Lecture 6 Navier's Equation : Date No.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}, \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\varepsilon_{ij} = \frac{1}{2}\cdot\left(u_{i,j} + u_{j,i}\right) \quad \cdots \quad (1)$$

$$\tau_{xx} = (\lambda + 2G)\,\varepsilon_{xx} + \lambda\,\varepsilon_{yy} + \lambda\,\varepsilon_{zz}$$
$$\tau_{yy} = \lambda\,\varepsilon_{xx} + (\lambda + 2G)\,\varepsilon_{yy} + \lambda\,\varepsilon_{zz}$$
$$\tau_{zz} = \lambda\,\varepsilon_{xx} + \lambda\,\varepsilon_{yy} + (\lambda + 2G)\,\varepsilon_{zz}$$
$$\tau_{xy} = 2G\,\varepsilon_{xy}, \quad \tau_{xz} = 2G\,\varepsilon_{xz}, \quad \tau_{yz} = 2G\,\varepsilon_{yz}$$
$$\tau_{ij} = \lambda\,\varepsilon_{\alpha\alpha}\,\delta_{ij} + 2\mu\,\varepsilon_{ij} \quad \cdots \quad (2)$$

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + F_z = 0$$

$$\tau_{ji,j} + F_i = 0 \quad \cdots \quad (3)$$

We are going to eliminate all variables except displacement.
put (1) into (2) and put the results into (3).

$$\frac{\partial}{\partial x}\left((\lambda+2G)\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} + \lambda\frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial y}\left(G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right) + F_x = 0.$$

$$\cancel{(\lambda+G)\frac{\partial^2 u}{\partial x^2}}\ \lambda\cdot\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right) + G\cdot\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial w}\right)$$
$$+ G\cdot\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_x = 0.$$

$$(\lambda+G)\cdot\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right) + G\cdot\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_x = 0.$$

similarly

$$(\lambda+G)\cdot\left(\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y \partial z}\right) + G\left(\frac{\partial v^2}{\partial x^2} + \frac{\partial v^2}{\partial y^2} + \frac{\partial v^2}{\partial z^2}\right) + F_y = 0.$$

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$$(\lambda+G)\left(\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 v}{\partial z\partial y}+\frac{\partial^2 w}{\partial z^2}\right)+G\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+F_z=0.$$

in tensor notation,

$$\mu\cdot u_{i,jj}+(\lambda+\mu)u_{j,ji}+F_i=0.$$

Stress equilibrium Equations in terms of displacement
= Navier's Equations.

There can be different forms for the same eqns.

① When F_x, F_y, F_z are known, these are the partial differential eqns for three unknowns.

② these form a set of differential eqns to solve elasticity problems. with given B.C.

$$\nabla\cdot u=\operatorname{div}\cdot\underline{u}=\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}.$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

$$(\lambda+G)\frac{\partial}{\partial x}(\nabla\cdot\underline{u})+G\cdot\nabla^2 u+F_x=0$$

$$(\lambda+G)\frac{\partial}{\partial y}(\nabla\cdot\underline{u})+G\cdot\nabla^2 v+F_y=0$$

$$(\lambda+G)\frac{\partial}{\partial z}(\nabla\cdot\underline{u})+G\cdot\nabla^2 w+F_z=0.$$

$$(\lambda+G)\nabla(\nabla\cdot\underline{u})+G\nabla^2\underline{u}+\underline{F}=0.$$

$$\tau_{ij}=\lambda\varepsilon_{\alpha\alpha}\delta_{ij}+2\mu\varepsilon_{ij}=\lambda u_{k,k}\cdot\delta_{ij}+\mu(u_{i,j}+u_{j,i})$$

$$\tau_{ij,j}+F_i=0.$$

dummy $\qquad \frac{\partial(\lambda u_{k,k}\delta_{ij})}{\partial x_j}=\lambda u_{k,ki}$

$$\mu(u_{i,jj}+u_{j,ij})+\lambda u_{k,ki}+F_i=0.$$

$$\mu u_{i,jj}+(\lambda+\mu)u_{j,ji}+F_i=0.$$

So far, this was displacement formulation → eliminate all ~~the~~ ~~...~~ variables except u_i

(변위방법)

☆ Stress formulation ← eliminate all variables except the stress.

