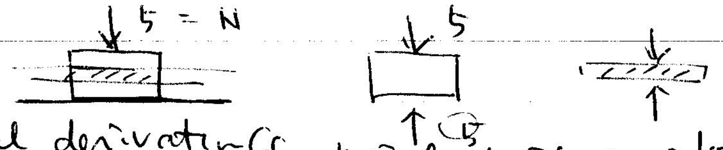
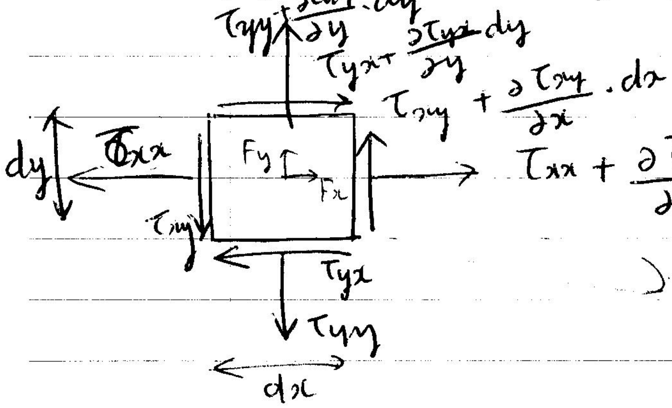
Lecture 4. Stress Equilibrium Eq.

Thus far, we have considered uniform stress.
However, the stress components vary from point to points in a stressed body, and these variations are governed by the equilibrium conditions of statics.



① elemental derivation (for physical clarity) → look at differential element.

Let's consider 2D first & expand it in 3D.



we assume, $\tau_{zz} = \tau_{yz} = \tau_{xz} = 0$.
$F_z = 0$.
⇒ plane stress condition.
$\tau_{xx}, \tau_{yy}, \tau_{xy}$ are independent of z
F_x, F_y

⟩ average stress acting at the center of faces.

body force intensity = (F_x, F_y).

body force : Force acting on elements of Volume of the body

surface force : force acting on surface elements.

(ex) gravitational force, electromagnetic force

(ex) stress due to mechanical contact, aerodynamic pressure acting on a body.

By writing $\Sigma F_x = 0$; assuming a unit depth,

$$F_x \cdot dx dy + \left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} dx\right) dy - \tau_{xx} \cdot dy + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx - \tau_{yx} dx = 0.$$

which when simplified becomes,

$$\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x\right) dx dy = 0.$$

since dx dy is not zero,

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$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0.$$

Similarly, $\Sigma F_y = 0$ yields,

$$\frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0.$$

By considering 3D counterparts,

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0.$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0. \qquad F_y = \frac{m \cdot g}{V}.$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + F_z = 0.$$

in tensor notations, $\sigma_{ji,j} + F_i = 0$. Eqns of stress Equilibrium

if we take $\Sigma M = 0$ from the lower left corner,

$$\left(\frac{\partial \tau_{yy}}{\partial y} dy\,dx\right)\frac{dx}{2} - \left(\frac{\partial \tau_{xx}}{\partial x} dx\,dy\right)\frac{dy}{2} + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) dy\,dx$$

$$- \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx\,dy + F_y\,dx\,dy\frac{dx}{2} - F_x\,dx\,dy\frac{dy}{2} = 0.$$

Neglecting terms containing triple products of dx & dy.

$\tau_{xy} = \tau_{yx}$, similarly $\tau_{zx} = \tau_{xz}$, $\tau_{zy} = \tau_{yz}$.

When a rock is not in a static equilibrium,

$$\sigma_{ji,j} + F_i = \rho \cdot \frac{\partial^2 u_i}{\partial t^2}$$

found by applying conservation of linear momentum.

Equation of motion. ←

later

same stress does not move
if different stress → gradient appear

9 stress component → 6 components.

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* We have three eqns for six variables.
→ additional eqns required for a complete solution of the stress ~~state~~ distribution throughout a body.

* these additional eqns are < strain-displacement ~~eq~~ relation, constitutive Eqn. (Hookes law)

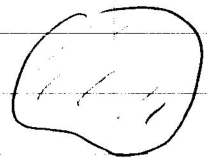
→ we need to know material property.

* Stress distribution in a body, in the terminology of mechanics of materials, is always statically indeterminate.

* ~~if stress~~ this eqn can be later expressed in terms of u, v, w.

② Using Divergence Theorem.

Denote the region of 3D space occupied by the body by B.
outer boundary of the body by ∂B.



$\frac{F}{\text{volume}}$

Total Force = body force + surface force.

total body force in x-direction $= \iiint_B F_x \, dV$.

total x component of force due to surface traction $= \iint_{\partial B} T_x \, dA$.

total inertia component in x direction $= \iiint_B \rho \cdot \frac{\partial^2 u_x}{\partial t^2} \, dV$

mass of small element $= \rho \, dV$.

Equating the total force to the total inertia term yields,

$$\iiint_B F_x \, dV + \iint_{\partial B} T_x \, dA = \iiint_B \rho \frac{\partial^2 u_x}{\partial t^2} \, dV.$$

To derive ~~the~~ more useful differential form of this equation, we first convert the surface integral into a volume integral over the entire body.

$\iiint u_{i,i}\, dV = \iint_{\partial B} u_i n_i\, dS$

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To do this, we invoke the divergence theorem. With any vector, $\underline{f}$, with components (f_x, f_y, f_z)

$$\iint_{\partial B} (f_x \cdot n_x + f_y \cdot n_y + f_z \cdot n_z)\, dA = \iint_{\partial B} \underline{f} \cdot \underline{n}\, dA = \iiint_B \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right) dV$$

where (n_x, n_y, n_z) are the components of the outward unit normal vector to the surface.

