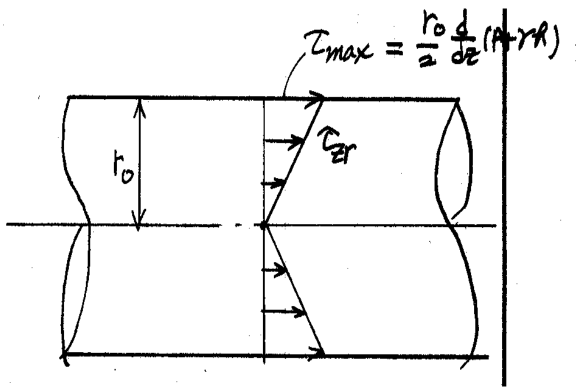
Combine (G) and (H)

$$\tau_{zr} = \frac{1}{2}\frac{d}{dz}\left(p+\gamma h\right)r \tag{6.45}$$

At center and walls

$$r = 0\,,\quad \tau_{zr} = 0$$

$$r = r_0\,,\quad \tau_{zr} = \frac{1}{2}\frac{d}{dz}\left(p+\gamma h\right)r_0 = \tau_{zr_{\max}}$$



6.6 Equations for Irrotational Motion

○ Newton's 2nd law $\rightarrow$ Momentum eq. $\rightarrow$ Eq. of motion

○ In Ch. 4, 1st law of thermodynamics $\rightarrow$ 1D Energy eq.

$\Rightarrow$ Bernoulli eq. for steady flow of an incompressible fluid <u>with zero friction (ideal fluid)</u>

○ In Ch. 6, Eq. of motion $\rightarrow$ Bernoulli eq.

Integration assuming irrotational flow

○ Irrotational flow = Potential flow

6.6.1 Velocity potential and stream function

If $\phi(x, y, z, t)$ is any <u>scalar quantity</u> having continuous first and second derivatives, then by a fundamental vector identity

$$\rightarrow curl(grad\ \phi) \equiv \underline{\nabla \times (\nabla \phi) \equiv 0} \qquad (6.46)$$

[Detail] vector identity

$$grad\ \phi = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

$$curl(grad\ \phi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} & \dfrac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial \phi^2}{\partial y \partial z} - \frac{\partial \phi^2}{\partial y \partial z}\right) + \vec{j}\left(\frac{\partial \phi^2}{\partial z \partial x} - \frac{\partial \phi^2}{\partial z \partial x}\right) + \vec{k}\left(\frac{\partial \phi^2}{\partial x \partial y} - \frac{\partial \phi^2}{\partial x \partial y}\right) \Rightarrow 0$$

By the way, for irrotational flow

Eq.(6.17) : $\nabla \times \vec{q} = 0$ (A)

Thus, from (6.46) and (A), we can say that for irrotational flow there must exist a scalar function ϕ whose gradient is equal to the velocity vector $\vec{q}$.

$$grad\ \phi = \vec{q} \qquad \text{(B)}$$

Now, let's define the positive direction of flow is the direction in which ϕ is decreasing, then

$$\boxed{\vec{q} = -grad\ \phi(x, y, z, t) = -\nabla \phi} \qquad (6.47)$$

where ϕ = **velocity potential**

$$\boxed{u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y},\ w = -\frac{\partial \phi}{\partial z}} \qquad (6.47a)$$

→ Velocity potential exists only for irrotational flows; however stream function is not subject to this restriction.

→ irrotational flow = potential flow for both compressible and incompressible fluids

(1) Continuity equation for incompressible fluid

Eq. (6.5): $\nabla \cdot \vec{q} = 0$ (C)

Substitute (6.47) into (C)

$$\therefore \nabla \cdot \left(-\nabla \phi\right) = -\nabla^2 \phi = 0 \qquad \rightarrow \text{ Laplace Eq.}$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \leftarrow \text{ Cartesian coordinates}$$

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \leftarrow \text{ Cylindrical coordinates}$$

[Detail] velocity potential in cylindrical coordinates

$$v_r = -\frac{\partial \phi}{\partial r},\ v_\theta = -\frac{\partial \phi}{r\partial \theta},\ v_z = -\frac{\partial \phi}{\partial z}$$

(2) For 2-D incompressible irrotational motion

• Velocity potential

$$u = -\frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial \phi}{\partial y}$$

• Stream function: Eq. (6.8)

$$u = -\frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \psi}{\partial x}$$

$$\therefore \left.\begin{aligned} \frac{\partial \psi}{\partial y} &= \frac{\partial \phi}{\partial x} \\ \frac{\partial \psi}{\partial x} &= -\frac{\partial \phi}{\partial y} \end{aligned}\right\} \rightarrow \text{ Cauchy-Riemann equation} \qquad (6.51)$$

Now, substitute stream function, (6.8) into irrotational flow, (6.17)

Eq. (6.17) : $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$ $\quad \leftarrow \left[rotation = 0 \quad \nabla \times \vec{q} = 0 \right]$

$$\therefore \; -\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x^2} \;\rightarrow\; \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \rightarrow \text{ Laplace eq.} \qquad \text{(D)}$$

Also, for 2-D flow, velocity potential satisfies the Laplace eq.

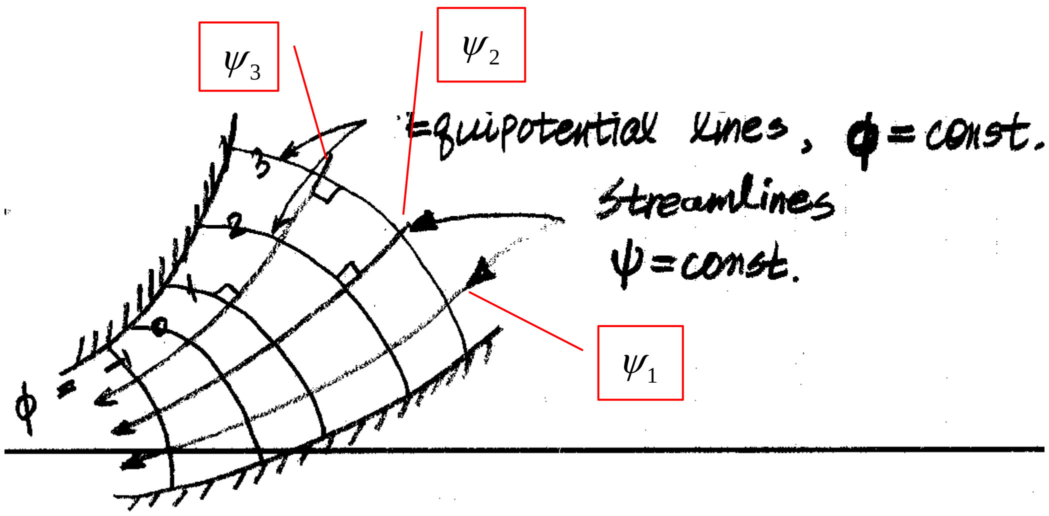
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{(E)}$$

→ Both ϕ and ψ satisfy the Laplace eq. for 2-D incompressible irrotational motion.

→ ϕ and ψ may be interchanged.

→ Lines of constant ϕ and ψ must form an orthogonal mesh system

→ Flow Net



(3) Flow net analysis

Along a streamline, $\psi =$constant.

