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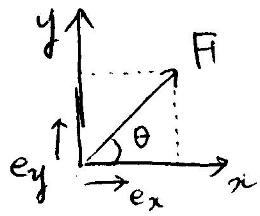
by R Zimmerman

mathematical aspects of rock mechanics.

stress — let's start by talking about forces.

elementary idea about vector is that it is a vector



$$\underline{F} = F_x \underline{e}_x + F_y \underline{e}_y = (\underline{F}\cdot\underline{e}_x)\underline{e}_x + (\underline{F}\cdot\underline{e}_y)\underline{e}_y = (F_x, F_y)$$

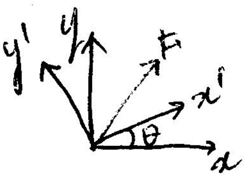
express in terms of F

How do I find this F_x?

$$F_x = \underline{F}\cdot\underline{e}_x = |F||e_x|\cos\theta.$$

What I am getting at is that ~.

what happens to this F if we change coordinate system?



$$F = F_x \underline{e}_x + F_y \underline{e}_y = F_{x'} \underline{e}_{x'} + F_{y'} \underline{e}_{y'}$$

$$(F_x \underline{e}_x + F_y \underline{e}_y)\cdot \underline{e}_{x'} = (F_{x'} \underline{e}_{x'} + F_{y'} \underline{e}_{y'})\,\underline{e}_{x'}$$

probably don't need to consider.

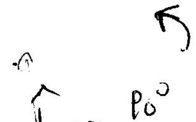
$$F_{x'} = \underline{F}\cdot\underline{e}_{x'} = \underline{F}'\underline{e}_{x'} \qquad = F_{x'}$$

$$F_x \underbrace{(e_x\cdot e_{x'})}_{\cos\theta} + F_y \underbrace{(e_y\cdot e_{x'})}_{\sin\theta}$$

$$F_{x'} = F_x\cos\theta + F_y\sin\theta.$$

Similarly, $F_{y'} = -F_x\sin\theta + F_y\cos\theta$

$$\begin{pmatrix}F_{x'}\\F_{y'}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}F_x\\F_y\end{pmatrix}$$



$90°$ $(1.0) \rightarrow \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\-1\end{pmatrix}$

$$\underline{F}' = \underline{\underline{R}}\,\underline{F}$$

two line is matrix.

all we have to do is to premultiply the rotation matrix.

$$\underline{\underline{R}} = \begin{pmatrix}\cos\measuredangle(e_x, e_{x'}) & \cos\measuredangle(e_y, e_{x'})\\ \cos\measuredangle(e_x, e_{y'}) & \cos\measuredangle(e_y, e_{y'})\end{pmatrix} \quad (e_z, e_{x'})$$

angle

$(e_x, e_{z'})$

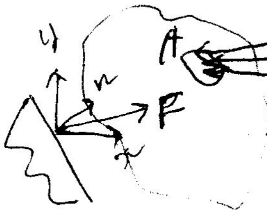
IP any mathematical entity, F, that obeys the relation $\underline{F}' = \underline{\underline{R}}\,\underline{F}$ when —.

intermediate concept to go from force to stress.

Traction.

consider a distributed force, $\underline{F}$, acting over a surface of area, A.

ski and shoes



average traction vector $= \underset{\sim}{T} = \dfrac{\underline{F}}{A}$.

= stress vector
= traction vector.

$$\underset{\sim}{T} = \lim_{\Delta A \to 0} \left(\frac{\Delta \underline{F}}{\Delta A} \right) \quad N/m^2 = Pa.$$

Let's think out ab^it

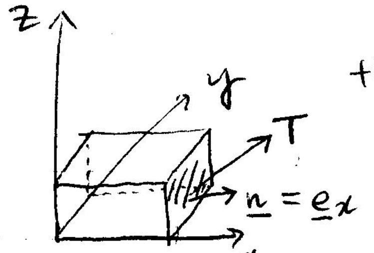
Traction depend on position and surface.

$\underset{\sim}{T}(\underset{\sim}{x}, \underset{\sim}{n})$ express in term of $\underline{x}$ and $\underline{n}$.

?

16 cy → cannot do well.

1823. Cauchy. invented the idea of stress.



the surfaces can be imaginary.

T (on face $\perp$ to e_x) $= (T_x, T_y, T_z)$

face acting x direction? definition Def: $(T_x, T_y, T_z)_{\text{on face} \perp \text{to } e_x} = (\tau_{xx}, \tau_{yx}, \tau_{zx})$

$$T(\underline{n} = e_x) = (\tau_{xx}, \tau_{yx}, \tau_{zx}) \rightarrow (\tau_{xx}, \tau_{xy}, \tau_{xz})$$

$$T(\underline{n} = e_z) = (\tau_{xz}, \tau_{yz}, \tau_{zz}) \rightarrow (\tau_{zx}, \tau_{zy}, \tau_{zz})$$

$$T(\underline{n} = e_y) = (\tau_{xy}, \tau_{yy}, \tau_{zy}) \rightarrow (\tau_{yx}, \tau_{yy}, \tau_{yz})$$

$$\underset{\sim}{\tau} = \begin{array}{c} \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \\ \begin{array}{ccc} T(e_x) & T(e_y) & T(e_z) \end{array} \end{array}$$

← stress matrix

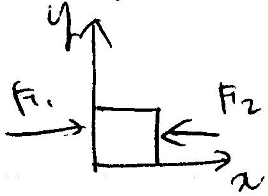
→ stress components.

normal stress components.