Equilibrium Eq. — 3
Constitutive Eq (Stress-strain) — 6
Compatibility Eq. 3.
} 12.

- First we eliminate strain components.
- From the last two equilibrium Eqns,

$$\frac{\partial \tau_{yz}}{\partial z} = -\left(\frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y\right)$$

$$\frac{\partial \tau_{yz}}{\partial y} = -\left(\frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + F_z\right)$$

By differentiating the first of these with respect to y & the second with respect to z, and adding, we obtain,

$$-2\frac{\partial^2 \tau_{yz}}{\partial y \partial z} = \frac{\partial^2 \tau_{yy}}{\partial y^2} + \frac{\partial^2 \tau_{zz}}{\partial z^2} + \frac{\partial}{\partial x}\left(\underline{\underline{\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}}\right) + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

from $\frac{\partial \tau_{xx}}{\partial x} + \underline{\underline{\frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}} + F_x = 0$

$$-2\cdot\frac{\partial^2 \tau_{yz}}{\partial y \partial z} = -\frac{\partial^2 \tau_{xx}}{\partial x^2} + \frac{\partial^2 \tau_{yy}}{\partial y^2} + \frac{\partial^2 \tau_{zz}}{\partial z^2} - \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad \cdots (*)$$

Now we select the ~~eqn~~ comp. eqns containing ε_{yy}, ε_{zz} & ε_{yz}. which were,

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\cdot\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}$$

express this in terms of stress.
with the aid of Generalized Hooke's Law.
recall that, $\varepsilon_{ij} = \frac{1+\nu}{E}\cdot\tau_{ij} - \frac{\nu}{E}\delta_{ij}\tau_{kk}$.
if we put $\tau_m = \frac{\tau_{kk}}{3}$.
scalar.

$$\varepsilon_{yy} = \frac{1+\nu}{E}\cdot\tau_{yy} - \frac{\nu}{E}\cdot 3\tau_m, \quad \varepsilon_{zz} = \frac{1+\nu}{E}\tau_{zz} - \frac{\nu}{E}\cdot 3\tau_m$$

$$\varepsilon_{yz} = \frac{1+\nu}{E}\tau_{yz}$$

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$$\frac{\partial^2}{\partial z^2}\left\{(1+\nu)\tau_{yy} - 3\nu\tau_m\right\} + \frac{\partial^2}{\partial y^2}\left\{(1+\nu)\tau_{zz} - 3\nu\tau_m\right\}$$

$$= \frac{\partial^2}{\partial y \partial z} 2(1+\nu)\tau_{yz}$$

From $*$), we can eliminate the terms containing τ_{yz},

$$\cancel{\frac{\partial^2}{\partial z^2}\left\{(1+\nu)\tau_{yy} - 3\nu\tau_m\right\}} = +(1+\nu)\left(-\frac{\partial^2\tau_{xx}}{\partial x^2} + \frac{\partial^2\tau_{yy}}{\partial y^2} + \frac{\partial^2\tau_{zz}}{\partial z^2} - \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)$$

$$(1+\nu)\left(\nabla^2 3\tau_m - \nabla^2\tau_{xx} - 3\cdot\frac{\partial^2\tau_m}{\partial x^2}\right) - \nu\left(3\nabla^2\tau_m - 3\frac{\partial^2\tau_m}{\partial x^2}\right)$$

$$= (1+\nu)\left(\frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} - \frac{\partial F_z}{\partial z}\right) \quad \cdots ①$$