Eq. for a streamline, Eq. (2.10)

$$\left.\frac{dy}{dx}\right|_{\psi=const.} = \frac{v}{u} \tag{6.54}$$

Along lines of constant velocity potential

$\rightarrow$ $d\phi = 0$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0 \tag{F}$$

Substitute Eq. (6.47a)

$$\left.\frac{dy}{dx}\right|_{\phi=const.} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} = -\frac{u}{v} \tag{6.55}$$

From Eqs. (6.54) and (6.55)

$$\left.\frac{dy}{dx}\right|_{\psi=const.} = -\left.\frac{dx}{dy}\right|_{\phi=const.} \tag{6.56}$$

$\rightarrow$ Slopes are the negative reciprocal of each other.

$\rightarrow$ Flow net analysis (graphical method) can be used when a solution of the Laplace equation is difficult for complex boundaries.

[Appendix I] Typical potential flow systems

1. Uniform flow

$$\rightarrow$$

$$u \quad \rightarrow \qquad u = \frac{\partial \phi}{\partial x} = U$$

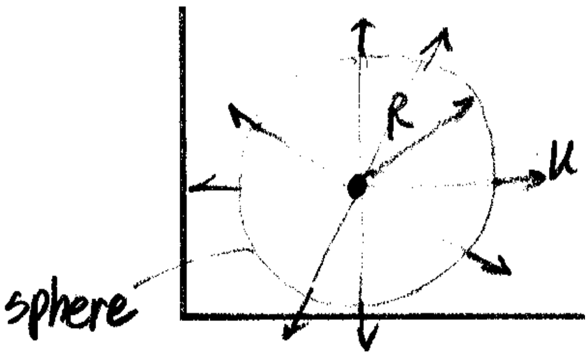
$$\rightarrow$$

$$\therefore \ \phi = Ux + const. \qquad \text{1-D}$$

$$\phi = U(lx + my + nz) \qquad \text{3-D}$$

where *l*, *m*, *n* = directional unit vectors

2. Source or Sink



let $\phi = -\dfrac{M}{R}$ (spherical source)

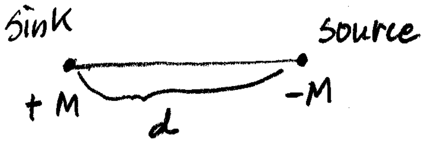
M = strength of sink or source (m^3 / s)

$u = \dfrac{\partial \phi}{\partial R}$ (spherical coordinates) $= \dfrac{M}{R^2}$

$v = w = 0$

3. Doublet

$\rightarrow$ sink plus source with the distance between, $d \rightarrow 0$



4. Vortex

In cylindrical coordinate: let $\phi = k\theta$

$$\begin{cases} u = 0 \\ v = -\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} = -\dfrac{k}{r} \\ w = 0 \end{cases}$$

By the way
$$\begin{cases} v = -\dfrac{\partial \psi}{\partial r} \\ \psi = -\displaystyle\int \dfrac{k}{r} dr = -k \ln r + C \end{cases}$$

$$\Gamma = \oint v ds = \int_0^{2\pi} v r d\theta = 2\pi k \text{ } (\because \text{ singularity at the origin})$$

[Appendix II] Potential flow problem

Find velocity potential ϕ $\begin{cases} \text{Find } \psi \rightarrow \text{Find flow pattern} \\ \text{Find velocity} \\ \text{Find kinetic energy} \end{cases}$

Find velocity, Find kinetic energy $\rightarrow$ Find pressure, force

$\uparrow$ Bernoulli eq.

6.6.2 The Bernoulli equation for irrotational incompressible fluids

(1) For irrotational incompressible fluids

Substitute Eq. (6.17) into Eq. (6.28)

Eq. (6.17) : $\nabla \times \vec{q} = 0$
$$\left.\begin{aligned} \frac{\partial w}{\partial y} &= \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} &= \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y} \end{aligned}\right\}$$
irrotational flow

Eq. (6.28): Navier-Stokes eq. (x -comp.) for incompressible fluid

$$\boxed{\frac{1}{2}\frac{\partial v^2}{\partial x}} \qquad \boxed{\frac{1}{2}\frac{\partial w^2}{\partial x}}$$

$$\frac{\partial u}{\partial t} + \underbrace{u\frac{\partial u}{\partial x}}_{\frac{1}{2}\frac{\partial u^2}{\partial x}} + \underbrace{v\frac{\partial u}{\partial y}}_{v\frac{\partial v}{\partial x}} + \underbrace{w\frac{\partial u}{\partial z}}_{w\frac{\partial w}{\partial x}} = -g\frac{\partial h}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2} + \underbrace{\frac{\partial^2 u}{\partial y^2}}_{\frac{\partial^2 v}{\partial y \partial x}} + \underbrace{\frac{\partial^2 u}{\partial z^2}}_{\frac{\partial^2 w}{\partial z \partial x}}\right]$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2}\right) = -g\frac{\partial h}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \qquad (6.57)$$

Substitute $q^2 = u^2 + v^2 + w^2$ and continuity eq. for incompressible fluid into Eq. (6.57)

Continuity eq., Eq. (6.5): $\nabla \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Then, viscous force term can be dropped.

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{q^2}{2}\right)=-g\frac{\partial h}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left[\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad \rightarrow\ x-\text{Eq.}$$

$$y-Eq. \qquad \frac{\partial v}{\partial t}+\frac{\partial}{\partial y}\left[\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0$$

$$z-Eq. \qquad \frac{\partial w}{\partial t}+\frac{\partial}{\partial z}\left[\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad (6.59)$$

Introduce velocity potential ϕ

$$\boxed{u=-\frac{\partial \phi}{\partial x}, v=-\frac{\partial \phi}{\partial y}, w=-\frac{\partial \phi}{\partial z}}$$

$$\frac{\partial u}{\partial t}=-\frac{\partial^2 \phi}{\partial t \partial x}, \quad \frac{\partial v}{\partial t}=-\frac{\partial^2 \phi}{\partial t \partial y}, \quad \frac{\partial w}{\partial t}=-\frac{\partial^2 \phi}{\partial t \partial z} \qquad (A)$$

Substituting (A) into (6.59) yields

$$\frac{\partial}{\partial x}\left[-\frac{\partial \phi}{\partial t}+\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad x-Eq.$$

$$\frac{\partial}{\partial y}\left[-\frac{\partial \phi}{\partial t}+\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad y-Eq.$$

$$\frac{\partial}{\partial z}\left[-\frac{\partial \phi}{\partial t}+\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad z-Eq. \qquad (B)$$

Integrating (B) leads to Bernoulli eq.

$$-\frac{\partial \phi}{\partial t}+\frac{q^2}{2}+gh+\frac{p}{\rho}=F(t) \qquad (6.60)$$

~ valid throughout the entire field of irrotational motion

For a steady flow; $\dfrac{\partial \phi}{\partial t} = 0$

$$\boxed{\frac{q^2}{2} + gh + \frac{p}{\rho} = const.} \qquad (6.61)$$

→ Bernoulli eq. for a steady, irrotational flow of an incompressible fluid

Dividing (6.61) by g (acceleration of gravity) gives the head terms

$$\frac{q^2}{2g} + h + \frac{p}{\gamma} = const.$$

$$\frac{q_1^2}{2g} + h_1 + \frac{p_1}{\gamma} = \frac{q_2^2}{2g} + h_2 + \frac{p_2}{\gamma} = H \qquad (6.62)$$

H = total head at a point; constant for entire flow field of irrotational motion

(for both along and normal to any streamline)

→ point form of 1- D Bernoulli Eq. for negligible friction

p, H, q = values at particular point → point values in flow field

[Cf] Eq. (4.26)

$$\frac{p_1}{\gamma} + h_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + h_2 + \frac{V_2^2}{2g} = H$$

H = constant along a stream tube

→ 1-D form of 1-D Bernoulli eq.

p, h, V = cross-sectional average values at each section → average values

•Assumptions made in deriving Eq. (6.62)

→ incompressibility + steadiness + irrotational motion+ constant viscosity (Newtonian fluid)

In Eq. (6.57), viscosity term dropped out because $\nabla \cdot \vec{q} = 0$ (continuity Eq.).