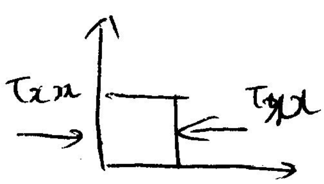
tangential tractions, or shear stress.

Using σ & τ is a bit obscure.

?



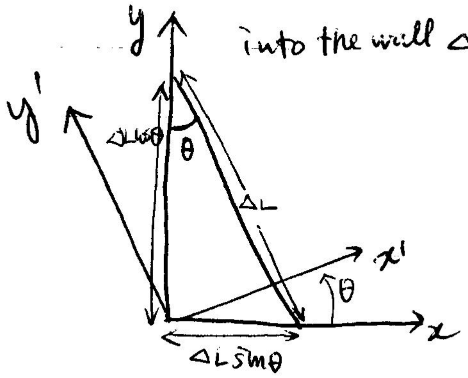
F_1 and F_2 has opposite sign.

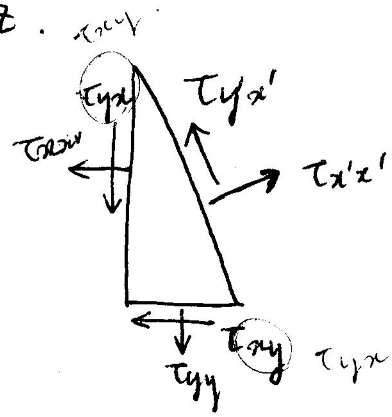


Stress follows different convention.

if $\underset{\sim}{n}$ is pointing in a positive coordinate direction, then the components of the stress on that surface are + if they ~~also~~ are point in the + coordinate direction.

into the wall Δz.





$$\sum F(\text{in } x' \text{ direction}) = 0$$

$\tau_{xx}\, \Delta z \cdot \Delta L \cos\theta \cos(180-\theta) = -\tau_{xx} \cos^2\theta\, \Delta z\, \Delta L$

$\tau_{yx} \cdot \Delta z \cdot \Delta L \cos\theta \cos(90+\theta) = -\tau_{yx} \cos\theta \sin\theta\, \Delta z \Delta L$

$\tau_{yy}\, \Delta z \cdot \Delta L \sin\theta \cos(90+\theta) = -\tau_{yy} \sin^2\theta\, \Delta z \Delta L.$

$\tau_{xy} \cdot \Delta z \cdot \Delta L \sin\theta \cos(180-\theta) = -\tau_{xy} \sin\theta \cos\theta\, \Delta z\, \Delta L$

$\tau_{x'x'}\, \Delta z \cdot \Delta L.$

$\tau_{xx} \cos^2\theta\, \Delta z \Delta L + \tau_{xy} \Delta z \Delta L + \tau_{yy} \sin$

drop out $\Delta z \Delta L$.

$$\tau_{x'x'} = \tau_{xx} \cos^2\theta + \tau_{yy} \sin^2\theta + (\tau_{xy} + \tau_{yx}) \sin\theta \cos\theta.$$

similarly

$$\tau_{x'y'} = -(\tau_{xx} - \tau_{yy}) \sin\theta \cos\theta + \tau_{xy} \cos^2\theta - \tau_{yx} \sin^2\theta.$$

$$\tau_{y'x'} = -(\tau_{xx} - \tau_{yy}) \sin\theta \cos\theta + \tau_{yx} \cos^2\theta - \tau_{xy} \sin^2\theta.$$

$$\tau_{y'y'} = \tau_{xx} \sin^2\theta + \tau_{yy} \cos^2\theta - (\tau_{xy} + \tau_{yx}) \sin\theta \cos\theta$$

$\underline{\underline{\tau}}' = \underline{\underline{R}}\, \underline{\underline{\tau}}\, \underline{\underline{R}}^T \cdot = \underline{\underline{R}}^2 \underline{\underline{\tau}}$ switching the rows and columns. you can't move them around.

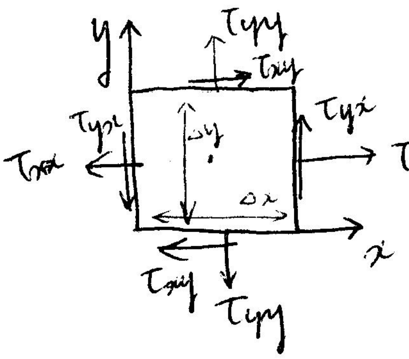
HW #1 $\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \longrightarrow$ Same !