Following similar steps, we have two other ~~express~~ relations

$$(1+\nu)\left(3\nabla^2\tau_m - \nabla^2\tau_{yy} - 3\frac{\partial^2\tau_m}{\partial y^2}\right) - \nu\left(3\nabla^2\tau_m - 3\frac{\partial^2\tau_m}{\partial y^2}\right)$$

$$= (1+\nu)\cdot\left(\frac{\partial F_y}{\partial y} - \frac{\partial F_x}{\partial x} - \frac{\partial F_z}{\partial z}\right)$$

$$(1+\nu)\left(3\nabla^2\tau_m - \nabla^2\tau_{zz} - 3\frac{\partial^2\tau_m}{\partial z^2}\right) - \nu\left(3\nabla^2\tau_m - 3\cdot\frac{\partial^2\tau_m}{\partial z^2}\right)$$

$$= (1+\nu)\left(\frac{\partial F_z}{\partial z} - \frac{\partial F_y}{\partial y} - \frac{\partial F_x}{\partial x}\right)$$

adding above Eqns. $3\nabla^2 - 3\nu\nabla^2$

$$3(1-\nu)\nabla^2\tau_m = (1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right).$$

$$\nabla^2\tau_m = \frac{(1+\nu)}{3(1-\nu)}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)$$

put this into ①, ~~and following~~

$$\frac{\partial^2 \tau_{xx}}{\partial x^2} + \frac{\partial^2 \tau_{xx}}{\partial y^2} + \nu\left(\frac{\partial^2 \tau_m}{\partial x^2} + \frac{\partial^2 \tau_m}{\partial y^2}\right) = \frac{1}{3}\left(\frac{\partial^2 \tau_{xx}}{\partial x^2} + \frac{\partial^2 \tau_{yy}}{\partial y^2} - \nu\left(\frac{\partial^2 \tau_{xx}}{\partial x^2} + \frac{\partial^2 \tau_{yy}}{\partial y^2}\right)\right)$$

$$\frac{1+\nu+1-\nu}{1+\nu}\frac{\partial^2 \tau_{xx}}{\partial x^2} = \frac{1-\nu}{1+\nu}\frac{\partial^2 \tau_{yy}}{\partial x^2}$$

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$$\nabla^2 \tau_{xx} + \frac{3}{1+\nu}\frac{\partial^2 \tau_m}{\partial x^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) - 2\frac{\partial F_x}{\partial x}.$$

$$\nabla^2 \tau_{yy} + \frac{3}{1+\nu}\frac{\partial^2 \tau_m}{\partial y^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) - 2\frac{\partial F_y}{\partial y}$$

$$\nabla^2 \tau_{zz} + \frac{3}{1+\nu}\frac{\partial^2 \tau_m}{\partial z^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) - 2\frac{\partial F_z}{\partial z}.$$

$\nabla^2(\tau_{xx}+\tau_{yy}) + \frac{1}{1+\nu}\nabla^2(\tau_{xx}+\tau_{yy}) = \frac{2(1+\nu)}{1-\nu}$; $\frac{2+\nu}{1+\nu}$; $-\frac{3\nu}{1-\nu} - 2$; $\frac{-2}{1-\nu} - 1$; $\frac{2+\nu}{1-\nu}$

Similarly we can obtain three eqns in term of the shear stress components.

$$\nabla^2 \tau_{yz} + \frac{3}{1+\nu}\frac{\partial^2 \tau_m}{\partial y \partial z} = -\left(\frac{\partial F_z}{\partial y} + \frac{\partial F_y}{\partial z}\right)$$

$$\nabla^2 \tau_{zx} + \frac{3}{1+\nu}\cdot\frac{\partial^2 \tau_m}{\partial z \partial x} = -\left(\frac{\partial F_x}{\partial z} + \frac{\partial F_z}{\partial x}\right)$$

$$\nabla^2 \tau_{xy} + \frac{3}{1+\nu}\cdot\frac{\partial^2 \tau_m}{\partial x \partial y} = -\left(\frac{\partial F_y}{\partial x} + \frac{\partial F_x}{\partial y}\right)$$

$-\frac{3\nu}{1-\nu} - 2 = \frac{-3\nu+2\nu-2}{1-\nu} = \frac{-\nu-2}{1-\nu}$

The six relations, are called 'compatibility equations in terms of stress' or 'Beltrami-Michell compatibility eqns'. 6 eqns can be further reduced to 3 independent 4th order eqns. → similar to compatibility Eq. this 3 + equilibrium Eqns 3 → for six τ_{ij}.