→ Thus, Eq. (6.62) can be applied to either a viscous or inviscid fluid.

• Viscous flow

Velocity gradients result in viscous shear.

→ Viscosity causes a spread of vorticity (forced vortex).

→ Flow becomes rotational.

→ *H* in Eq. (6.62) varies throughout the fluid field.

→ Irrotational motion takes place only in a few special cases (irrotational vortex).

potential flow

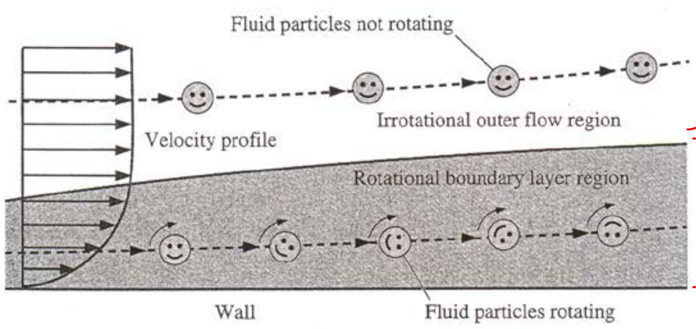


FIGURE 4–24
The difference between rotational and irrotational flow: fluid elements in a rotational region of the flow rotate, but those in an irrotational region of the flow do not.

rotational flow

•Irrotational motion can never become rotational as long as only gravitational and pressure force acts on the fluid particles (without shear forces).

→ In real fluids, nearly irrotational flows may be generated if the motion is primarily a result of pressure and gravity forces.

[Ex] free surface wave motion generated by pressure forces (Fig. 6.8)

flow over a weir under gravity forces (Fig. 6.9)

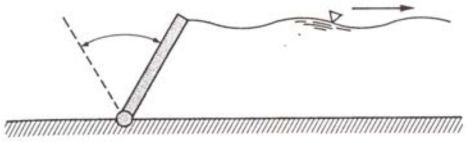


FIG. 6–8. Wave generation by pressure forces.

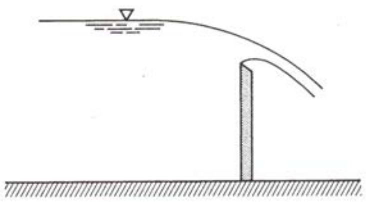


FIG. 6–9. Flow over a weir under gravity forces.

•Vortex motion

i) Forced vortex - rotational flow

~ generated by the transmission of tangential shear stresses

→ rotating cylinder

ii) Free vortex - irrotational flow

~ generated by the gravity and pressure

→ drain in the tank bottom, tornado, hurricane

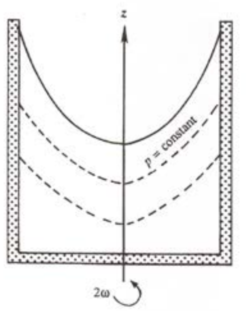


Figure 5.2 Constant pressure surfaces in a solid-body rotation generated in a rotating tank containing liquid.

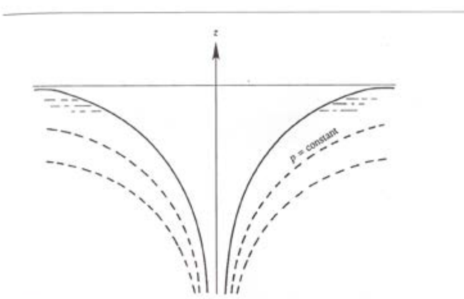


Figure 5.3 Irrotational vortex in a liquid.

•Boundary layer flow (Ch. 8)

i) Flow within thin boundary layer - viscous flow- rotational flow

→ use boundary layer theory

ii) Flow outside the boundary layer - irrotational (potential) flow

→ use potential flow theory

6.7 Equations for Frictionless Flow

6.7.1 The Bernoulli equation for flow along a streamline

For <u>inviscid flow</u> $(\mu = 0)$

→ Assume no frictional (viscous) effects but compressible fluid flows

→ Bernoulli eq. can be obtained by <u>integrating Navier-Stokes equation</u> along a streamline.

Eq. (6.24a): N-S eq. for compressible fluid ($\mu = 0$)

$$\rho\vec{g} - \nabla p + \cancel{\mu\nabla^2\vec{q}} + \cancel{\frac{\mu}{3}\nabla(\nabla\cdot\vec{q})} = \rho\frac{\partial\vec{q}}{\partial t} + \rho\left(\vec{q}\cdot\nabla\right)\vec{q}$$

$$\vec{g} - \frac{\nabla p}{\rho} = \frac{\partial\vec{q}}{\partial t} + \left(\vec{q}\cdot\nabla\right)\vec{q} \tag{6.63}$$

→ **Euler's equation of motion** for inviscid (ideal) fluid flow

$\boxed{\vec{g} = -g\nabla h}$

Substituting (6.26a) into (6.63) leads to

$$-g\nabla h - \frac{\nabla p}{\rho} = \frac{\partial\vec{q}}{\partial t} + \left(\vec{q}\cdot\nabla\right)\vec{q} \tag{6.64}$$

$\boxed{\vec{i}dx + \vec{j}dy + \vec{k}dz}$

Multiply $d\vec{r}$ (element of streamline length) and <u>integrate along the streamline</u>

$$-g\int \nabla h\cdot d\vec{r} - \int \frac{1}{\rho}\nabla p\cdot d\vec{r} = \int \left(\frac{\partial\vec{q}}{\partial t}\right)\cdot d\vec{r} + \int \left[\left(\vec{q}\cdot\nabla\right)\vec{q}\right]\cdot d\vec{r} + C\left(t\right)$$

$$-gh - \int \frac{dp}{\rho} = \int \left(\frac{\partial\vec{q}}{\partial t}\right)\cdot d\vec{r} + \int \underbrace{\left[\left(\vec{q}\cdot\nabla\right)\vec{q}\right]\cdot d\vec{r}}_{\boxed{\mathrm{I}}} + C\left(t\right) \tag{6.66}$$

$$I = \left[\left(\vec{q}\cdot\nabla\right)\vec{q}\right]\cdot d\vec{r} = d\vec{r}\cdot\left[\left(\vec{q}\cdot\nabla\right)\vec{q}\right] = \vec{q}\cdot\left[\left(\underbrace{d\vec{r}\cdot\nabla}_{\boxed{\mathrm{II}}}\right)\vec{q}\right]$$