R $\underline{\underline{\tau}} = \underline{\underline{\tau}}^T$ i.e. $\underline{\underline{\tau}}$ is symmetric. $\tau_{xy} = \tau_{yx}$

$\rightarrow$ only 6 independent component in 3D $\quad \tau_{xz} = \tau_{zx}$

3 " 2D. $\quad \tau_{yz} = \tau_{zy}$.

different way of looking at it. cancel out the two of the denominator
Does everyone believe that?
delta x over 2.



$$\Sigma \text{ mont} = I \cdot \dot{\omega} = I\alpha = I\ddot{\theta}$$

$$\tau_{yx} \Delta y \Delta z \frac{\Delta x}{2} - \tau_{xy} \Delta x \Delta z \frac{\Delta y}{2}$$

$$\tau_{yx} \Delta y \Delta z \frac{\Delta x}{2} - \tau_{xy} \Delta x \Delta z \frac{\Delta y}{2} = \frac{m}{12}(\Delta x^2 + \Delta y^2)$$

$$\therefore \tau_{yx} = \tau_{xy} + \rho \frac{\Delta x \Delta y \Delta z}{12}((\Delta x)^2 + (\Delta y)^2)\dot{\omega}$$

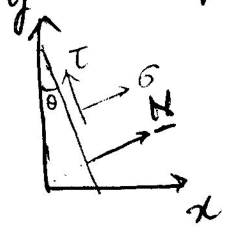
moment of inertia
$= \frac{m \cdot (\Delta x^2 + \Delta y^2)}{12}$

$$\tau_{yx} - \tau_{xy} = \frac{\rho}{12}((\Delta x)^2 + (\Delta y)^2)\dot{\omega}$$

when $\Delta x, \Delta y \to 0$

$$\tau_{yx} = \tau_{xy}.$$

as long as we assume that $\tau_{yx} = \tau_{xy}$, then our stresses will automatically satisfies the law of conservation of angular momentum.



assume that we find that the state of stress in the (x, y) coordinate system.

Q: What are the traction component acting on this plane?

Def: normal traction on this plane is "$\hat{\sigma}$"
shear " "τ".

$$\sigma = \tau_{x'x'}, \quad \tau = \tau_{y'x'} = \tau_{x'y'}.$$

$$\sigma = \tau_{xx}\cos^2\theta + \tau_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta.$$

$$\tau = (\tau_{yy} - \tau_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\cos^2\theta = \frac{1}{2} + \frac{\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1}{2} - \frac{\cos 2\theta}{2}$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta, \quad \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

$$\sigma = \tau_{xx}\frac{1}{2} + \tau_{xx}\frac{1}{2}\cos 2\theta + \tau_{yy}\frac{1}{2} - \tau_{yy}\frac{1}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \frac{1}{2}(\tau_{xx}+\tau_{yy}) + \left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau = -\left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

Q: Is there a plane on which σ or τ are maximum?

take the derivative with respect to θ.

$$\frac{d\sigma}{d\theta} = -(\tau_{xx}-\tau_{yy})\sin 2\theta + 2\tau_{xy}\cos 2\theta$$

factoring out by 2

$$\frac{d\sigma}{d\theta} = 2\left[-\left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta\right] = \underline{\underline{2\tau}}\ !$$

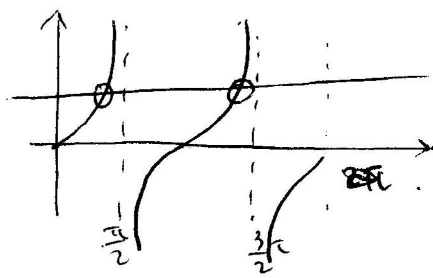
as a side issue to all the way.

$$\frac{\tau_{xx}-\tau_{yy}}{2}\sin 2\theta = \tau_{xy}\cos 2\theta$$

$$\frac{2\tau_{xy}}{\tau_{xx}-\tau_{yy}} = \tan 2\theta$$ take inverse tangent.

$$2\theta = \arctan\left(\frac{2\tau_{xy}}{\tau_{xx}-\tau_{yy}}\right)$$

$$\theta = \frac{1}{2}\arctan\left(\frac{2\tau_{xy}}{\tau_{xx}-\tau_{yy}}\right)$$



it is going to be two data.
2 roots differ by π.
$2\theta = 2\theta_1$ & $2\theta = 2\theta_1 + \pi$.
$\theta = \theta_1$ & $\theta_1 + \frac{\pi}{2}$

I will say more about that as we go on.

2 solution differ from eachother by 90°.
$\rightarrow$ principal stress direction.

HW #2. for the value see p.10.

$$\sigma_1 = \sigma_{max} = \frac{\tau_{xx}+\tau_{yy}}{2} + \sqrt{\left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)^2 + \tau_{xy}^2}, \quad \tau = 0$$

$$\sigma_2 = \sigma_{min} = \quad " \quad - \sqrt{\quad}, \quad \tau = 0$$

keep in the back of your mind.