For example) in plane strain, $\tau_{zz} = \nu(\tau_{xx} + \tau_{yy})$.

$$\nabla^2 \cdot \nu(\tau_{xx} + \tau_{yy}) \neq 0 = -\frac{1}{1-\nu}\cdot\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right).$$

$$\left\{\begin{aligned} &\nabla^2(\tau_{xx}+\tau_{yy}) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) && \text{— comp Eq in tens of stress.} \\ &\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0 && \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + F_y = 0. && \Big\} \text{ Stress Equil Eqs.} \end{aligned}\right.$$

$\tau_{xx}, \tau_{xy}, \tau_{yy}$ can be obtained.

plane stress. $$\nabla^2(\tau_{xx} + \tau_{yy}) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

i) 2D plane stress

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \cdot \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \quad \cdots \text{①}$$

from stress-strain relationship.

$$\varepsilon_{xx} = \frac{1}{E} \cdot (\tau_{xx} - \nu \tau_{yy}), \quad \varepsilon_{yy} = \frac{1}{E}(\tau_{yy} - \nu \tau_{xx})$$

$$\varepsilon_{xy} = \frac{1}{2G} \tau_{xy} = \frac{1+\nu}{E} \cdot \tau_{xy}$$

substituting in ①

$$\frac{\partial^2 (\tau_{xx} - \nu \tau_{yy})}{\partial y^2} + \frac{\partial^2 (\tau_{yy} - \nu \tau_{xx})}{\partial x^2} = 2(1+\nu) \frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

Equil Eqs for 2D.

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0 \quad \text{— w.r.t } x$$

$$+\Bigg(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + F_y = 0 \quad \text{— w.r.t } y$$

$$\frac{\partial^2 \tau_{xx}}{\partial x^2} + \frac{\partial^2 \tau_{yx}}{\partial x \partial y} + \frac{\partial F_x}{\partial x} = 0$$

$$\frac{\partial^2 \tau_{xy}}{\partial y \partial x} + \frac{\partial^2 \tau_{yy}}{\partial y^2} + \frac{\partial F_y}{\partial y} = 0$$

$$2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \tau_{xx}}{\partial x^2} - \frac{\partial^2 \tau_{yy}}{\partial y^2} - \frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y}$$

$$\frac{\partial^2 (\tau_{xx} - \nu \tau_{yy})}{\partial y^2} + \frac{\partial^2 (\tau_{yy} - \nu \tau_{xx})}{\partial x^2} + (1+\nu) \frac{\partial^2 \tau_{xx}}{\partial x^2} + (1+\nu) \frac{\partial^2 \tau_{yy}}{\partial y^2} = -(1+\nu)\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\right)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\tau_{xx} + \tau_{yy}) = -(1+\nu)\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\right)$$

$$\nabla^2 (\tau_{xx} + \tau_{yy}) = -(1+\nu)\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\right)$$

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ii) in 2D plane strain,

$$\varepsilon_{xx} = \frac{1}{E}\{(1-\nu^2)\tau_{xx} - \nu(1+\nu)\tau_{yy}\}$$

$$\varepsilon_{yy} = \frac{1}{E}\{(1-\nu^2)\tau_{yy} - \nu(1+\nu)\tau_{xx}\}$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\tau_{xy}.$$

$$\frac{\partial^2}{\partial y^2}\{(1-\nu^2)\tau_{xx} - \nu(1+\nu)\tau_{yy}\} + \frac{\partial^2}{\partial x^2}\{(1-\nu^2)\tau_{yy} - \nu(1+\nu)\tau_{xx}\}$$

$$+ (1+\nu)\frac{\partial^2\tau_{xx}}{\partial x^2} + (1+\nu)\frac{\partial^2\tau_{yy}}{\partial y^2} = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

$$(1+\nu)(1-\nu)\left(\frac{\partial^2\tau_{xx}}{\partial x^2} + \frac{\partial^2\tau_{yy}}{\partial y^2}\right) = -(1+\nu)\left(\frac{\partial F}{\partial x} + \frac{\partial F_y}{\partial y}\right).$$

$$\nabla^2(\tau_{xx} + \tau_{yy}) = -\frac{1}{1-\nu}\left(\frac{\partial F}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