By the way,

$$II = d\vec{r} \cdot \nabla = \frac{\partial(\ )}{\partial x} dx + \frac{\partial(\ )}{\partial y} dy + \frac{\partial(\ )}{\partial z} dz$$

$$\therefore \ \left(d\vec{r} \cdot \nabla \right) \vec{q} = \frac{\partial \vec{q}}{\partial x} dx + \frac{\partial \vec{q}}{\partial y} dy + \frac{\partial \vec{q}}{\partial z} dz = d\vec{q}$$

$$I = \vec{q} \cdot d\vec{q} = d\left(\frac{q^2}{2} \right)$$

$$\therefore \int \left[\left(\vec{q} \cdot \nabla \right) \vec{q} \right] \cdot d\vec{r} = \int d\left(\frac{q^2}{2} \right) = \frac{q^2}{2}$$

Thus, Eq. (6.66) becomes

$$\int \frac{dp}{\rho} + gh + \frac{q^2}{2} + \int \left(\frac{\partial q}{\partial t} \right) \cdot d\vec{r} = -C(t) \tag{6.67}$$

For steady motion, $\dfrac{\partial \vec{q}}{\partial t} = 0; C(t) \to C$

$$\int \frac{dp}{\rho} + gh + \frac{q^2}{2} = const. \quad \text{along a streamline} \tag{6.68}$$

For incompressible fluids, $\rho = \text{const.}$

$$\frac{p}{\rho} + gh + \frac{q^2}{2} = const.$$

Divide by g

$$\boxed{\frac{p}{\gamma} + h + \frac{q^2}{2g} = C \qquad \text{along a streamline}} \tag{6.69}$$

→ Bernoulli equation for steady, frictionless, incompressible fluid flow

→ Eq. (6.69) is identical to Eq. (6.22). Constant C is varying from one streamline to another in a rotational flow, Eq. (6.69); it is invariant throughout the fluid for irrotational flow, Eq. (6.22).

6.7.2 Summary of Bernoulli equation forms

• Bernoulli equations for steady, incompressible flow

1) For irrotational flow

$$H = \frac{p}{\gamma} + h + \frac{q^2}{2g} = \text{constant throughout the flow field} \tag{6.62}$$

2) For frictionless flow (rotational)

$$H = \frac{p}{\gamma} + h + \frac{q^2}{2g} = \text{ constant along a streamline} \tag{6.69}$$

3) For 1-D frictionless flow (rotational)

$$H = \frac{p}{\gamma} + h + Ke\frac{V^2}{2g} = \text{ constant along finite pipe} \tag{4.25}$$

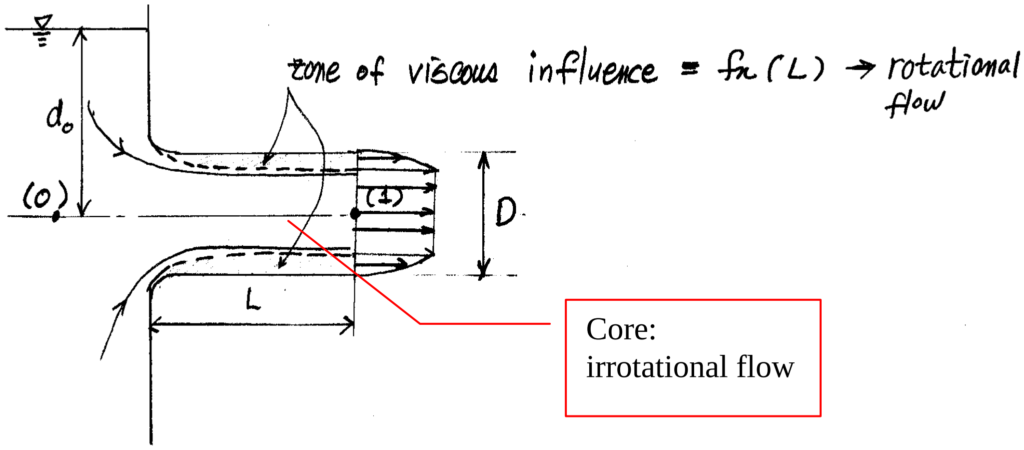
4) For steady flow with friction

~ include head loss h_L

$$\frac{p_1}{\gamma} + h_1 + \frac{q_1^2}{2g} = \frac{p_2}{\gamma} + h_2 + \frac{q_2^2}{2g} + h_L$$

6.7.3 Applications of Bernoulli's equation to flows of real fluids

(1) Efflux from a short tube



1) Zone of viscous action (boundary layer): frictional effects cannot be neglected.

2) Flow in the reservoir and central core of the tube: primary forces are pressure and gravity forces. → irrotational flow

Apply Bernoulli eq. along the centerline streamline between (0) and (1)

$$\frac{p_0}{\gamma} + z_0 + \frac{q_0^2}{2g} = \frac{p_1}{\gamma} + z_1 + \frac{q_1^2}{2g}$$

p_0 = hydrostatic pressure $= \gamma d_0$, $p_1 = p_{atm} \rightarrow p_{1_{gage}} = 0$

$z_0 = z_1$

$q_0 = 0$ (neglect velocity at the large reservoir)

$$\therefore \frac{q_1^2}{2g} = d_0$$

$q_1 = \sqrt{2gd_0}$ → Torricelli's result (6.74)

If we neglect thickness of the zone of viscous influence

$$Q = \frac{\pi D^2}{4} q_1$$

(2) Stratified flow

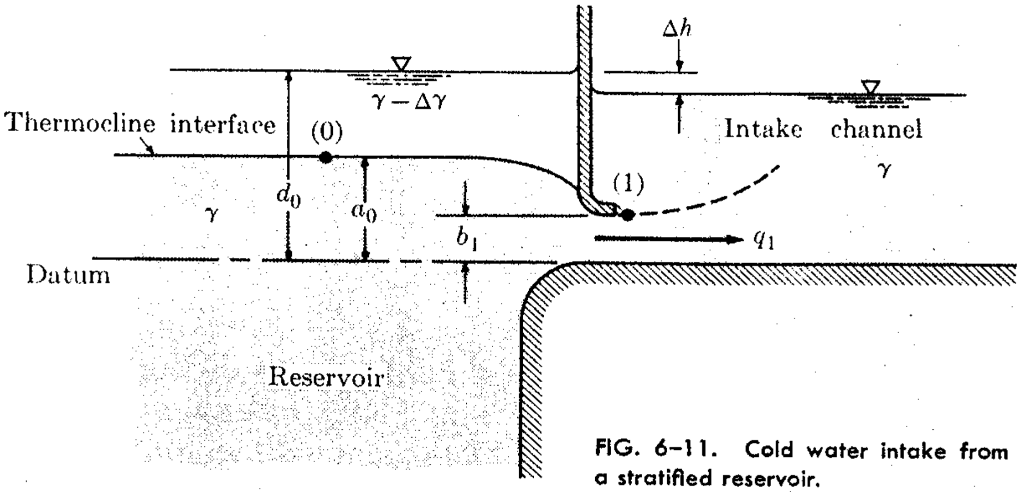


FIG. 6-11. Cold water intake from a stratified reservoir.

During summer months, large reservoirs and lakes become thermally stratified.

→ At thermocline, temperature changes rapidly with depth.