Q: On what plane is τ a maximum or minimum?

$$\frac{d\tau}{d\theta} = -(\tau_{xx}-\tau_{yy})\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

if I keep man... that!

$$-\frac{(\tau_{xx}-\tau_{yy})}{2\tau_{xy}} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

$$= -\frac{1}{\tan 2\theta_{max}}, \quad \Rightarrow$$ these two angles are perpendicular each other.

$\theta, \theta+90$. $$\tan(90+\theta) = \frac{\sin(90+\theta)}{\cos(90+\theta)} = \frac{\cos\theta}{-\sin\theta} = -\frac{1}{\tan\theta}$$

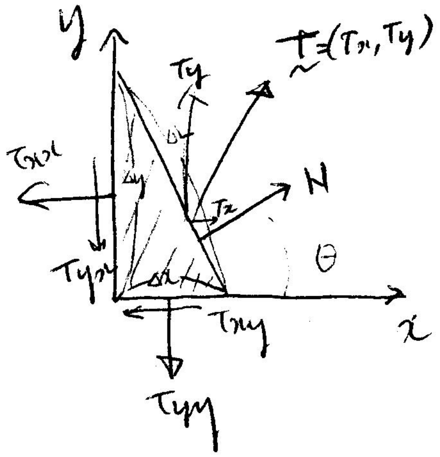
$$\tan\theta \tan(90+\theta) = -1.$$

R: place on which τ is a maximum is at $45°$ rotation from the places on which σ is a maximum (min)

if we take one new coordinate system to be aligned with the 2 principal stress directions, then the stress tensor looks like.

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

this may mislead to believe there is no shear in a rock, but there is!



$$\Sigma F_x = -\tau_{xx}\Delta y\Delta z - \tau_{xy}\Delta x\Delta z + T_x \Delta L\Delta z = 0.$$

$$\tau_{xx}\Delta y\Delta z + \tau_{xy}\Delta x\Delta z = T_x \Delta L\Delta z$$

$$\tau_{xx}\frac{\Delta y}{\Delta L} + \tau_{xy}\frac{\Delta x}{\Delta L} = T_x.$$

$$\tau_{xx}\cos\theta + \tau_{xy}\sin\theta = T_x.$$

$$\underline{N} = (\cos\theta, \sin\theta) = (n_x, n_y)$$

$$T_x = \tau_{xx} n_x + \tau_{xy} n_y.$$

$$T_y = \tau_{yx} n_x + \tau_{yy} n_y.$$

all we get is, in fact, —.

it never turns out that way.

explain that out.

$$\begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \end{pmatrix}.$$

$$\boxed{\underline{T} = \underline{\underline{\tau}} \cdot \underline{n}}$$ ☆ Cauchy's formula.

Q: Can we find a plane on which T is purely parallel ~~normal~~ to N?

$\underline{T} = \lambda \underline{n}$

Does that help us? It does help us!

because $\underline{T} = \underline{\underline{\tau}} \underline{n}$

$\underline{\underline{\tau}} \cdot \underline{n} = \lambda \cdot \underline{n} = \lambda \underline{\underline{I}} \underline{n}$, $(\underline{\underline{\tau}} - \lambda \underline{\underline{I}}) \underline{n} = 0$.

$Ax = b$. $x = A^{-1} b$. $\Rightarrow$ $n = (\tau - \lambda \underline{\underline{I}})^{-1} \cdot 0 = 0$.

what was wrong was that we assumed that A^{-1} existed.

$\Rightarrow (\tau - \lambda \underline{\underline{I}})^{-1}$ does not exist. $\rightarrow$ $\det(\underline{\underline{\tau}} - \lambda \underline{\underline{I}}) = 0$.

$\det(\tau - \sigma I) = 0$.

We want to find a σ such that $\det(\tau - \sigma I) = 0$.

Because $\underline{\underline{\tau}}$ is symmetric (from linear algebra)

① There will always be 3 real roots for σ.

② each of these 3 values of σ has its own $\underline{n}$.

③ all 3 of these n vector are normal to each other.

$\underline{\underline{\tau}} - \sigma \underline{\underline{I}} = \begin{pmatrix} \tau_{xx} - \sigma & \tau_{xy} \\ \tau_{yx} & \tau_{yy} - \sigma \end{pmatrix}$, $\det(\tau - \sigma I) = (\tau_{xx} - \sigma)(\tau_{yy} - \sigma) - \tau_{xy}^2 = 0$.

$$\sigma^2 - \underline{\underline{(\tau_{xx} + \tau_{yy})}} \sigma + \underline{\underline{(\tau_{xx}\tau_{yy} - \tau_{xy}^2)}} = 0.$$

$$\sigma = \frac{\sigma_{xx} + \tau_{yy}}{2} \pm \sqrt{\left(\frac{\tau_{xx} - \tau_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

Same as before, but much easier!

Invariant (trace of $\underline{\underline{\tau}}$) $\det(\tau)$ #3 prove $\det(\tau)$ is invariant.

σ_1, σ_2 are not dependent on coordinate system! Otherwise physically it does not make sense.

