23 Analytic Functions, Line integral

23.1 Hyperbolic Functions.

(11)

$$\cosh z = \frac{1}{2}(e^z + e^{-z}), \quad \sinh z = \frac{1}{2}(e^z - e^{-z})$$

(12)

$$(\cosh z)' = \sinh z \qquad (\sinh z)' = \cosh z$$

Definition of other hyperbolic functions.

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}$$

(13)

$$\text{sech } z = \frac{1}{\cosh z}, \quad \text{csch } z = \frac{1}{\sinh z}$$

Complex trigonometric and hyperbolic functions are related.

(14)

$$\cosh iz = \cos z, \qquad \sinh iz$$

Proof.

$$\cosh iz = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$$

$$\sin iz = \frac{1}{2}(e^{iz} - e^{-iz}) = i\frac{1}{2i}(e^{iz} - e^{-iz}) = i \sin z.$$

$$\cosh z : \text{ even }, \ \sinh z : \text{ odd}$$

(15)

$$\cos iz = \cosh z, \qquad \sin iz = i \sinh z$$

Proof.

$$\cos iz = \frac{1}{2}(e^{-z} + e^z) = \cosh z.$$

$$\sin iz = \frac{1}{2i}(e^{-z} - e^z) = \frac{i}{2}(e^z - e^{-z}) = i \sinh z$$

Conformal Mapping by sin z , cos z , and cosh z

Sine

$$w = \sin z$$

$$u = Re(\sin z) = \sin x \cosh y$$

$$v = Im(\sin z) = \cos x \sinh y$$

x=const

(16)

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = \cosh^2 y - \sinh^2 y = 1 \qquad \text{(Hyperbolas)}$$

y=const

(17)

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = \sin^2 x + \cos^2 x = 1 \qquad \text{(Ellipses)}$$

$$\text{Exceptions.} \qquad x = \pm\pi/2 \to u \le -1 \quad \& \quad u \ge 1. \quad (v = 0)$$

upper and lower sides of the rectangle are mapped onto semi-ellipses.

$$x = \pm\frac{\pi}{2} \Rightarrow \left(\begin{array}{c} -\cosh 1 \le u \le -1 \\ 1 \le u \le \cosh u \end{array} \right.$$

Cosine

$$w = \cos z = \sin(z + \pi/2)$$

translation to the right through $\pi/2$

23.2 Logarithm. General Power.

natural logarithm of $z = x + iy \to \ln z$.
: defined as the inverse of the exponential function.
i.e, $w = \ln z$ is defined for $z \neq 0$

$$e^w = z.$$

Set $\qquad w = u + iv$ and $z = re^{i\theta}$

$$e^w = e^{u+iv} = re^{i\theta}$$

$$\therefore e^u = r, \qquad v = \theta.$$

(1)

$$\ln z = \ln r + i\theta \qquad (r = |z| > 0, \theta = \arg z)$$

The complex natural logarithm $\ln z (z \neq 0)$ is infinitely many-valued.

Principal value of $\ln z$

(2)

$$\mathrm{Ln}\ z = \ln |z| + i \mathrm{Arg}\ z. \qquad (z \neq 0)$$

(3)

$$\ln z = \mathrm{Ln}\ z \pm 2n\pi i \qquad (n = 1, 2, \cdots)$$

If z is positive real, then Arg $z = 0$, and Ln z becomes identical with the real natural logarithm.

(4) (a) $\ln(z_1 z_2) = \ln z_1 + \ln z_2$ (b) $\ln(z_1/z_2) = \ln z_1 - z_2$

Example 1.

$$z_1 = z_2 = e^{\pi i} = -1$$

$$\mathrm{Ln}\ z_1 = \mathrm{Ln}\ z_2 = \pi i.$$

$$\ln(z_1 z_2) = \ln 1 = 2\pi i : \ \mathrm{Ln}\ (z_1 z_2) = \mathrm{Ln}\ 1 = 0$$

(5a)

$$e^{\ln z} = z$$

since $\qquad \arg(e^z) = y \pm 2n\pi$

(5b)

$$\ln(e^z) = z \pm 2n\pi i \qquad (n = 0, 1, \cdots)$$

(6)

$$(\ln z)' = 1/z \qquad [n = 1, 2, \cdots \text{ fixed, } z \text{ not negative real or zero}]$$

$$u = \ln r = \ln |z| = \frac{1}{2} \ln(x^2 + y^2), \qquad v = \arg z = \arctan y/x + c.$$

Cauchy-Riemann equations.

$$u_x = \frac{x}{x^2 + y^2} = v_y = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}$$

$$u_y = \frac{y}{x^2 + y^2} = -v_x = \frac{-1}{1 + (y/x)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{y}{x^2 + y^2}$$

$$(\ln z)' = u_x + iv_x = \frac{x}{x^2 + y^2} + i \frac{1}{1 + (y/x)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{x - iy}{x^2 + y^2} = \frac{1}{z}$$

Conformal Mapping by $\ln z$ **.**

Principal of Inverse Mapping : The mapping by the inverse $z = f^{-1}(w)$ of $w = f(z)$ is obtained by interchanging the roles of the z-plane and the w-plane in the mapping by

$w = f(z)$.