•Selective withdrawal: Colder water is withdrawn into the intake channel with a velocity q_1 (uniform over the height b_1) in order to provide cool condenser water for thermal (nuclear) power plant.

Apply Bernoulli eq. between points (0) and (1)

$$\frac{p_0}{\gamma} + a_0 + \frac{q_0^{\,2}}{2g} = \frac{p_1}{\gamma} + b_1 + \frac{q_1^{\,2}}{2g}$$

$$q_0 \cong 0$$

$$p_0 = \text{hydrostatic pressure} = (\gamma - \Delta\gamma)(d_0 - a_0)$$

$$p_1 = \gamma(d_0 - \Delta h - b_1)$$

$$\therefore \quad \frac{q_1^{\,2}}{2g} = \Delta h - \frac{\Delta\gamma}{\gamma}(d_0 - a_0)$$

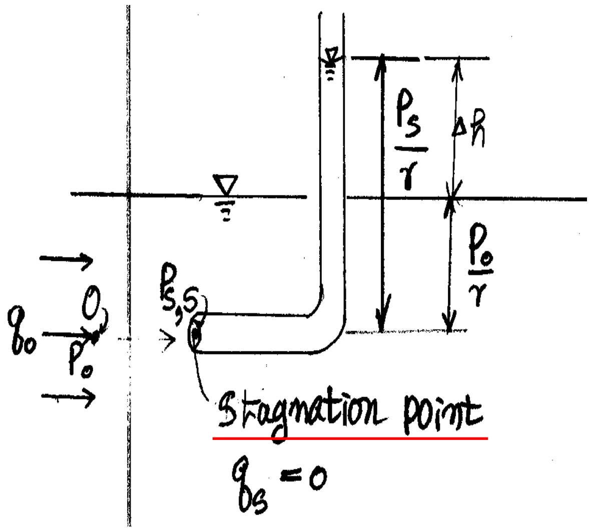
$$q_1 = \left[2g \left\{ \Delta h - \frac{\Delta \gamma}{\gamma} \left(d_0 - a_0 \right) \right\} \right]^{\frac{1}{2}} \tag{6.77}$$

For <u>isothermal (unstratified)</u> case, $a_0 = d_0$

$q_1 = \sqrt{2g\Delta h}$ → Torricelli's result

(3) Velocity measurements with the Pitot tube (Henri Pitot, 1732)

→ Measure velocity from stagnation or impact pressure



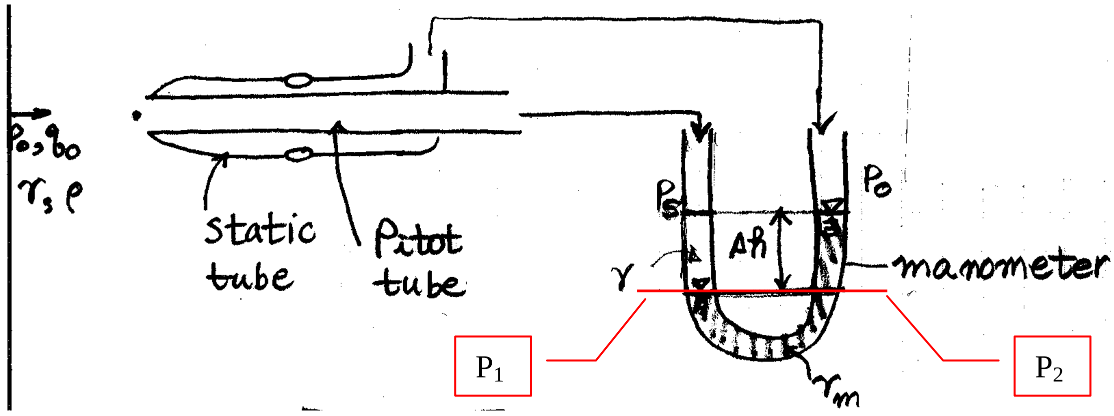
$$\frac{p_0}{\gamma} + h_0 + \frac{q_0^2}{2g} = \frac{p_s}{\gamma} + h_s + \cancel{\frac{q_s^2}{2g}}$$

$h_0 = h_s$, $q_s = 0$

$$\therefore \ \frac{q_0^2}{2g} = \frac{p_s - p_0}{\gamma} = \Delta h$$

$$q_0 = \sqrt{2g\Delta h}$$

•Pitot-static tube



$$q_0 = \sqrt{\frac{2(p_s - p_0)}{\rho}} \qquad \text{(A)}$$

By the way,

$$p_1 = p_s + \gamma \Delta h = p_2 = p_0 + \gamma_m \Delta h$$

$$p_s - p_0 = \Delta h(\gamma_m - \gamma) \qquad \text{(B)}$$

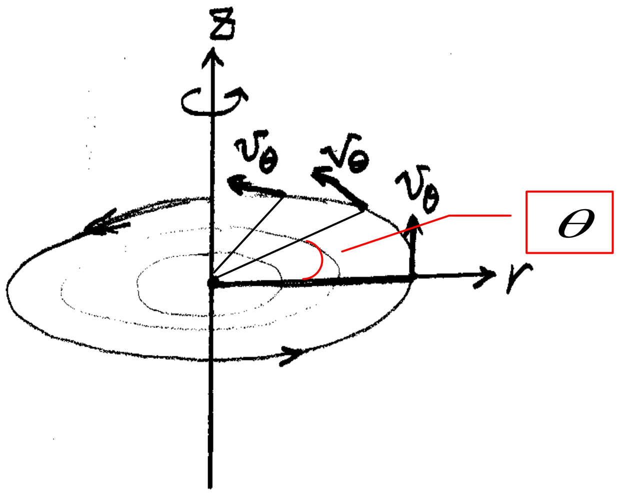
Combine (A) and (B)

$$q_0 = \sqrt{\frac{2\Delta h(\gamma_m - \gamma)}{\rho}}$$

6.8 Vortex Motion

•vortex = fluid motion in which streamlines are concentric circles

For steady flow of an incompressible fluid, apply Navier-Stokes equations in cylindrical coordinates



Assumptions:

$$\frac{\partial(\ )}{\partial t}=0$$

$$v_\theta \neq 0$$

$$v_r=0; \quad v_z=0; \quad \frac{\partial v_\theta}{\partial z}=0$$

$$\frac{\partial p}{\partial \theta}=0$$

$$\frac{\partial p}{\partial z}=\frac{\partial p}{\partial h} \quad (h\text{=vertical direction})$$

Continuity Eq.: Eq. (6.30)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r v_r}\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(v_\theta\right)+\frac{\partial}{\partial z}\left(\cancel{v_z}\right)=0$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left(v_\theta\right)=0\rightarrow\frac{\partial v_\theta}{\partial\theta}=0$$

Navier-Stokes Eq.: Eq. (6.29)

1) r -comp.

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}}+v_r\cancel{\frac{\partial v_r}{\partial r}}+\frac{v_\theta}{r}\cancel{\frac{\partial v_r}{\partial\theta}}-\frac{v_\theta{}^2}{r}+v_z\cancel{\frac{\partial v_r}{\partial z}}\right)$$

$$=-\frac{\partial p}{\partial r}+\mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left[\cancel{rv_r}\right]\right)+\frac{1}{r^2}\cancel{\frac{\partial^2 v_r}{\partial\theta^2}}-\frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial\theta}}+\cancel{\frac{\partial^2 v_r}{\partial z^2}}\right\}+\cancel{\rho g_r}$$

$$\frac{v_\theta{}^2}{r}=\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{6.83a}$$

2) θ-comp.