That is where "invariant" comes!

$$\sigma^2 - \text{trace}(\tau)\sigma + \det(\underline{\underline{\tau}}) = 0$$

why invariant important? eg failure criteria.

$f(\sigma_1, \sigma_2, \sigma_3)$ ~~e~~ better to write like $f(I_1, I_2, I_3)$

because failure does not depend on coordinate system.

hw #4. $\tau = \begin{pmatrix} 4 & 1 \\ \phi & 4 \end{pmatrix}$

find σ_1, σ_2 & n_1, n_2.

using 1) transformation eqns.

2) eigenvalue method.

homework #1. Calculation of $\underline{\underline{\tau}}' = \underline{\underline{R}}\,\underline{\underline{\tau}}\,\underline{\underline{R}}^T$

$$\begin{pmatrix} \tau_{x'x'} & \tau_{x'y'} \\ \tau_{y'x'} & \tau_{y'y'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\,\tau_{xx} + \sin\theta\,\tau_{yx} & \cos\theta\,\tau_{xy} + \sin\theta\,\tau_{yy} \\ -\sin\theta\,\tau_{xx} + \cos\theta\,\tau_{yx} & -\sin\theta\,\tau_{xy} + \cos\theta\,\tau_{yy} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta\,\tau_{xx} + \cos\theta\sin\theta\,\tau_{yx} + \cos\theta\sin\theta\,\tau_{xy} + \sin^2\tau_{yy} & -\cos\theta\sin\theta\,\tau_{xx} - \sin^2\tau_{yx} + \cos^2\tau_{xy} + \cos\theta\sin\theta\,\tau_{yy} \\ -\cos\theta\sin\theta\,\tau_{xx} + \cos^2\tau_{yx} - \sin^2\tau_{xy} + \cos\theta\sin\theta\,\tau_{yy} & \sin^2\theta\,\tau_{xx} - \cos\theta\sin\theta\,\tau_{yx} - \cos\theta\sin\theta\,\tau_{xy} + \cos^2\tau_{yy} \end{pmatrix}$$

$$\therefore \left\{ \begin{aligned} \tau_{x'x'} &= \tau_{xx}\cos^2\theta + \tau_{yy}\sin^2\theta + (\tau_{xy}+\tau_{yx})\cos\theta\sin\theta \\ \tau_{x'y'} &= -(\tau_{xx}-\tau_{yy})\cos\theta\sin\theta + \tau_{xy}\cos^2\theta - \tau_{yx}\sin^2\theta \\ \tau_{y'x'} &= -(\tau_{xx}-\tau_{yy})\cos\theta\sin\theta + \tau_{yx}\cos^2\theta - \tau_{xy}\sin^2\theta \\ \tau_{y'y'} &= \tau_{xx}\sin^2\theta + \tau_{yy}\cos^2\theta - (\tau_{xy}+\tau_{yx})\cos\theta\sin\theta \end{aligned} \right\} \cdots ①$$

above equations are the same as the one solved by force equilibrium.

homework #2. Prove that $\det(\tau)$ is invariant.

$$\det(\tau) \text{ in } xy \text{ coordinate} = \det\begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix} = \tau_{xx}\tau_{yy} - \tau_{xy}^2$$

$$\det(\tau) \text{ in } x'y' \text{ coordinate} = \det\begin{pmatrix} \tau_{x'x'} & \tau_{x'y'} \\ \tau_{x'y'} & \tau_{y'y'} \end{pmatrix} = \tau_{x'x'}\tau_{y'y'} - \tau_{x'y'}^2$$

$$= \{\tau_{xx}\cos^2\theta + \tau_{yy}\sin^2\theta + 2\tau_{xy}\cos\theta\sin\theta\}\{\tau_{xx}\sin^2\theta + \tau_{yy}\cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta\}$$
$$- \{-(\tau_{xx}-\tau_{yy})\cos\theta\sin\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)\}^2$$

$$= \cancel{\tau_{xx}^2\cos\theta\sin^2\theta} + \tau_{xx}\tau_{yy}\sin^4\theta + \cancel{2\tau_{xx}\tau_{xy}\cos\theta\sin^3\theta} + \tau_{xx}\tau_{yy}\cos^4\theta + \cancel{\tau_{yy}^2\cos^2\theta\sin^2\theta} + \cancel{2\tau_{yy}\tau_{xy}\cos^3\theta\sin\theta}$$
$$\cancel{-2\tau_{xx}\tau_{xy}\cos^3\theta\sin\theta} \cancel{-2\tau_{yy}\tau_{xy}\sin^3\theta\cos\theta} - 4\tau_{xy}^2\cos^2\theta\sin^2\theta \cancel{-\tau_{xx}^2\cos^2\theta\sin^2\theta} + 2\tau_{xx}\tau_{yy}\cos^2\theta\sin^2\theta$$
$$\cancel{-\tau_{yy}^2\cos^2\theta\sin^2\theta} - \tau_{xy}^2\cos^4\theta - \tau_{xy}^2\sin^4\theta + 2\tau_{xy}^2\cos^2\theta\sin^2\theta + \cancel{2\tau_{xx}\tau_{xy}\cos^3\theta\sin\theta} \cancel{-2\tau_{yy}\tau_{xy}\cos^3\theta\sin\theta}$$
$$\cancel{-2\tau_{xx}\tau_{xy}\cos\theta\sin^3\theta} + \cancel{2\tau_{yy}\tau_{xy}\cos\theta\sin^3\theta}$$

