25 Cauchy's Integral Formula

25.1 Independence of Path

Theorem 1 (Independence of path)
If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

Proof.
c_2^* : the path c_2 with the orientation reversed.
$c_1 + c_2^*$: simple closed path.
Cauchy's theorem applies.
(2')

$$\int_{c_1} f dz + \int_{c_2^*} f dz = 0 \Rightarrow \int_{c_1} f dz = -\int_{c_2^*} f dz$$

(2)

$$\therefore \int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$

This proves the theorem for paths that have only the endpoints in common.

Principle of Deformation of Path.
Hence we may impose a continuous deformation of the path of an integral, keeping the ends fixed. As long as our deforming path always contains only points at which $f(z)$ is analytic, the integral retains the same value. This is called the principle of path.

Existence of Indefinite Integral.

Theorem 2 (Existence of an infinite integral)

If $f(z)$ is analytic in a simply connected domain D, then there exists an indefinite integral $F(z)$ of $f(z)$ in D-thus, $F'(z) = f(z)$- which is analytic in D, and for al paths in D joining any two points z_0 and z_1 in D, the integral of $f(z)$ from z_0 to z_1 can be evaluated by

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0). \quad [F'(z) = f(z)]$$

Proof. If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D
(3)

$$F(z) = \int_{z_0}^{z_1} f(z^*)dz^*,$$

which is uniquely determined. We show that this $F(z)$ is analytic in D and $F'(z) = f(z)$

(4)

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(z^*)dz^* - \int_{z_0}^{z} f(z^*)dz^*\right] = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^*)dz^*$$

$$(4) - f(z) : \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^*)dz^* - f(z) \qquad ---a)$$

Show that R.H.S approaches zero as $\Delta z \to 0$
$f(z)$ is a constant because z is kept fixed

$$\int_{z}^{z+\Delta z} f(z)dz^* = f(z)\int_{z}^{z+\Delta z} dz^* = f(z)\Delta z.$$

$$\text{Thus} \qquad f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)dz^* \qquad ---b)$$

$$b) \to a) \qquad \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(z^*)-f(z)]dz^*$$

Since $f(z)$ is analytic, it is continuous. An $\varepsilon > 0$ being given, we can thus find a $\delta > 0$ such that $|f(z^*) - f(z)| < \varepsilon$ when $|z^* - z| < \delta$. Hence, letting $|\Delta z| < \delta$, we see that the ML-inequality yields

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| = \frac{1}{\Delta z}\left|\int_{z}^{z+\Delta z} [f(z^*)-f(z)]dz^*\right| \leq \frac{1}{|\Delta z|}\varepsilon|\Delta z| = \varepsilon.$$

By the definition of limit and derivative,

$$F'(z) = \lim_{\Delta z \to 0} \frac{F(+\Delta z) - f(z)}{\Delta z} = f(z)$$

Since z is any point in D, this implies that $F(z)$ is analytic in D and is an indefinite integral on antiderivative of $f(z)$ in D, written

$$F(z) = \int f(z)dz$$

Also of $G'(z) = f(z)$, then $F'(z) - G'(z) \equiv 0$ in $D :$ hence $F(z) - G(z)$ is constant in D. Two indefinite integrals of $f(z)$ can differ only by a constant. This proves theorem.

Cauchy's Theorem for Multiply Connected Domains.
For a doubly connected domain D
(5)

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$

Proof.

$$D_1 : \int_{c_{1_0}} f(z)dz + \int_{\tilde{c_2}} f(z)dz + \int_{c_2^*} f(z)dz + \int_{\tilde{c_2}} f(z)dz = 0 \qquad ---1)$$

since $f(z)$ is analytic in D_1

$$D_2 : \int_1^* f(z)dz - \int_{\tilde{c_2}} f(z)dz + \int_{c_{2_0}} f(z)dz - \int_{\tilde{c_1}} f(z)dz = 0 \qquad ---2)$$

$$1) + 2); \quad \int_{z_{1_0}} f(z)dz + \int_{c_{1^*}} f(z)dz + \int_{c_{2^*}} f(z)dz + \int_{c_{2_0}} f(z)dz = 0$$

$$C_{1_0} + C_{1^*} = C_1(ccw), \quad C_{2^*} + C_{2_0} = C_2(cw)$$

$$\int_{c_1} f(z)dz - \int_{c_2} f(z)dz = 0 \quad \text{in both ccw.}$$

$$\therefore \int_{c_1} f(z)dz = \int_{c_2} f(z)dz.$$

Example 1. A basic result : Integral of integer power.