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}}+\cancel{v_r}\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial\theta}}-\cancel{\frac{v_r v_\theta}{r}}+\cancel{v_z}\frac{\partial v_\theta}{\partial z}\right)$$

$$=-\frac{1}{r}\cancel{\frac{\partial p}{\partial\theta}}+\mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left[rv_\theta\right]\right)+\frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial\theta^2}}-\frac{2}{r^2}\cancel{\frac{\partial v_r}{\partial\theta}}+\cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right\}+\cancel{\rho g_\theta}$$

$$\therefore\ 0=\frac{\mu}{\rho}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rv_\theta\right)\right] \tag{6.83 b}$$

3) z-comp.

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}}+\cancel{v_r}\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\cancel{\frac{\partial v_z}{\partial\theta}}-\cancel{\frac{v_r v_\theta}{r}}+v_z\cancel{\frac{\partial v_z}{\partial z}}\right)$$

$$= -\frac{\partial p}{\partial z} + \mu \left\{ \frac{1}{r}\frac{\partial}{\partial r}\left(r \cancel{\frac{\partial v_z}{\partial r}} \right) + \frac{1}{r^2} \cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}} \right\} + \rho g_z$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z = -\frac{1}{\rho}\frac{\partial p}{\partial h} - g \qquad (6.83\text{ c})$$

Integrate θ-Eq. w.r.t. r

$$C_1 = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)$$

$$rC_1 = \frac{\partial}{\partial r}(rv_\theta)$$

Integrate again

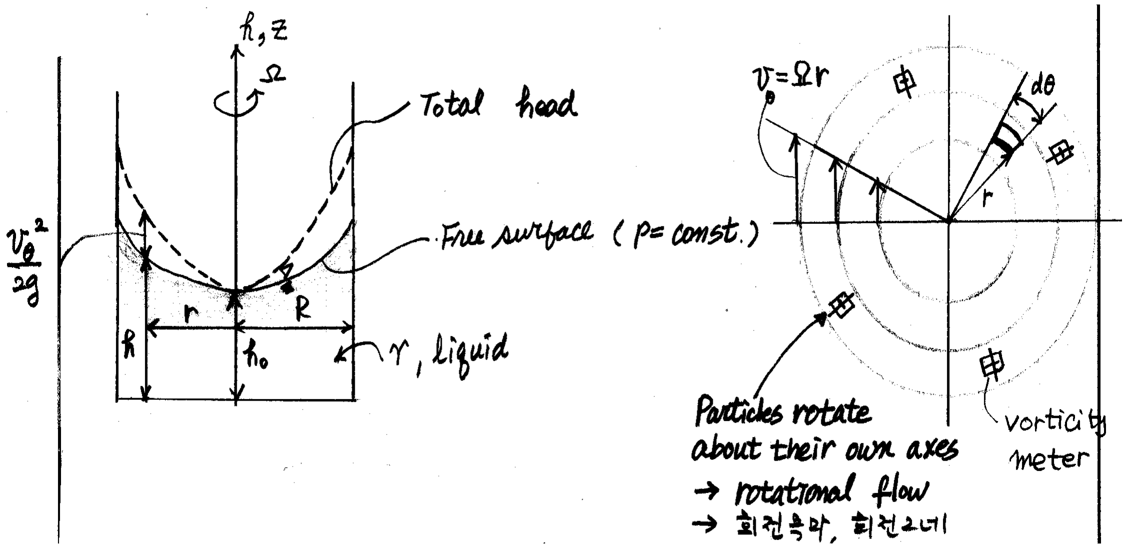
$$\left.\begin{array}{ll} \dfrac{r^2}{2}C_1 + C_2 = rv_\theta & (A) \\ v_\theta = \dfrac{C_1}{2}r + \dfrac{C_2}{r} & (B) \end{array}\right\} \text{need 2 BCs} \qquad (6.84)$$

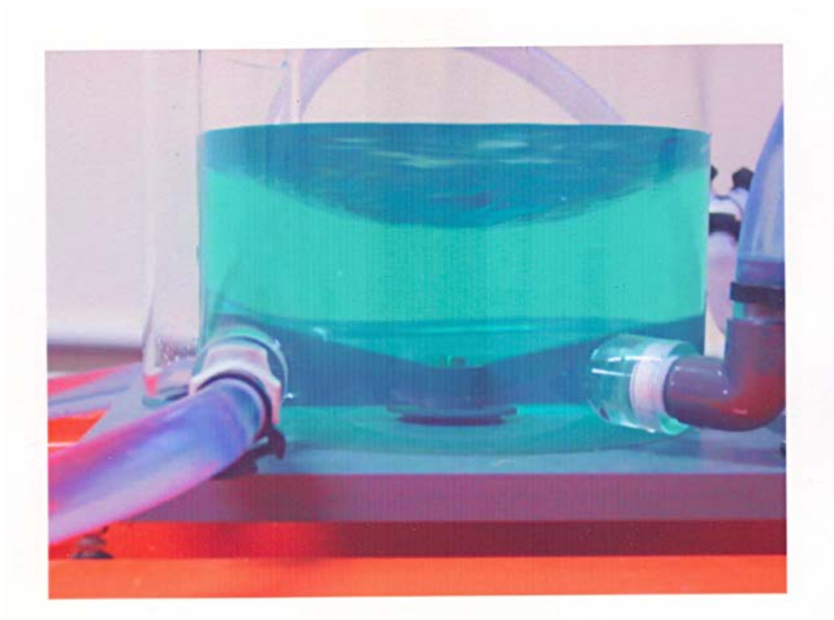
z-Eq.

$$\frac{\partial p}{\partial h} = -\rho g = -\gamma$$

$p = -\gamma h \qquad \rightarrow$ hydrostatic pressure distribution

6.8.1 Forced Vortex - rotational flow





Consider cylindrical container of radius R is rotated at a constant angular velocity Ω about a vertical axis

Substitute BCs into Eq. (6.84)

$$\text{i)}\ \ r = 0,\quad v_\theta = 0 \qquad \rightarrow (A):\ 0 + C_2 = 0 \qquad \therefore\ C_2 = 0$$

ii) $r = R,\ v_\theta = R\Omega \quad \rightarrow (B): R\Omega = \frac{C_1}{2} R \quad \therefore\ C_1 = 2\Omega$

Eq. (B) becomes

$$\boxed{v_\theta = \frac{2\Omega}{2} r = \Omega r} \qquad \rightarrow \text{ solid-body rotation}$$

$$r - Eq.: \frac{\Omega^2 r^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} \quad \rightarrow \quad \frac{\partial p}{\partial r} = \rho\Omega^2 r \qquad \text{(C)}$$

$$z - Eq.: \frac{\partial p}{\partial h} = -\gamma \qquad \text{(D)}$$

Consider total derivative dp

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial h} dh = \rho\Omega^2 r dr - \gamma dh$$

Integrate once

$$p = \rho\Omega^2 \frac{r^2}{2} - \gamma h + C_3$$

Incorporate B.C. to decide C_3

$r = 0;\quad h = h_0$ and $p = p_0$

$p_0 = 0 - \gamma h_0 + C_3 \quad \therefore\ C_3 = p_0 + \gamma h_0$

$$\boxed{p - p_0 = \rho\frac{\Omega^2 r^2}{2} - \gamma\left(h - h_0\right)}$$

At free surface

$$p = p_0$$

$$\boxed{h = h_0 + \frac{\Omega^2}{2g} r^2} \quad \rightarrow \text{ paraboloid of revolution}$$

•Rotation components in cylindrical coordinates

Eq. (6.18):

$$\omega_z = \frac{1}{2}\left(-\cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}} + \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} \right)$$

$$= \frac{1}{2}\left(\frac{r\Omega}{r} + \frac{\partial}{\partial r}(r\Omega) \right) = \frac{1}{2}(\Omega + \Omega) = \Omega$$

vorticity $= 2\omega_z = 2\Omega \neq 0$

→ rotational flow

→ Forced vortex is generated by the transmission of tangential shear stresses.