$$= \tau_{xx}\tau_{yy}\sin^4\theta + \tau_{xx}\tau_{yy}\cos^4\theta - \tau_{xy}^2\cos^4\theta - \tau_{xy}^2\sin^4\theta - 2\tau_{xy}^2\cos^2\theta\sin^2\theta + 2\tau_{xx}\tau_{yy}\cos^2\theta\sin^2\theta$$

$$= \tau_{xx}\tau_{yy}(\sin^2\theta + \cos^2\theta)^2 - \tau_{xy}^2\cdot(\cos^2\theta + \sin^2\theta)^2$$

$$= \tau_{xx}\tau_{yy} - \tau_{xy}^2$$

$\therefore \det(\tau)$ is invariant!

homework #4) $\tau = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ find $\sigma_1, \sigma_2, n_1, n_2$.

$\tau \cdot n = \sigma \cdot n \rightarrow (\tau - \sigma I)n = 0$.

$\det(\tau - \sigma I) = 0$, $\det\begin{pmatrix} 4-\sigma & 1 \\ 1 & 4-\sigma \end{pmatrix} = 0$

$\sigma^2 - 8\sigma + 15 = 0$, $\sigma = 3$ or $5 \rightarrow \sigma_1 = 5, \sigma_2 = 3$

when $\sigma = 3$, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} n_x \\ n_y \end{pmatrix} = 0$. $n_x + n_y = 0$, $n_y = -n_x$. $\sqrt{n_x^2 + n_y^2} = 1$.

$n_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

when $\sigma = 5$, $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} n_x \\ n_y \end{pmatrix} = 0$. $n_x = n_y$, $\sqrt{n_x^2 + n_y^2} = 1$

$n_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Furthermore $n_1 \cdot n_2 = 0$ meaning $\underline{n}_1$ and $\underline{n}_2$ are perpendicular each other.

$$\sigma_\theta = \frac{1}{2}(\tau_{xx}+\tau_{yy}) + \frac{\tau_{xx}-\tau_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

From $\sin 2\theta = \pm \dfrac{2\tau_{xy}}{\sqrt{4\tau_{xy}^2+(\tau_{xx}-\tau_{yy})^2}}$, $\cos 2\theta = \pm \dfrac{\tau_{xx}-\tau_{yy}}{\sqrt{4\tau_{xy}^2+(\tau_{xx}-\tau_{yy})^2}}$

$$\begin{aligned}
\sigma_\theta &= \frac{1}{2}(\tau_{xx}+\tau_{yy}) \pm \frac{1}{2}\frac{(\tau_{xx}-\tau_{yy})^2}{\sqrt{4\tau_{xy}^2+(\tau_{xx}-\tau_{yy})^2}} \pm 2\cdot\frac{\tau_{xy}^2}{\sqrt{4\tau_{xy}^2+(\tau_{xx}-\tau_{yy})^2}} \\
&= \frac{1}{2}(\tau_{xx}+\tau_{yy}) \pm \frac{1}{2}\frac{4\tau_{xy}^2+(\tau_{xx}-\tau_{yy})^2}{\sqrt{4\tau_{xy}^2+(\tau_{xx}-\tau_{yy})^2}} \\
&= \frac{1}{2}(\tau_{xx}+\tau_{yy}) \pm \sqrt{\left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)^2+\tau_{xy}^2}
\end{aligned}$$

$$\begin{aligned}
\tau &= -\frac{(\tau_{xx}-\tau_{yy})}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \\
&= \mp\left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)\frac{2\tau_{xy}}{\sqrt{4\tau_{xy}^2+(\tau_{xx}-\tau_{yy})^2}} \pm \tau_{xy}\cdot\frac{(\tau_{xx}-\tau_{yy})}{\sqrt{4\tau_{xy}^2+(\tau_{xx}-\tau_{yy})^2}} \\
&= 0
\end{aligned}$$

