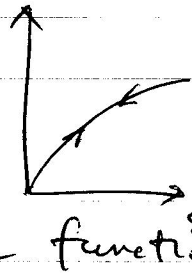
March 30 2009

Lecture 5. Constitutive Equations

No.

- We need a constitutive model.
  i.e., a relationship between stress and strain.

elastic behavior $\tau = f(\varepsilon)$



- no dependence on time
- irreverisible.

linear elastic $\underline{\underline{\tau}}$ is a linear function of $\underline{\underline{\varepsilon}}$.

In most applications, we assume that rock is a linear elastic materials.

① Isotropic case. uniaxial tension.

gets longer i) $\varepsilon_{xx} = \frac{1}{E}\tau_{xx}$ E : Young's Modulus

gets thinner ii) $\varepsilon_{yy} = -\frac{\nu}{E}\tau_{xx}$ ν : poisson's ratio.

iii) $\varepsilon_{zz} = -\frac{\nu}{E}\tau_{xx}$

what strains are caused by τ_{yy}?

$$\varepsilon_{yy} = -\frac{1}{E}\tau_{yy}, \quad \varepsilon_{xx} = \varepsilon_{zz} = -\frac{1}{E}\nu\tau_{yy}.$$

If we have $\tau_{xx}, \tau_{yy}, \tau_{zz}$ acting at the same time,

$$\varepsilon_{xx} = \frac{1}{E}\tau_{xx} - \frac{1}{E}\nu\tau_{yy} - \frac{\nu}{E}\tau_{zz}$$

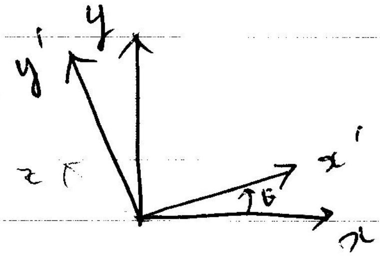
$$= \frac{1}{E}\left(\tau_{xx} - \nu\left(\tau_{yy} + \tau_{zz}\right)\right).$$

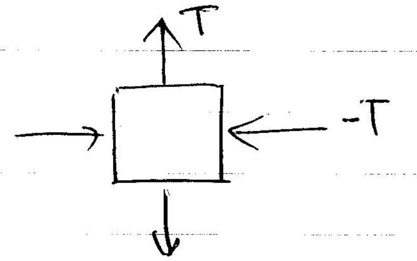
$$\varepsilon_{yy} = \frac{1}{E}\left(-\nu\tau_{xx} + \tau_{yy} - \nu\tau_{zz}\right)$$

$$\varepsilon_{zz} = \frac{1}{E}\left(-\nu\tau_{xx} - \nu\tau_{yy} + \tau_{zz}\right).$$

here, we assumed that normal stress cause only normal strain. However, this assumption is released for fully anisotropic material.

in order to get the relationships between shear strain & stress, we use some tricks.





$$\underline{\tau} = \begin{pmatrix} -T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

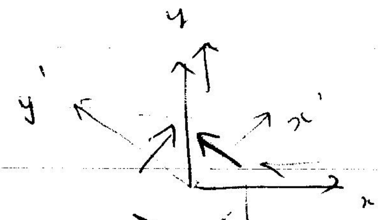
$$R = \begin{pmatrix} \cos\theta & +\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R^T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\tau' = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

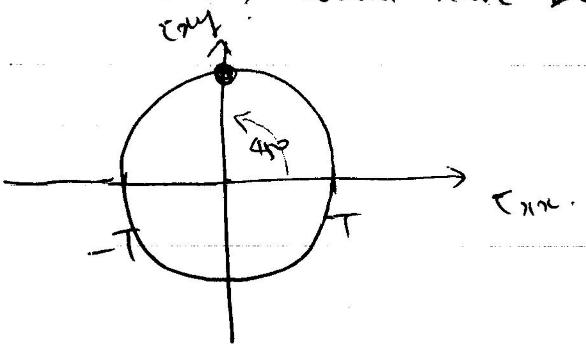
$$= \begin{pmatrix} -T\cos\theta & T\sin\theta & 0 \\ T\sin\theta & T\cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (s^2-c^2)T & 2csT & 0 \\ 2csT & (c^2-s^2)T & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

look at $\theta = 45^\circ$. $\sin\theta = \cos\theta = 1/\sqrt{2}$

$$\tau' = \begin{pmatrix} 0 & T & 0 \\ T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$



this would have been found using Mohr-Circle.