*assumption : differentiability of the function
smoothness of the outer boundary ∂B.

To apply the divergence theorem to the surface integral before, we use $T_i = \sigma_{ij} \cdot n_i = \sigma_{ji} n_j$

$$T_x = \tau_{xx} n_x + \tau_{yx} \cdot n_y + \tau_{zx} \cdot n_z.$$

$$\iint_{\partial B} T_x\, dB = \iint_{\partial B} (\tau_{xx} n_x + \tau_{yx} n_y + \tau_{zx} \cdot n_z)\, dA = \iiint_B \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dV$$

$$\iiint_B F_x\, dV + \iiint_B \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dV = \iiint \rho \cdot \frac{d^2 u}{\partial t^2}\, dV.$$

$$\iiint_B \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x - \rho \cdot \frac{\partial^2 u_x}{\partial t^2}\right] dV = 0.$$

in order for the integral of the bracketed term to vanish over any arbitrary region, the integrand must be zero.

$\therefore$

Similarly.

this can be also expressed in vector / matrix form.

$$\text{gradient} = \nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

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if we premultiply the stress matrix by the transpose of this gradient vector.

$$\nabla^T \underline{\underline{\tau}} + \underline{F} = \rho \frac{\partial^2 \underline{u}}{\partial t^2}$$

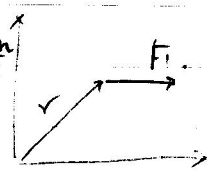
$$\left(\frac{\partial}{\partial x} \ \frac{\partial}{\partial y} \ \frac{\partial}{\partial z}\right)\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} + (F_x \ F_y \ F_z) = \rho\left(\frac{\partial^2 u_x}{\partial t^2} \ \frac{\partial^2 u_y}{\partial t^2} \ \frac{\partial^2 u_z}{\partial t^2}\right)$$

$$\underline{\mathrm{div.}\ \underline{\underline{\tau}} + \underline{F} = \rho \ddot{\underline{u}}}$$

For moment,

$$\underbrace{\iint_{\partial B} r \times T \, dB}_{\varepsilon_{rmn} x_m T_n} + \underbrace{\iiint_{B} r \times F \, dV}_{\varepsilon_{rmn} x_m F_n} = \underbrace{\iiint r \times m \frac{d^2u}{dt^2} dV}_{m \varepsilon_{rmn} x_m \frac{\partial^2 u_n}{\partial t^2}} + \iiint \underbrace{m_r}_{\text{body moment}} dV = 0$$

ex) electromagnetic



$$\iint_{\partial B} \varepsilon_{rmn} x_m \sigma_{jn} n_j \, dB$$

$r \times F$ = moment

$\varepsilon_{ijk} r_j F_k$

$$= \iiint_B \frac{\partial \varepsilon_{rmn} x_m \sigma_{jn}}{\partial x_j} dV = \iiint\left(\varepsilon_{rmn} \delta_{mj} \sigma_{jn} + \varepsilon_{rmn} x_m \sigma_{jn,j}\right) dV$$

$$\iiint\left[\varepsilon_{rmn} x_m \underbrace{\left(\sigma_{jn,j} + F_n - m \frac{\partial^2 u_n}{\partial t^2}\right)}_{=0} + \varepsilon_{rmn} \delta_{mj} \sigma_{jn} + m_r\right] dV = 0$$

$$\iiint\left(\varepsilon_{rmn} \delta_{mj} \sigma_{jn} + m_r\right) dV = 0$$

$$\varepsilon_{rmn} \sigma_{mn} + m_r = 0$$

$$\begin{cases} \sigma_{23} - \sigma_{32} + m_1 = 0 \\ \sigma_{13} - \sigma_{31} + m_2 = 0 \\ \sigma_{12} - \sigma_{21} + m_3 = 0 \end{cases}$$

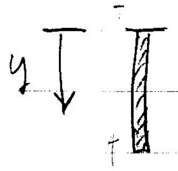
when $m_i = 0$, $\longrightarrow$ $\tau_{ij} = \tau_{ji}$

when there is no body moment, stress is symmetry.

the physical meaning of stress symmetry is moment equilibrium

$$\tau_{ij,j} + F_i = m \frac{\partial^2 u_i}{\partial t^2}.$$

1D example)



$$\cancel{\frac{\partial \tau_{xy}}{\partial x}} + \frac{\partial \tau_{yy}}{\partial y} + \cancel{\frac{\partial \tau_{zy}}{\partial z}} + F_y = 0$$

$$\tau_{yy} = -F_y \cdot y + C_1$$

$$F_y = \frac{mg}{V} = \rho g$$

$$\tau_{yy} = -\rho g y + C_1$$

BC. $y = 0$, $\tau_{yy} = 0 \rightarrow C_1 = 0$

$$\tau_{yy} = -\rho g y$$ ⊖ compressive

$$= -2500\ kg/m^3 \times 9.8\ m/sec^2 \times y$$

$$= 2500\ Pa \cdot m \cdot y$$

$$= \underline{0.025 \times y\ MPa}$$ $\underline{\text{insitu stress}}$

for calulation of displacement.

$$\tau_{yy} = E \cdot \varepsilon_{yy} = E \frac{\partial u_y}{\partial y}$$

$$E \frac{\partial u_y}{\partial y} = -\rho g y + \cancel{C_1}$$

$$u_y = \frac{1}{E}\left(-\frac{\rho g}{2} y^2 + C_2\right)$$ $y = 1 \rightarrow u = 0$

$$C_2 = \frac{\rho g}{2}$$

$$u_y = \frac{\rho g}{2E}\left(-y^2 + 1\right)$$