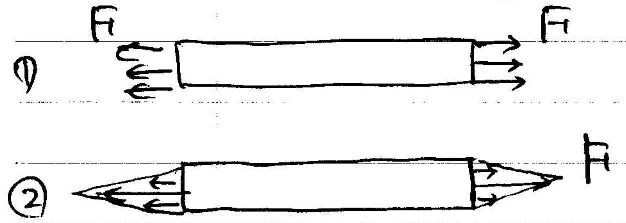
- Eqns do not contain "E". → Stress distribution is the same for all isotropic material with the same ν.
- In the case of constant body force (which is usually the case), compatibility Eqn in terms of stress without body force hold for both plane strain & stress. Stress distributions are the same with the same shape of the boundary & external force.

$$\nabla^2(\tau_{xx} + \tau_{yy}) = 0.$$

* Saint-Venant's Principle – End effect.

" the stresses due to two statically equivalent loadings applied over a small area are significantly different only in the vicinity the area on which the loadings are applied," and at distances which are large in comparison with the linear dimensions of the area on which the loadings are applied, the effect due to these two loadings are the same ".

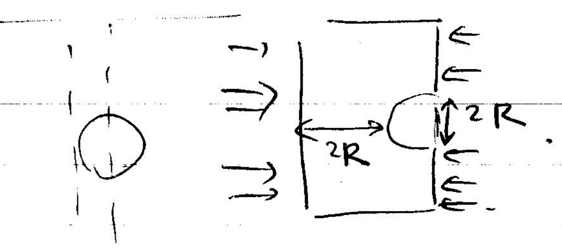
$\tau_{xx} = p, \quad \tau_{xy} = \tau_{yz} = \tau_{zx} = \tau_{yy} = \tau_{zz} = 0$.



① satisfy stress equilibrium eqns & compat eqns.
we have the exact solution, furthermore, the uniqueness theorem asserts that it is the only solution.

② According to Saint-Venant Principle, stresses for ② closely approximate the actual stress distribution except near the end of the bar.

- local change doesn't make a global change as long as the resultant forces are the same.
- fundamentally local stress tends to diffuses.



× summary of today's lecture 3-3-4

× mid-term exam oral 22nd 14:30 - 15:00 - 15:30 16:30 - 17:30

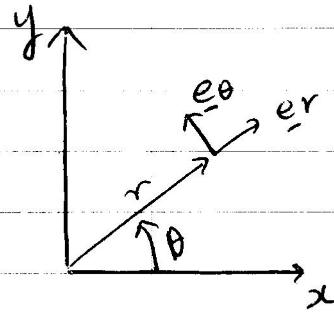
~~in~~ error in the textbook

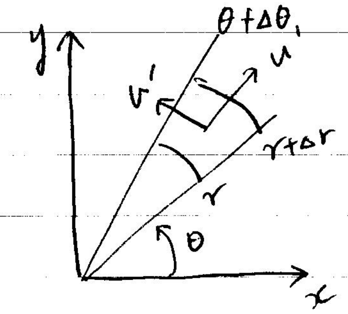
☆ Stress & Strain in polar & cylindrical coord.

it is convenient to use polar (2D) or cylindrical (3D) coordinates for.

- tunnels, boreholes or cylindrical rock core.

Cylindrical coor. = polar + z-axis.





infinitesimal element in polar coordinate.

$$x = r\cos\theta \,,\quad y = r\sin\theta$$
$$r = \sqrt{x^2+y^2} \,,\quad \theta = \arctan\left(\frac{y}{x}\right)$$

At each point, we can imagine a local pair of $\underline{e}_r$ & $\underline{e}_\theta$ pointing in the r & θ directions, respectively.

$\{\underline{e}_r, e_\theta\}$ thought of as $\{\underline{e}_x, \underline{e}_y\}$.

$$\tau_{rr} = \tau_{xx}\cos^2\theta + 2\tau_{xy}\sin\theta\cos\theta + \tau_{yy}\sin^2\theta$$
$$\tau_{\theta\theta} = \tau_{xx}\sin^2\theta - 2\tau_{xy}\sin\theta\cos\theta + \tau_{yy}\cos^2\theta$$
$$\tau_{r\theta} = (\tau_{yy}-\tau_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

displacement in polar coordinates, (u', v').