Do the same thing for $\varepsilon_{x'x'}$. (when $\varepsilon_{xy}=0$)

$$\varepsilon_{x'x'} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + 0$$

$$\varepsilon_{y'y'} = \varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta + 0$$

$$\varepsilon_{x'y'} = \varepsilon_{y'x'} = -\varepsilon_{xx}\sin\theta\cos\theta + \varepsilon_{yy}\sin\theta\cos\theta + 0$$

recall that, if $\tau_{zz}=0$,

$$\varepsilon_{xx} = \frac{1}{E}(\tau_{xx} - \nu\tau_{yy})$$

$$\varepsilon_{yy} = \frac{1}{E}(\tau_{yy} - \nu\tau_{xx})$$

$$\varepsilon_{x'x'} = \cos^2\theta\,\frac{1}{E}(\tau_{xx} - \nu\tau_{yy}) + \sin^2\theta\cdot\frac{1}{E}(\tau_{yy} - \nu\tau_{xx})$$

if $\theta = 45°$, $\tau_{xx} = -T$, $\tau_{yy} = T$.

$$\varepsilon_{x'x'} = 0,\quad \varepsilon_{y'y'} = 0$$

$$\varepsilon_{x'y'} = -\frac{1}{2E}(\tau_{xx} - \nu\tau_{yy}) + \frac{1}{2E}(\tau_{yy} - \nu\tau_{xx})$$

$$= \frac{1}{2E}T\cdot 2(1+\nu) = \frac{1+\nu}{E}\cdot T$$

Since $\tau_{x'y'} = T$. engineering strain

$$\varepsilon_{x'y'} = \frac{1+\nu}{E}\cdot\tau_{x'y'},\quad \gamma_{x'y'} = 2\varepsilon_{x'y'}$$

$$\tau_{x'y'} = \frac{1}{G}\cdot\gamma_{x'y'} \qquad \tau_{x'y'} = \frac{E}{2(1+\nu)}\cdot\gamma_{x'y'}$$

$$\therefore G = \frac{E}{2(1+\nu)}$$

Finally

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{pmatrix} \begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix} \quad \cdots (5\text{-}1)$$

Constitutive Eq. for isotropic linear elastic rock.

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$E \geq 0$, $G \geq 0$, $\frac{E}{2(1+\nu)} > 0 \rightarrow \underline{\underline{\nu > -1}}$.

with hydrostatic stress

$\tau_{xx} = \tau_{yy} = \tau_{zz} = -p$, $\tau_{xy} = \tau_{xz} = \tau_{yz} = 0$.

$\varepsilon_{xx} = \frac{1}{E} \cdot (\tau_{xx} - \nu\tau_{yy} - \nu\tau_{zz}) = \frac{1}{E} \cdot (-p + 2\nu p) = \frac{-p}{E}(1-2\nu)$

$= \varepsilon_{zz} = \varepsilon_{yy}$

$\varepsilon_{bulk} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = -\frac{3}{E}p(1-2\nu) = \frac{\Delta V}{V}$

$\hookrightarrow \frac{1}{K}$. bulk modulus.

$\varepsilon_v = -\frac{1}{K}p$.

compressibility, $\beta = \frac{1}{K} = -\frac{\Delta V}{V} \cdot \frac{1}{p} = -\frac{1}{V} \cdot \frac{\Delta V}{p}$..

$K > 0 \rightarrow \nu > \frac{1}{2}$.

$\therefore$ $E > 0$, $-1 < \nu < \frac{1}{2}$.

constraint of material property = bound of elastic parameters.