(6)

$$\oint (z - z_0)^m dz = \begin{cases} 2\pi i & (m = -1) \\ 0 & (m \neq -1 \text{ and integer}) \end{cases}$$

for ccw integration around any simple closed path containing z_0 in its interior.

25.2 Cauchy's Integral Formula

Theorem 3 (Cauchy's integral formula)
Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0,
(1)

$$\oint_c \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) \qquad \text{(Cauchy's integral formula)}$$

the integration being taken ccw.
(1*)

$$f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} dz \qquad \text{(Cauchy's integral formula)}$$

Proof.

$$f(z) = f(z_0) + [f(z) - f(z_0)]$$

(2)

$$\oint_c \frac{f(z)}{z - z_0} dz = f(z_0) \oint_c \frac{dz}{z - z_0} + \oint_c \frac{f(z) - f(z_0)}{z - z_0} dz$$

$$\oint \frac{dz}{z-z_0} = 2\pi i \quad \text{(Example 6 in sec.13.2)}$$

1^{st} term on the R.H.S

$$\therefore f(z_0)\oint_c \frac{dz}{z-z_0} = 2\pi i f(z_0)$$

C is replaced by a small circle k of radius ρ by the principle of deformation of path . Hence an $\varepsilon > 0$ being given, we can find a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ for all z in the disk $|z - z_0| < \delta$.

$$\therefore \left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \frac{\varepsilon}{\rho}$$

By the ML-inequality,

$$\left|\oint_k \frac{f(z)-f(z_0)}{z-z_0}dz\right| < \frac{\varepsilon}{\rho}2\pi\rho = 2\pi\varepsilon$$

Since $\varepsilon(> 0)$ can be chosen arbitrarily small, it follows that the above integral must have the value zero.

Example 2. Cauchy's integral formula.

$$\oint_c \frac{e^z}{z-2}dz = 2\pi i e^z|_{z=2} = 2\pi i e^2 \approx 46.4268i$$

for any contour enclosing $z_0 = 2$.

Example 3. Cauchy's integral formula.

$$\oint_c \frac{z^3-6}{2z-i}dz = \oint_c \frac{\frac{1}{2}z^3-3}{z-\frac{i}{2}}dz - 2\pi i(\frac{1}{2}z^3-3)|_{z=i/2} = \pi/8 - 6\pi i \quad (z_0 = i/2 \text{ inside } C)$$

Example 4. Integration around different contours.

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z-1)(z+1)}$$

Solution.
(a) circle $|z-1| = 1$, encloses $z_0 = 1$

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{z+1}\cdot\frac{1}{z-1} \;;\; f(z) = \frac{z^2+1}{z+1}$$

(b) gives the same as (a) by the principle of deformation of path
(c) $z_0 = -1$

$$g(z) = \frac{z^2+1}{z-1}\cdot\frac{1}{z+1} \;:\; \text{ thus } f(z) = \frac{z^2+1}{z-1}$$

(d) 0. $g(z)$ is analytic

Example 5. use of partial fractions.

$$g(z) = \frac{\tan z}{z^2 - 1} \qquad : \text{ the circle } C : |z| = 3/2 \ \ (\text{ccw})$$

Solution. $\tan z$ is not analytic at $\pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside the contour.

$$(z^2 - 1)^{-1} = 1/(z-1)(z+1) \text{ is not analytic at 1 and -1}$$

$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$

$$\begin{aligned} \oint \frac{\tan z}{z^2-1} dz &= \frac{1}{2}\left[\oint \frac{\tan z}{z-1} dz - \oint \frac{\tan z}{z+1} dz\right] \\ &= \frac{1\pi i}{2}[\tan 1 - \tan(-1)] = 2\pi i \tan 1 \approx 9.785 i \end{aligned}$$

Multiply connected domain.
For instance, if $f(z)$ is analytic on C_1 and C_2 and in the ring-shaped domain bounded by C_1 and C_2 and z_0 is any point in that domain, then
(3)

$$f(z_0) = \frac{1}{2\pi i} \oint_{c_1} \frac{f(z)}{z - z_0} dz + \frac{1}{2\pi i} \oint_{c_2} \frac{f(z)}{z - z_0} dz,$$

where the outer integral C_1 is taken ccw and the inner clockwise.