16 March 2009

Lecture 3. Cartesian Tensor notation.

easy to see patterns. using indicial notation.

now coordinate is 1, 2, 3 instead of x, y, z.

$(x, y, z) \longrightarrow (x_1, x_2, x_3)$

$(u, v, w) \longrightarrow (u_1, u_2, u_3)$

① vector

$\underline{x} = (x, y, z) = (x_1, x_2, x_3) = \underline{\underline{x_i}} \quad (i = 1, 2, 3)$

shorthand of (x_1, x_2, x_3)

$\underline{x}^* = \underline{x} + \underline{u} = (x+u, y+v, z+w) \Rightarrow x^*_i = x_i + u_i$

② 2nd order tensor

$$\underline{\underline{T}} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = T_{ij} \quad \begin{pmatrix} i : 1 \to 3 \\ j : 1 \to 3 \end{pmatrix}$$

Advantage : $\vec{A}$ $\underline{\underline{A}}$ are not needed.

elegant ~~short~~ ~~descrip~~ representation.

*) Dot Product.

$\underline{x} = (x_1, x_2, x_3) \qquad \underline{y} = (y_1, y_2, y_3)$

$\underline{x} \cdot \underline{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 = \sum_{i=1}^{3} x_i y_i$

summation convention ~~notation~~

$\equiv x_i y_i$

Einstein used this first, and often called Einstein convention.

Summation convention : repetition of an index in a term will denote a summation with respect to that index over its range.

~~du~~ ~~dum~~ dummy index : an index that is summed over

free index : one that is not summed.

\# of free index denotes the order of tensor.

$x_i y_i = x_k y_k$

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$$

MooKeuk

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x_i

1 free index → vector = 1st order tensor

τ_{ij}

2 free index → 2nd order tensor

If we have a matrix $\underline{A}$ in cartesian notation,
this is A_{ij}

$A_{ji} \rightarrow \underline{\underline{A}}^T$.

Recall, $\underline{T}_{\epsilon} = \underline{\underline{\tau}} \cdot \underline{n}$

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} \tau_{11}n_1 + \tau_{12}n_2 + \tau_{13}n_3 \\ \tau_{21}n_1 + \tau_{22}n_2 + \tau_{23}n_3 \\ \tau_{31}n_1 + \tau_{32}n_2 + \tau_{33}n_3 \end{pmatrix}$$

$$= \begin{pmatrix} \sum^{3} \tau_{1j} n_j \\ \sum \tau_{2j} n_j \\ \sum \tau_{3j} n_j \end{pmatrix} \Rightarrow \tau_{ij} n_j$$

$\rightarrow T_i = \underline{\tau_{ji}} n_j$

$\therefore$ $T_i = \tau_{ij} n_j$ dummy index.

need to match free index part.

this is a vector, so $\tau_{ij} n_j = 2$ (X)

① free indices must match up on both sides of an eqn.

② A dummy index cannot be repeated more than twice in a term that is a product $\tau_{ii} n_i$ (X)

Q. What is the order of the following tensors? and what are the free indices?

u_i	i	1
$u_i u_j$	i, j	2
$u_i u_i$	~~i~~	0
$u_i u_i u_k$	k	1
$\tau_{ij} x_j$	i	1

$\tau_{ij} x_k$ i, j, k 3

$\varepsilon_{bulk} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{ii}$

*) Multiplication of a matrix Date (ij) . . No.

$$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{2} A_{1k}B_{k1} & \sum_{k=1}^{2} A_{1k}B_{k2} \\ \sum A_{2k}B_{k1} & \sum A_{2k}B_{k2} \end{pmatrix}$$

$$\underline{\underline{AB = A_{ik}B_{kj}}} = \underline{\underline{C_{ij}}}$$. same for $\overset{3D.}{(1, 2, 3)}$

2nd order

$$T_{ij}n_j = \underline{\tau}\,\underline{n} \neq \underline{n}\,\underline{\tau}$$

in matrix .

But in tensor notation, $T_{ij}n_j = n_jT_{ij}$

because it's the location of indices that matters.

a bit similar to scalar.

if $A_{ik}B_{kj}$ ~~exp~~ means $\underline{A}\,\underline{B}$, what do the following things mean in regular matrix notation,

$$A_{ik}B_{jk} = \underline{A}\,\underline{B}^T$$
$$A_{ki}B_{kj} = \underline{A}^T\underline{B}$$
$$A_{ki}B_{jk} = \underline{A}^T\underline{B}^T$$
$$B_{kj}A_{ik} = \underline{A}\,\underline{B}$$

german mathematician.