$\rightarrow$ closely related to the condition that any deformation of a rock result in ~~positive~~ energy being stored in the rock, rather than released by the rock.

$\rightarrow$ strain energy fn positive definite.

cork, $\nu = 0$.
rubber, $\nu \doteqdot 0.5$

negative ν.

~~ν can be ta~~ note that these constraints are relaxed for anisotropic ~~rock~~ material.

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✩ Various forms of constitutive Eqns.

inverse form of 5-1.

$$\underline{\underline{\varepsilon}} = [S]\,\underline{\underline{\tau}} \quad \rightarrow \quad \underline{\underline{\tau}} = \ ?\ \underline{\underline{\varepsilon}}$$

$$\varepsilon_{ij} = S_{ijkl}\,\tau_{kl} \quad \rightarrow \quad \tau_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

$$\begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{pmatrix}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \frac{E}{2(1+\nu)} = G.$$

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \frac{E\nu}{(1+\nu)(1-2\nu)} + \frac{E(1-2\nu)}{(1+\nu)(1-2\nu)} = \lambda + 2G.$$

$$\boxed{\begin{aligned} \tau_{xx} &= (\lambda+2G)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz} \\ \tau_{yy} &= \lambda\varepsilon_{xx} + (\lambda+2G)\varepsilon_{yy} + \lambda\varepsilon_{zz} \\ \tau_{zz} &= \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda+2G)\varepsilon_{zz} \\ \tau_{xy} &= 2G\varepsilon_{xy},\ \tau_{yz} = 2G\varepsilon_{yz},\ \tau_{zx} = 2G\varepsilon_{zx}. \end{aligned}} \quad 5\text{-}2$$

in tensor notation

$$\tau_{ij} = \lambda\varepsilon_{\alpha\alpha}\delta_{ij} + 2\mu\varepsilon_{ij} \qquad (\mu = G)$$

also,
$$\begin{aligned} \tau_{xx} &= \lambda\varepsilon_v + 2G\varepsilon_{xx} \\ \tau_{yy} &= \lambda\varepsilon_v + 2G\varepsilon_{yy} \\ \tau_{zz} &= \lambda\varepsilon_v + 2G\varepsilon_{zz} \end{aligned}$$

Summing up, $3\tau_m = \tau_{xx}+\tau_{yy}+\tau_{zz} = 3\lambda\varepsilon_v + 2G(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})$
$= (3\lambda+2G)\varepsilon_v$.

$$\tau_m = \underbrace{\left(\frac{3\lambda+2G}{3}\right)}_{K}\varepsilon_v = K\varepsilon_v.$$

cf.

E, ν, G (μ), K, λ. only two of them are independent.

$E = ?\ (\lambda, G) \rightarrow \dfrac{G(3\lambda+2G)}{\lambda+G}$

$\nu = ?$ 〃 $\rightarrow \dfrac{\lambda}{2(\lambda+G)}$.

$\lambda = K - \frac{2G}{3}$ Date No. $\frac{G(3\lambda+2G)}{\lambda+G}$ $\frac{3G}{E} = G$ $2(1+\nu)$

- incompressible solid

$$\beta = 0\ ,\ K = \frac{1}{\beta} \to \infty\ ,\ \lambda \to \infty,\ E \to 3G\ ,\ \nu \to \frac{1}{2}$$

note that rigid material $E, G \to \infty$.

- Compressible fluid → shear modulus → 0.

$$G \to 0,\quad \nu \to 1/2\ ,\quad E \to 0\ ,\quad \lambda = K.$$

$$\nu = \frac{3K-2G}{6K+2G}. \qquad 2G(1+\nu)$$

unit of E, G.