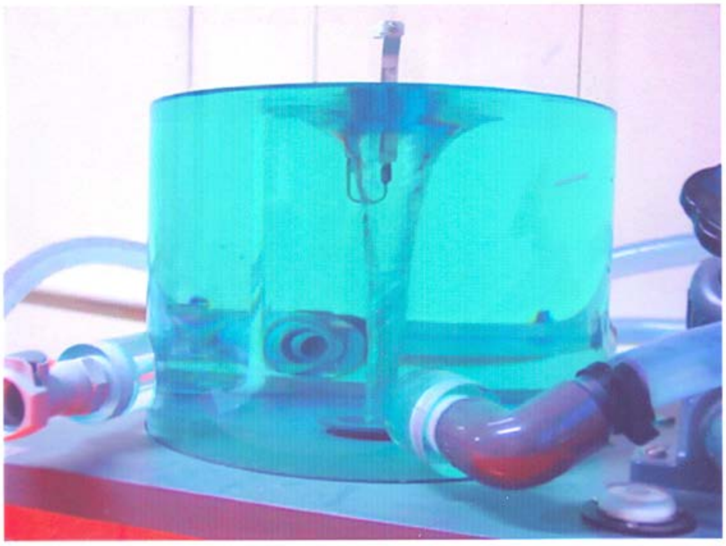
•Total head

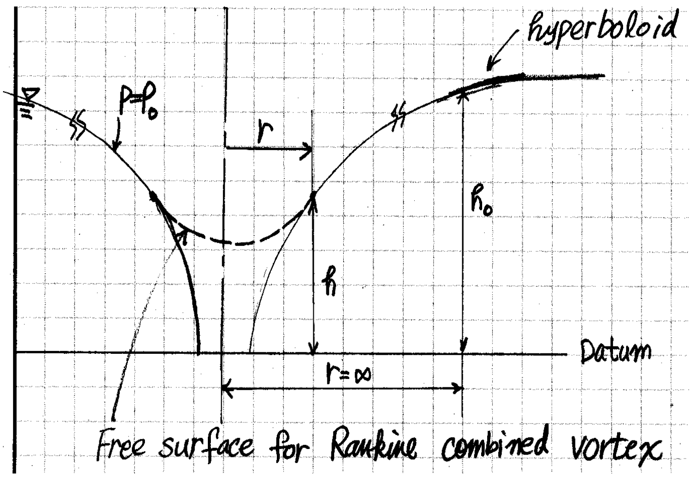
$$H = \frac{p}{\gamma} + h + \frac{{v_\theta}^2}{2g} \neq \text{const.}$$

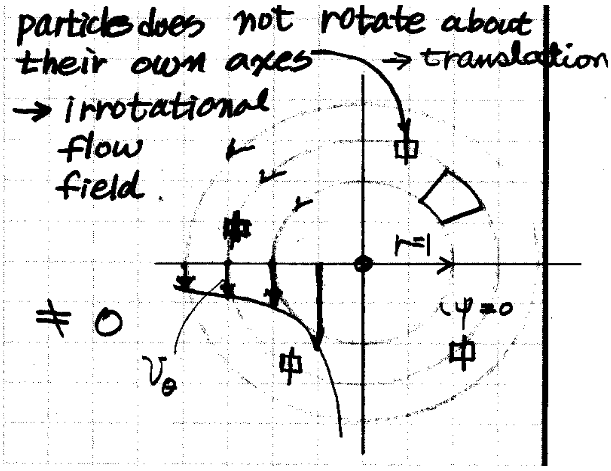
→ increases with radius

6.8.2 Irrotational or free vortex

Free vortex: drain hole vortex, tornado, hurricane, morning glory spillway







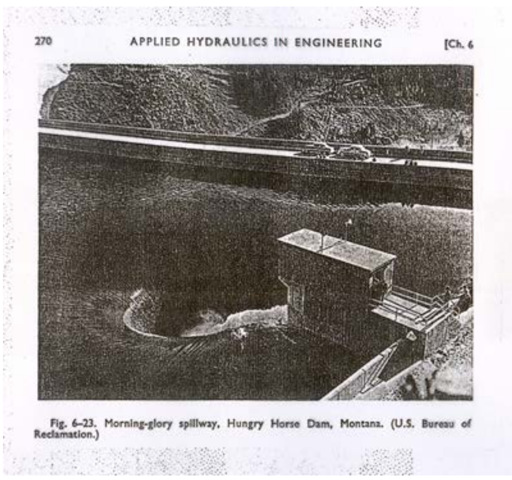


Fig. 6–23. Morning-glory spillway, Hungry Horse Dam, Montana. (U.S. Bureau of Reclamation.)

For irrotational flow,

$$\frac{p}{\gamma}+h+\frac{v_\theta^{\,2}}{2g}=\text{const.} \qquad \rightarrow \text{throughout the fluid field}$$

Differentiate w.r.t r

$$\frac{1}{\gamma}\frac{\partial p}{\partial r}+\cancel{\frac{\partial h}{\partial r}}+\frac{1}{g}v_\theta\frac{\partial v_\theta}{\partial r}=0$$

z coincides with h

$$\left(\frac{\partial h}{\partial r}=\frac{\partial h}{\partial \theta}=0,\ \frac{\partial h}{\partial z}=1\right)$$

$$\therefore\ \frac{\partial p}{\partial r}=-\rho v_\theta\frac{\partial v_\theta}{\partial r} \qquad \text{(A)}$$

Eq (6.83a): r -Eq. of N-S Eq.

$$\frac{\partial p}{\partial r}=\rho\frac{v_\theta^{\,2}}{r} \qquad \text{(B)}$$

Equate (A) and (B)

$$-\rho v_\theta\frac{\partial v_\theta}{\partial r}=\rho\frac{v_\theta^{\,2}}{r} \quad\rightarrow\quad -\frac{\partial v_\theta}{\partial r}r=v_\theta$$

Integrate using separation of variables

$$\int\frac{1}{v_\theta}\partial v_\theta=\int-\frac{1}{r}\partial r$$

$$\ln v_\theta=-\ln r+C$$

$$\ln v_\theta+\ln r=\ln\left(v_\theta r\right)=C$$

$v_\theta r=C_4$ ~ constant angular momentum

$$\boxed{v_\theta = \frac{C_4}{r}}$$

[Cf] Forced vortex

$$v_\theta = \Omega r$$

•Radial pressure gradient

(B):

$$\frac{\partial p}{\partial r} = \rho \frac{v_\theta{}^2}{r} = \rho \frac{(v_\theta r)^2}{r^3} = \rho \frac{C_4{}^2}{r^3}$$