*) Kronecker delata, δ_{ij}

Recall the '~~one~~ identity matrix", $\underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\underline{I}\,\underline{x} = \underline{x}$ for any vector $\underline{x}$

Q) How do we write this in tensor form?

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

0

1

Def) $\begin{pmatrix} \delta_{11} = 1, & \delta_{12} = 0, & \delta_{13} = 0 \\ \delta_{21} = 0, & \delta_{22} = 1, & \delta_{23} = 0 \\ \delta_{31} = 0, & \delta_{32} = 0, & \delta_{33} = 1 \end{pmatrix}$

or $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

$\therefore \; \underline{\underline{I}}\,\underline{x} = \delta_{ij} x_j = x_i$

$\delta_{11} A_{1k}$

$\delta_{i1} A_{1k} + \delta_{i2} A_{2k} + \delta_{i3} A_{3k}$

$= A_{11} \; A_{12} \; A_{13}$

$A_{21} \; A_{22} \; A_{23}$

$A_{31} \; A_{32} \; A_{33}$

$\delta_{ij} A_{jk} = \underline{A_{ik}}$.

replacement rule.

$\underline{\underline{I}}$ is symmetric

$\therefore \; \delta_{ij} = \delta_{ji}$ → simple rule that is going to be helpful.

$\delta_{ij} x_i = x_j$.

$\delta_{ij} \delta_{jk} = \delta_{ik}$.

$\delta_{ij} \tau_{ij} = \delta_{ji} \tau_{ij} = \tau_{jj} . = \tau_{ii}$.

* permutation symbol, ε_{rst} .

Def.) $\begin{cases} \varepsilon_{111} = \varepsilon_{222} = \varepsilon_{333} = \varepsilon_{112} = \varepsilon_{121} = \varepsilon_{211} = \varepsilon_{221} = \varepsilon_{331} = \cdots = 0 \\ \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 \\ \varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1 \end{cases}$

other words, ε_{ijk} vanishes whenever the values of any two indices coincide. $\varepsilon_{ijk} = 1$ when the subscript permute as 1, 2, 3 ; $\varepsilon_{ijk} = -1$ otherwise.

useful for many expressions.

ex) $\det(a_{ij}) = \det(A) = \varepsilon_{rst} a_{r1} a_{s2} a_{t3}$. → homework

$u \times v =$

$= z_i = \varepsilon_{ijk} u_j v_k$.

* contraction

Def) If we have 2 indices in a tensor, say i and j, and we let "$i = j$", this is called a 'contraction' and reduces the order of the tensor by order of two.

$A_{ij} = B_{ik} C_{kj}$

when $i = j$, $A_{ii} = B_{ik} C_{ki} \rightarrow$ scalar.

Q) what is $A_{ij} B_{ij}$ in direct notation?

no free indices $\rightarrow$ zeroth order tensor = scalar.

$A_{ij} B_{jk} = \underline{\underline{A}}\,\underline{\underline{B}}$

$A_{ij} B_{kj} = \underline{\underline{A}}\,\underline{\underline{B}}^T$, $\rightarrow$ let $k = i$ contracting.

$A_{ij} B_{ij} = \underline{\text{trace of } AB^T}$.

*) Differentiation.

Def) We shall use comma to denote partial differentiation.

given, $f(x_1, x_2, x_3)$.

① $\frac{\partial f}{\partial x_1} = f_{,1}$, $\frac{\partial f}{\partial x_2} = f_{,2}$, $\frac{\partial f}{\partial x_3} = f_{,3}$

$\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) =$ gradient of $f = \nabla f = \underline{\underline{f_{,i}}} = \frac{\partial f}{\partial x_i}$.

② $f_{,ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} \end{pmatrix}$.

$f_{,ij} = f_{,ji}$.

③ $f_{,ii} = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} = \nabla^2 f$ laplacian.

*) the $\varepsilon - \delta$ indentity

Kronecker delta & permutation symbols are connected by the identity.

$\varepsilon_{ijk} \varepsilon_{ist} = \delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks}$

*) coming back to stress & strain

τ_{ij} , $T_i = \tau_{ij} n_j$

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

Eigen value problems.

$$(\tau_{ij} - \sigma \delta_{ij}) n_j = 0 .$$

Tensor Transformation

$$\tau_{ij} = \beta_{im} \beta_{jn} \tau_{mn} .$$

$$\equiv \underline{\underline{\tau}}' = \underline{\underline{R}}\, \underline{\underline{\tau}}\, \underline{\underline{R}}^T .$$

Homework.
Dong Keun's question.

* Compatibility Equation   Date 3/23 No.

From $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

six equations for the strain components as a fn of only three displacement components → if we specify u, v and w as a fn of x, y, z
→ we may derive strain components.