Pa, MPa. $\underline{14.5\ \text{psi} = 0.1\ \text{MPa} = 1\ \text{bar}.}$

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* addition)

From $\tau_{ij} = \lambda \varepsilon_{\alpha\alpha}\delta_{ij} + 2\mu\varepsilon_{ij}$ ... ①

this can be expressed for ε_{ij}.

$$\varepsilon_{ij} = \frac{1}{2\mu}\left(\tau_{ij} - \lambda \varepsilon_{\alpha\alpha}\delta_{ij}\right). \quad \text{... ②}$$

$$\tau_{\alpha\alpha} = 3\lambda\varepsilon_{\alpha\alpha} + 2\mu\varepsilon_{\alpha\alpha}$$

From ①, we contract τ_{ij}.

$$\tau_{kk} = 3\lambda\varepsilon_{kk} + 2\mu\varepsilon_{kk} = (3\lambda+2\mu)\varepsilon_{kk}.$$

$$\varepsilon_{kk} = \frac{1}{3\lambda+2\mu}\tau_{kk}$$

put this into ②.

$$\varepsilon_{ij} = \frac{1}{2\mu}\cdot\left(\tau_{ij} - \lambda\cdot\frac{1}{3\lambda+2\mu}\tau_{\alpha\alpha}\delta_{ij}\right) = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\tau_{\alpha\alpha}\delta_{ij} + \frac{1}{2\mu}\tau_{ij}$$

Stress-strain relationship. ← I don't see the merit of using this expression.

$$\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\tau_{kk}$$

e.g.)
$$\varepsilon_{11} = \left(\frac{1}{2\mu} - \frac{1}{2\mu}\frac{\lambda}{(3\lambda+2\mu)}\right)\tau_{11} - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\tau_{22}+\tau_{33})$$

$$= \frac{1}{2\mu}\cdot\frac{2(\lambda+\mu)}{3\lambda+2\mu}\tau_{11} - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\tau_{22}+\tau_{33})$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}. \qquad = \frac{1}{E}\tau_{11} - \frac{\nu}{E}(\tau_{22}+\tau_{33})$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)}$$

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☆ Special cases.

i) hydrostatic stress, $\sigma_1 = \sigma_2 = \sigma_3 = p$.

- when rock is surrounded by a fluid under a pressure of magnitude, P.

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{P}{3K}.$$

$$\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{P}{K} = \beta P.$$

(β: compressibility of the rock.)

ii) uniaxial stress, $\sigma_1 \neq 0$, $\sigma_2 = \sigma_3 = 0$.

- laboratory testing of a rock.
- pillar in underground mine

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx}, \quad \varepsilon_{yy} = -\frac{\nu}{E}\tau_{xx} = \varepsilon_{zz}.$$

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{1-2\nu}{E}\cdot\tau_{xx}$$

the volume ↓ for compressive stress.
↑ for (+).

iii) uniaxial strain $\varepsilon_1 \neq 0$, $\varepsilon_2 = \varepsilon_3 = 0$

- when fluid is withdrawn from a reservoir

from 5-2. (lateral strain is inhibited by the rock)

$$\sigma_1 = (\lambda + 2G)\,\varepsilon_1$$

$$\sigma_2 = \sigma_3 = \lambda\varepsilon_1 = \nu/(1-\nu)\,\sigma_1$$

$$= \lambda/(\lambda+2G)\,\varepsilon_1 = \frac{2G\nu}{1-2\nu}\cdot\left(\frac{2G\nu}{1-2\nu} + \frac{2G}{1}\right)$$

$$= \frac{2G\nu}{1-2\nu}\Big/\left(\frac{2G-2G\nu}{1-2\nu}\right)$$

$$= \frac{\nu}{1-\nu}$$

- often used as a simple model for calculating in situ stress below earth's surfaces.
- $K_0 = \frac{\nu}{1-\nu}$ ~~earth~~ coefficient of earth pressure at rest in geotechnical literature.

assumption: rock is homogeneous, linearly isotropic continuum
under gravity alone with confined displacement.

iv) (biaxial stress or) plane stress. $\sigma_3 = 0$, $\begin{pmatrix}\sigma_1 \neq 0 \\ \sigma_2 \neq 0\end{pmatrix}$

from 5-1)