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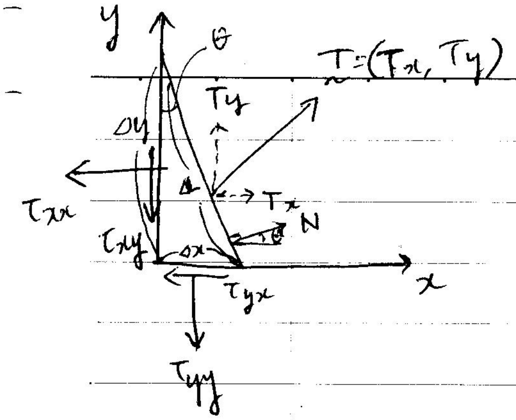

$$\Sigma F_x = -\tau_{xx}\cdot\Delta y\cdot\Delta z - \tau_{yx}\,\Delta x\,\Delta z + T_x\cdot\Delta L\,\Delta z = 0.$$

$$\tau_{xx}\,\Delta y\,\Delta z + \tau_{yx}\cdot\Delta x\,\Delta z = T_x\cdot\Delta L\,\Delta z.$$

$$\downarrow$$

$$\tau_{xx}\cdot\frac{\Delta y}{\Delta L} + \tau_{yx}\cdot\frac{\Delta x}{\Delta L} = T_x$$

$$\tau_{xx}\cdot\cos\theta + \tau_{yx}\cdot\sin\theta = T_x.$$

$$\underset{\sim}{N} = (\cos\theta, \sin\theta) = (n_x, n_y).$$

$$T_x = \tau_{xx}\cdot n_x + \tau_{yx}\cdot n_y$$

$$T_y = \tau_{xy}\cdot n_x + \tau_{yy}\cdot n_y$$

$$\begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{pmatrix} \tau_{xx} & \tau_{yx} \\ \tau_{xy} & \tau_{yy} \end{pmatrix}\begin{pmatrix} n_x \\ n_y \end{pmatrix} \qquad \underbrace{\underline{T} = \underline{\underline{\tau}}\cdot\underline{n}}_{\text{Cauchy's formula}}, \qquad T_i = \tau_{ji}\cdot n_j$$

Knowing the component $\underline{\underline{\tau}}$, we can write down at once the traction vector (stress vector) acting on any surface with unit outer normal vector $\underline{n}$ whose components are (n_x, n_y, n_z).
Cauchy's formula assures us that the nine components of stresses τ_{ij} are necessary and sufficient to define the traction across any surface element in a body.
Hence, the stress state in a body is characterized completely by a set of quantities τ_{ij}.

Q: Can we find a plane on which T is purely parallel to $\underline{n}$?

$\underline{T} = \lambda\cdot\underset{\sim}{n}$ → this doesn't help us.

Because $\underline{T} = \underline{\underline{\tau}}\cdot\underline{n}$

$$\underline{T} = \underline{\tau}\cdot\underline{n} = \lambda\cdot\underline{n} = \lambda\cdot\underset{\approx}{I}\cdot n, \Rightarrow (\underline{\underline{\tau}} - \lambda\underline{\underline{I}})\underline{n} = 0.$$

eigen value, eigen vector

$$\underset{\sim}{n} = (\underset{\approx}{\tau} - \lambda\underline{\underline{I}})^{-1}\cdot 0 = 0, \Rightarrow ???$$ something was wrong.

$(\underset{\approx}{\tau} - \lambda\underline{\underline{I}})^{-1}$ does not exist. → $\det(\underline{\underline{\tau}} - \lambda\underline{\underline{I}}) = 0$.

We want to find ~~σ~~ τ such that $\det(\tau - \lambda I) = 0$.

$\det(\tau - \sigma I) = 0$.

Scalar.

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Because $\underline{\underline{\tau}}$ is symmetric (from linear algebra),

in 3D.
① There will always be 3 real roots for τ
② each of these 3 values of τ has its own $\underset{\sim}{n}$
③ all 3 of these n vectors are normal to each other.

$$\underline{\underline{\tau}} - \sigma \underline{\underline{I}} = \begin{pmatrix} \tau_{xx} - \sigma & \tau_{yx} \\ \tau_{xy} & \tau_{yy} - \sigma \end{pmatrix}, \quad \det(\tau - \sigma I) = (\tau_{xx} - \sigma)(\tau_{yy} - \sigma) - \tau_{xy}^2 = 0.$$

$$\sigma^2 - (\tau_{xx} + \tau_{yy})\sigma + (\tau_{xx}\tau_{yy} - \tau_{xy}^2) = 0.$$

$$\sigma = \frac{\tau_{xx} + \tau_{yy}}{2} \pm \sqrt{\left(\frac{\tau_{xx} - \tau_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

same as before, but much easier!.

invariant (trace of $\underline{\underline{\tau}}$) invariant ($\det(\tau)$).

σ_1, σ_2 are not dependent on coordinate system!
Otherwise physically it does not make sense.
That's where the word 'invariant' comes!

$$\sigma^2 - \mathrm{trace}(\underline{\underline{\tau}})\sigma + \det(\underline{\underline{\tau}}) = 0.$$

why invariant important? e.g. failure criteria
$f(\sigma_1, \sigma_2, \sigma_3)$ → better to write like $f(I_1, I_2, I_3)$

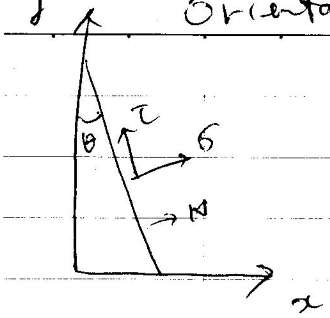
in 3D, $\sigma^3 - I_1\sigma^2 - I_2\sigma - I_3 = 0$.

$$I_1 = \tau_{xx} + \tau_{yy} + \tau_{zz}$$
$$I_2 = \tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2 - \tau_{xx}\tau_{yy} - \tau_{xx}\tau_{zz} - \tau_{yy}\tau_{zz}$$
$$I_3 = \tau_{xx}\tau_{yy}\tau_{zz} + 2\tau_{xy}\tau_{xz}\tau_{yz} - \tau_{xx}\tau_{yz}^2 - \tau_{yy}\tau_{xz}^2 - \tau_{zz}\tau_{xy}^2$$

#3 $\tau = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ find σ_1, σ_2, & n_1, n_2.

using 1) transformation eqn.
2) eigenvalue method.