•Total derivative

$$\boxed{\frac{\partial p}{\partial h} = -\gamma}$$

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial h} dh = \rho \frac{C_4{}^2}{r^3} dr - \gamma dh$$

Integrate once

$$p = -\rho \frac{C_4{}^2}{2r^2} - \gamma h + C_5 \tag{6.93}$$

B.C.: $r = \infty$: $h = h_0$ and $p = p_0$

Substitute B.C. into Eq. (6.93)

$$p_0 = -\gamma h_0 + C_5$$

$$C_5 = p_0 + \gamma h_0$$

$$\boxed{p - p_0 = \gamma\left(h_0 - h\right) - \rho\frac{C_4^{\ 2}}{2r^2}} \tag{6.94}$$

[Cf] Forced vortex: $p - p_0 = \frac{\rho}{2}\Omega^2 r^2 + \gamma\left(h_0 - h\right)$

•Locus of free surface is given when $p = p_0$

$$h = h_0 - \frac{C_4^{\ 2}}{2gr^2}$$ → hyperboloid of revolution

[Cf] Forced vortex: $h = h_0 + \frac{\Omega^2}{2g}r^2$

•Circulation

$\boxed{ds = rd\theta}$

$$\Gamma = \oint \vec{q}\cdot d\vec{s} = \int_0^{2\pi} \underline{v_\theta}rd\theta = \left[C_4\theta\right]_0^{2\pi} = 2\pi C_4 \neq 0$$

$\boxed{v_\theta r = C_4}$

→ Even though flow is irrotational, circulation for a contour enclosing the origin is not zero because of the singularity point.

•Stream function, ψ

$\boxed{C_4 = \frac{\Gamma}{2\pi}}$

$$v_\theta = \frac{\partial\psi}{\partial r} = \frac{C_4}{r} = \frac{\Gamma}{2\pi r}$$

$$\psi = \frac{\Gamma}{2\pi}\int\frac{dr}{r} = \frac{\Gamma}{2\pi}\ln r \tag{6.97}$$

where Γ = vortex strength

•Vorticity component ω_z

$$\omega_z = -\frac{1}{r}\cancel{\frac{\partial v_r}{\partial \theta}} + \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}$$

Substitute $v_\theta = \dfrac{C_4}{r}$

$$\omega_z = \frac{C_4}{r^2} + \frac{\partial}{\partial r}\left(\frac{C_4}{r}\right) = \frac{C_4}{r^2} - \frac{C_4}{r^2} = 0$$

→ Irrotational motion

At $r = 0$ of drain hole vortex, either fluid does not occupy the space or fluid is rotational (forced vortex) when drain in the tank bottom is suddenly closed.

→ Rankine combined vortex

→ fluid motion is ultimately dissipated through viscous action

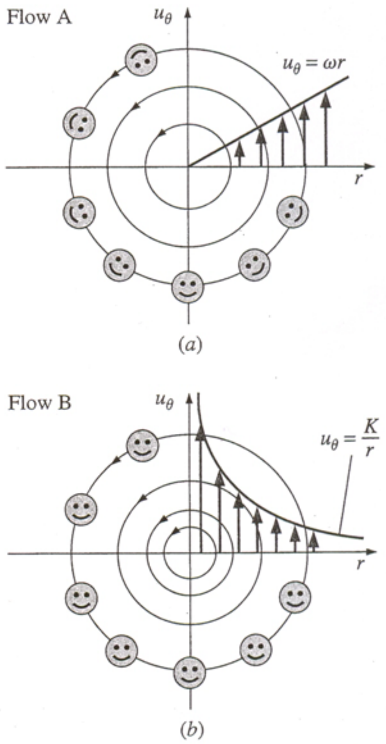


FIGURE 4–28
Streamlines and velocity profiles for (a) flow A, solid-body rotation and (b) flow B, a line vortex. Flow A is rotational, but flow B is irrotational everywhere except at the origin.

FIGURE 4–29
A simple analogy: (a) *rotational* circular flow is analogous to a roundabout, while (b) *irrotational* circular flow is analogous to a Ferris wheel.
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Homework Assignment # 4

Due: 1 week from today

6-4. Consider an incompressible two-dimensional flow of a viscous fluid in the xy-plane in which the body force is due to gravity. (a) Prove that the divergence of the vorticity vector is zero. (This expresses the conservation of vorticity, $\nabla \cdot \zeta = 0$.) (b) Show that the Navier-Stokes equation for this flow can be written in terms of the vorticity as $d\zeta / dt = \nu \nabla^2 \zeta$. (This is a "diffusion" equation and indicates that vorticity is diffused into a fluid at a rate which depends on the magnitude of the kinematic viscosity.) Note that $d\zeta / dt$ is the substantial derivative defined in Section 2-1.

6-5. Consider a steady, incompressible laminar flow between parallel plates as shown in Fig. 6-4 for the following conditions: a =0.03 m, U =0.3 m/sec, $\mu = 0.476$ N·sec/m^2, $\partial p / \partial x$ =625 N/m^3 (pressure increases in $+x$-direction). (a) Plot the velocity distribution, u, in the z-direction. (b) In which direction is the net fluid motion? (c) Plot the distribution of shear stress τ_{zx} in the z-direction.

6-7. An incompressible liquid of density ρ and viscosity μ flows in a thin film down glass plate inclined at an angle α to the horizontal. The thickness, a, of the liquid film normal to the plate is constant, the velocity is everywhere parallel to the plate, and the flow is steady. Neglect viscous shear between the air and the moving liquid at the free surface. Determine the variation in longitudinal velocity in the direction normal to the plate, the shear stress at the plate, and the average velocity of flow.

6-11. Consider steady laminar flow in the horizontal axial direction through the annular space between two concentric circular tubes. The radii of the inner and outer tube are r_1 and r_2, respectively. Derive the expression for the velocity distribution in the direction as a function of viscosity, pressure gradient $\partial p / \partial x$, and tube dimensions.

6-15. The velocity potential for a steady incompressible flow is given by $\Phi = (-a/2)(x^2 + 2y - z^2)$, where a is an arbitrary constant greater than zero. (a) Find the equation for the velocity vector $\vec{q} = \vec{i}u + \vec{j}v + \vec{k}w$. (b) Find the equation for the streamlines in the xz $(y = 0)$ plane. (c) Prove that the continuity equation is satisfied.

6-21. The velocity variation across the radius of a rectangular bend (Fig.6-22) may be approximated by a free vortex distribution $v_\theta r = const.$ Derive an expression for the pressure difference between the inside and outside of the bend as a function of the discharge Q, the fluid density ρ, and the geometric parameters R and b, assuming frictionless flow.

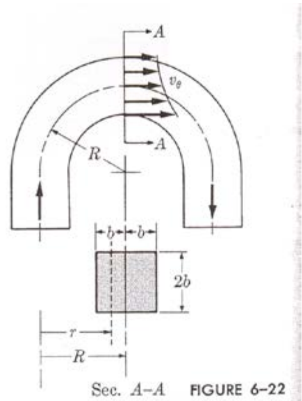

FIGURE 6–22