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2), \quad \varepsilon_2 = \frac{1}{E}\cdot(\sigma_2 - \nu\sigma_1), \quad \varepsilon_3 = -\frac{\nu}{E}(\sigma_1 + \sigma_2)$$

- when a thin plate is loaded by forces acting in its own plane.
- ~~expansion~~ $\overset{(+)}{}$ in out-of-plane direction, for (−) stress.
  $(-)$ 〃 for (+) stress

from 5-2)

$$\sigma_1 = \frac{4G(\lambda+G)}{\lambda+2G}\varepsilon_1 + \frac{2G\lambda}{\lambda+2G}\varepsilon_2, \quad \varepsilon_3 =$$

$$\sigma_2 = \frac{2G\lambda}{\lambda+2G}\varepsilon_1 + \frac{4G(\lambda+G)}{\lambda+2G}\varepsilon_2,$$

v) plane strain $\varepsilon_3 = 0$, $\varepsilon_1 \neq 0$, $\varepsilon_2 \neq 0$.

from 5-2)

$$\sigma_1 = (\lambda+2G)\varepsilon_1 + \lambda\varepsilon_2, \quad \sigma_2 = (\lambda+2G)\varepsilon_2 + \lambda\varepsilon_1$$

$$\sigma_3 = \lambda(\varepsilon_1+\varepsilon_2) = \nu(\sigma_1+\sigma_2).$$

from 5-1) $\sigma_3 = \nu(\sigma_1+\sigma_2)$.

☆ in order for ε_3 to be zero, we need a non-zero σ_3 to counteract the Poisson effect due to the two in-plane stresses.

inverse form.

$$\varepsilon_1 = \frac{1-\nu^2}{E}\sigma_1 - \frac{\nu(1+\nu)}{E}\sigma_2$$

$$\varepsilon_2 = \frac{1-\nu^2}{E}\sigma_2 - \frac{\nu(1+\nu)}{E}\sigma_1$$

- when analysing the stresses around boreholes or tunnels.

iv) + v) expression.

$$\varepsilon_1 = \frac{(K+1)}{8G}\sigma_1 + \frac{(K-3)}{8G}\sigma_2$$

$$\varepsilon_2 = \frac{(K-3)}{8G}\sigma_1 + \frac{(K+1)}{8G}\sigma_2 .$$

coefficient K, $\left\langle \begin{array}{ll} K = 3-4\nu & \text{plane strain} \\ K = \dfrac{3-\nu}{1+\nu} & \text{plane stress} \end{array} \right.$

$$= 3-4\nu/(1+\nu) = \frac{3-4\nu}{1+\nu} .$$

$$3-4\nu = \frac{3-\frac{\nu}{1-\nu}}{1+\frac{\nu}{1-\nu}} = \frac{\frac{3-4\nu}{1-\nu}}{\frac{1}{1-\nu}} = 3-4\nu .$$

vi) constant strain along z-axis = generalized plane strain
when w is independent of x and y, that both u and v are endependent of z,

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = \varepsilon \ (\text{constant}) .$$

from 5-1)

$$\varepsilon_{xx} = \frac{1}{E}\cdot(\sigma_{xx} - \nu(\sigma_y+\sigma_z))$$

$$\varepsilon_{zz} = \frac{1}{E}(\sigma_z - \nu(\sigma_x+\sigma_y)) = \varepsilon = \text{constant}.$$

$$\sigma_z = E\varepsilon + \nu(\sigma_x+\sigma_y)$$

$$\varepsilon_{xx} = \frac{1}{E}\cdot\left((1-\nu^2)\sigma_{xx} - \nu(1+\nu)\tau_{yy}\right) - \nu\varepsilon$$

$$\varepsilon_{yy} = \frac{1}{E}\left((1-\nu^2)\tau_{yy} - \nu(1+\nu)\tau_{xx}\right) - \nu\varepsilon .$$

$$2G\varepsilon_{xy} = \tau_{xy} .$$

both geometric form & external loads exerted on the object do not change along z-axis.