However, we might think of six strain components as fn of x, y & z → six equations for three unknowns, u, v & w.

This system of solution does not in general possess a solution for u, v & w unless six strain components are somehow related.

In otherwords, all six strain components cannot be arbitrarily prescribed if we are to maintain single-valued, continuous displacement fn.

- Let's consider the simpler, but essentially equivalent, situation of a single function f of two variables, x and y.

Assume that

$$M(x,y) = \frac{\partial f}{\partial x}, \quad N(x,y) = \frac{\partial f}{\partial y}$$

M & N cannot be specified arbitrarily, even if they are both assumed to be continuous.

∵ mixed derivative of f must be equal.

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$=$$

∴ M & N are not completely independent, but must satisfy the pde $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

ex) $\frac{\partial f}{\partial x} = x^2 + 2y$, $\frac{\partial f}{\partial y} = 2y^2$, we must satisfy $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y}$.

- Let's consider 2D first,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \cdots ① \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \cdots ③$$

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$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \cdots ②$$

differentiate ① twice with respect to y, and ② twice wrt x, and add.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

differentiate ③ wrt x & y, we get

$$\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2} \cdot \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Since the order of differentiation for single valued, continuous function is immaterial, we see that

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \cdot \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \quad ④\text{-}1 \rightarrow \text{case for 2D.}$$

By similar approaches, we can develop five additional eqs for 3D.

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \quad 4\text{-}2$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \quad 4\text{-}3$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \quad 4\text{-}4$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \quad 4\text{-}5$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z}\right) \quad 4\text{-}6$$

Compatibility Eq or Saint Venant ———.

- The strain component must satisfy these expressions in order that ~~⊕~~ solutions for the displacement components exist.
- Six compatibility eqn are equivalent to three independent fourth-order equations.

4-1 with respect to z twice $\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\left(\frac{\partial \varepsilon_{yy}}{\partial z^2}+\frac{\partial \varepsilon_{zz}}{\partial y^2}\right)+\frac{\partial^2}{\partial y^2}\right($

ex) $$\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 \varepsilon_{xx}}{\partial y^2}+\frac{\partial^2 \varepsilon_{yy}}{\partial x^2}\right)=2\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}\right)$$

4-4 ~~twice~~ wrt y, z

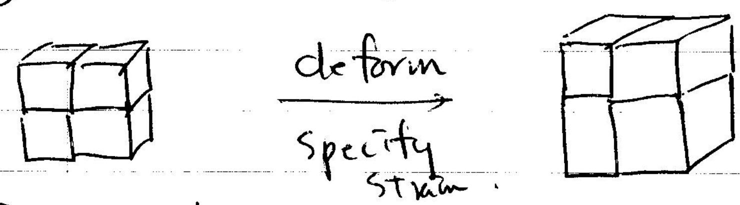
$$\frac{\partial^2}{\partial y \partial z}\left(\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z}\right)=\frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{yz}}{\partial x}+\frac{\partial \varepsilon_{zx}}{\partial y}+\frac{\partial \varepsilon_{xy}}{\partial z}\right)$$

$$=\frac{1}{2}\left\{-\frac{\partial^2}{\partial x^2}\left(\cancel{\frac{\partial^2 \varepsilon_{yy}}{\partial z^2}}+\cancel{\frac{\partial^2 \varepsilon_{zz}}{\partial y^2}}\right)+\frac{\partial^2}{\partial y^2}\left(\cancel{\frac{\partial^2 \varepsilon_{zz}}{\partial x^2}}+\frac{\partial^2 \varepsilon_{xx}}{\partial z^2}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 \varepsilon_{xx}}{\partial y^2}+\cancel{\frac{\partial^2 \varepsilon_{yy}}{\partial x^2}}\right)\right\}$$

$$=\frac{\partial^4}{\partial y^2 \partial z^2}(\varepsilon_{xx})$$

✓ indicial notation

-- geometrical significance



imagine a cube divided up into infinitesimal cube

shouldn't be any space.

if we specify 'u' there won't be.

* give one example —

* displacement based equation — don't use compa eq.

stress based equation — use compa. eq.

tensor or indicial notation,

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_m}+\frac{\partial^2 \varepsilon_{km}}{\partial x_i \partial x_j}-\frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_m}-\frac{\partial^2 \varepsilon_{jm}}{\partial x_i \partial x_k}=0 \qquad \varepsilon_{ij,km}+\varepsilon_{km,ij}-\varepsilon_{ik,jm}-\varepsilon_{jm,ik}=0$$

→ 81 eqs. But only six are distinct.