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u' \\ v' \end{pmatrix} \quad u' = u\cos\theta + v\sin\theta \,,\quad v' = -u\sin\theta + v\cos\theta \,.$$
$$u = u'\cos\theta - v'\sin\theta \,,\quad v = u'\sin\theta + v'\cos\theta \,.$$

Straightforward so far. But more complicated for strains.
← involve <u>differentiation</u>. $\underline{e}_r, \underline{e}_\theta$ ~~at~~ different directions at different locations. point in

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2)^{\frac{1}{2}} = \frac{x}{(x^2+y^2)^{\frac{1}{2}}} = \frac{x}{r} = \cos\theta$$
$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2)^{\frac{1}{2}} = \frac{y}{(x^2+y^2)^{\frac{1}{2}}} = \frac{y}{r} = \sin\theta$$

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$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(\arctan\left(y/x\right)\right) = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \frac{-y}{r^2} = \frac{-\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}\left(\arctan\left(y/x\right)\right) = \frac{1/x}{\left(1+\left(y/x\right)^2\right)} = \frac{x}{r^2} = \frac{\cos\theta}{r}.$$

apply chain rule to ε_{xx},

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \cos\theta\cdot\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}.$$

from before. $u' = u\cos\theta + v\sin\theta$

$$\frac{\partial u}{\partial r} = \cos\theta\frac{\partial u'}{\partial r} - \sin\theta\frac{\partial v'}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \cos\theta\frac{\partial u'}{\partial \theta} - u'\sin\theta - \sin\theta\frac{\partial v'}{\partial \theta} - v'\cos\theta$$

$$\varepsilon_{xx} = \frac{\partial u'}{\partial r}\cos^2\theta - \left[\frac{\partial v'}{\partial r} + \frac{1}{r}\frac{\partial u'}{\partial \theta} - \frac{v'}{r}\right]\sin\theta\cos\theta + \left(\frac{1}{r}\frac{\partial v'}{\partial \theta} + \frac{u'}{r}\right)\sin^2\theta$$

rotate $\varepsilon_{rr}, \varepsilon_{\theta\theta}$ in $-\theta$

$$\varepsilon_{xx} = \varepsilon_{rr}\cos^2\theta - 2\varepsilon_{r\theta}\sin\theta\cos\theta + \varepsilon_{\theta\theta}\sin^2\theta.$$

By comparing,

$$\varepsilon_{rr} = \frac{\partial u'}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v'}{\partial \theta} + \frac{u'}{r}, \quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial v'}{\partial r} + \frac{1}{r}\frac{\partial u'}{\partial \theta} - \frac{v'}{r}\right)$$

if we add z-axis. perpendicular to r, θ,

$$\varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial w'}{\partial r} + \frac{\partial u'}{\partial z}\right), \quad \varepsilon_{z\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial w'}{\partial \theta} + \frac{\partial v'}{\partial z}\right).$$

$$\varepsilon_{zz} = \frac{\partial w'}{\partial z}.$$

ex) volumetric strain

$$= \varepsilon_v = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = \frac{\partial u'}{\partial r} + \frac{u'}{r} + \frac{1}{r}\frac{\partial v'}{\partial \theta} + \frac{\partial w'}{\partial z}.$$

Equations of stress equilibrium.

$$
\begin{cases}
\dfrac{\partial \tau_{rz}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta z}}{\partial \theta} + \dfrac{\partial \tau_{zz}}{\partial z} + \dfrac{\tau_{rz}}{r} + \rho F_z = 0. \\[2ex]
\dfrac{\partial \tau_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta r}}{\partial \theta} + \dfrac{\partial \tau_{zr}}{\partial z} + \dfrac{(\tau_{rr} - \tau_{\theta\theta})}{r} + \rho F_r = 0 \\[2ex]
\dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta\theta}}{\partial \theta} + \dfrac{\partial \tau_{r\theta}}{\partial z} + \dfrac{2\,\tau_{r\theta}}{r} + \rho F_\theta = 0
\end{cases}
$$