Natural logarithm

$w = e^z$ maps a fundamental strip onto the w-plane without $z = 0$ Ln z maps the z-plane [with $z = 0$ omitted and cut along the negative real axis, where $\theta = Im(\text{Ln } z)$ jumps by 2π] onto the horizontal strip $-\pi < v \leq \pi$ of the w-plane

$$w = \text{Ln } z + 2\pi i \quad \rightarrow \quad \text{the strip } \pi < v \leq 3\pi$$

$$w = \ln z = \text{Ln } z \pm 2n\pi i \qquad (n = 0, 1, 2, \cdots)$$

: cover the whole w-plane without overlapping.

General Powers.

(7)

$$z^c = e^{c \ln z} \qquad (c, \text{ complex }, z \neq 0)$$

Principal value of z^c : $z^c = e^{c\text{Ln}z}$

$$\text{If } c = 1/n \text{ where } n = 2, 3, \cdots.$$
$$z^c = \sqrt[n]{z} = e^{(1/n)\ln z}$$

Example 2. General power.

$$i^i == e^{i \ln z} = \exp(i \ln i) = \exp[i(\frac{\pi}{2}i \pm 2n\pi i)] = e^{-(\pi/2) \mp 2n\pi}$$

$$\text{principal value } (n = 0) : e^{-\pi/2}$$

$$\begin{aligned}
(1+i)^{2-i} &= \exp[(2-i)\ln(1+i)] \\
&= \exp[(2-i)\ln\sqrt{2} + \frac{1}{4}\pi i \pm 2n\pi i] \\
&= 2e^{\pi/4 \pm 2n\pi}[\sin(\frac{1}{2}\ln 2) + i\cos(\frac{1}{2}\ln 2)]
\end{aligned}$$

For any complex number a ,
(8)

$$a^z = e^{z \ln a}$$

23.3 Line Integral in the Complex Plane

indefinite integral : a function whose derivative equals a given analytic function in a region.

Complex definite integral : line integrals.

$$\int_c f(z)dz$$

$f(z)$: integrand.
C : path of integration.

(1)

$$z(t) = x(t) + iy(t) \qquad\qquad (a \leq t \leq b)$$

$$: \text{ parametric representation of } a \text{ curve } C$$

increasing t : positive sense on C, (1) orients C.
smooth curve, C : C has a continuous and nonzero derivative $\dot{z} = dz/dt$ at each point.

Definition of the Complex Line Integral
- We partition the interval
$a \leq t \leq b$ in (1) $t_0(= a), t_1, \cdots, t_{n-1}, t_n(= b)$.
where $t_0 < t_1, \cdots < t_n$.
- Corresponding subdivision of C by points.
$z_0, z_1, \cdots, z_{n-1}, z_n(= Z)$.
where $z_j = z_{(tj)}$
- We choose an arbitrary point, a point $z_{m-1} < \zeta_m < z_m$

(2)

$$S_n = \sum_{m=1}^{n} f(\zeta_m)\triangle z_m \text{ where } \triangle z_m = z_m - z_{m-1}$$

$$\lim_{\Delta z_m \to 0} S_n = \int_c f(z)dz \qquad c: \text{ path of integration.}$$

$$\text{or } \oint_c f(z)dz \qquad \text{if } c \text{ is a closed path.}$$

General Assumption : All paths of integration for complex line integrals are assumed to be "piecewise smooth", that is, they consist of finitely many smooth curves joined end to end.

Three Basic Properties Directly Implied by the Definition.
1. Linearity

(3)

$$\int_c [k_1 f_1(z) + k_2 f_2(z)]dz = k_1 \int_c f_1(z)dz + k_2 \int_c f_2(z)dz$$

2. Sense reversal

(4)

$$\int_{z_0}^{z} f(z)dz = -\int_{z}^{z_0} f(z)dz.$$

3. Partitioning of path

(5)

$$\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz.$$

Existence of the Complex Line Integral

$$f(z) = u(x,y) + iv(x,y)$$

$$\text{set } \zeta_m = \xi_m + i\eta_m \text{ and } \Delta z_m = \Delta x_m + i\Delta y_m$$

from (2)

(6)

$$S_n = \sum (u + iv)(\Delta x_m + i\Delta y_m)$$

where

$$u = u(\xi_m, \eta_m), v = v(\xi_m, \eta_m)$$

$$S_n = \sum u\Delta x_m - \sum v\Delta y_m + i[\sum u\Delta y_m + \sum v\Delta x_m]$$

- sums are real
- since f is continuous, u and v are continuous

(7)

$$\lim_{n\to\infty} S_n = \int_c f(z)dz = \int_c udx - \int_c vdy + i[\int_c udy + \int_c vdy]$$