Orientation & characteristics of principal stress Date No.



assume that we know the state of stress in the (x, y) coordinate.

Q: What are the traction compoment acting on this plane?

normal traction on this plane : σ

shear " : τ.

$$\sigma = \tau_{xx}\cos^2 + \tau_{yy}\sin^2\theta + 2\,\tau_{xy}\sin\cos\theta$$

$$\tau = (\tau_{yy}-\tau_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta).$$

By double angle ~~theorem~~ formulas,

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}, \quad \sin 2\theta = 2\sin\cos$$

$$\begin{cases} \sigma = \tau_{xx}\cdot\frac{1}{2} + \tau_{xx}\cdot\frac{1}{2}\cos 2\theta + \tau_{yy}\frac{1}{2} - \tau_{yy}\frac{1}{2}\cos 2\theta + \tau_{xy}\cdot\sin 2\theta. \\ \quad = \frac{1}{2}\cdot(\tau_{xx}+\tau_{yy}) + \frac{1}{2}(\tau_{xx}-\tau_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau = -\frac{1}{2}(\tau_{xx}-\tau_{yy})\sin 2\theta + \tau_{xy}\cdot\cos 2\theta. \end{cases}$$

Q: Is there a plane on which σ or τ are maximum?

take the derivative with respect to θ,

$$\frac{\partial\sigma}{\partial\theta} = -(\tau_{xx}-\tau_{yy})\sin 2\theta + 2\tau_{xy}\cos 2\theta$$

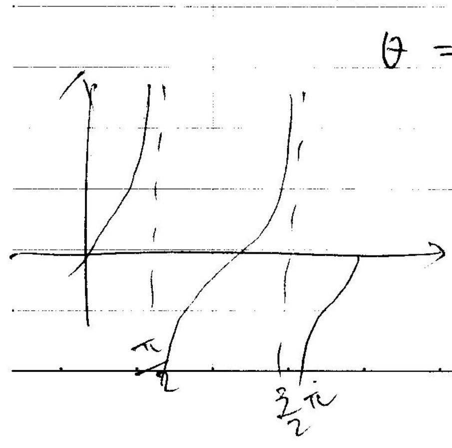
$$= 2\cdot\left(-\left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta\right) = \underline{\underline{2\tau}} = 0.$$

!

$$\frac{\tau_{xx}-\tau_{yy}}{2}\sin 2\theta = \tau_{xy}\cos 2\theta.$$

$$\frac{2\tau_{xy}}{\tau_{xx}-\tau_{yy}} = \underline{\underline{\tan 2\theta}}.$$

$$\theta = \frac{1}{2}\arctan\left(\frac{2\tau_{xy}}{\tau_{xx}-\tau_{yy}}\right).$$



it is going to be 2 data.

2 roots differ by π.

$2\theta = 2\theta_1, \quad 2\theta = 2\theta_1 + \pi.$

$\theta = \theta_1$ & $\theta_1 + \frac{\pi}{2}$

2 solution differ from each other by $90°$.

→ principal stress direction.

$$\sin 2\alpha = \pm \frac{2\tau_{xy}}{\sqrt{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}},$$

$$\cos 2\alpha = \pm \frac{\sigma_x - \sigma_y}{\sqrt{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}}.$$

$$\tau = 0, \quad \sigma_{max} = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau = 0 \quad \sigma_{min} = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

Q: On what plane, is τ a maximum or minimum?

$$\frac{\partial \tau}{\partial \theta} = -(\tau_{xx} - \tau_{yy}) \cos 2\theta - 2\tau_{xy} \sin 2\theta = 0$$

$$-\frac{(\tau_{xx} - \tau_{yy})}{2\tau_{xy}} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = -\frac{1}{\tan 2\theta_{max}}.$$

$$\tan 2\theta \times \tan 2\theta_{max} = -1,$$

$\tan(90+\theta) = \cdots = -\frac{1}{\tan\theta}$. θ & θ_{max}, $2\theta = 2\theta_{max} + 90°$

$$\theta = \theta_{max} + 45°.$$

A: Place on which τ is maximum is at $45°$ rotation from the place on which σ is a maximum.

if we take one new coordinate system to be aligned with the 2 principal stress directions, then the stress tensor looks like

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

this may mislead to believe there is no shearing in a rock, but there is!