Lecture 7. Anisotropic materials No. ①

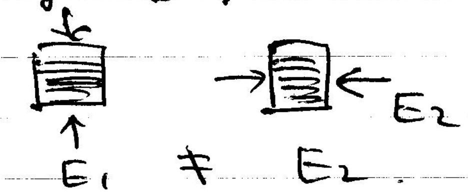
In contrast to isotropic rock, anisotropic rock has more than 2 elastic constants.

① difficult to characterize
② difficult to solve Boundary value problem.

(Barla, 1974)

- class A : exhibit anisotropic properties despite apparent isotropy — some intact granite
- class B : anisotropic properties with clear visual evidence
  - ex) metamorphic rocks
  - sedimentary rocks
  - fractured rock mass with fractures.

Anisotropy can be an issue. eg) shale.



* Generalized Hooke law ~~for anisotropic material~~.

$\tau_{ij} = C_{ijkl} \cdot \varepsilon_{kl}$
(9) (9)

C_{ijkl} need 81 component.

① symmetry of τ_{ij} → $C_{ijkl} = C_{jikl}$
② symmetry of ε_{kl} → $C_{ijkl} = C_{ijlk}$
→ C_{ijkl} 36 component

this can be reduced more by using 'storage of strain energy' (or existence of strain energy).

Strain Energy, $W = \frac{1}{2}\tau_{ij}\varepsilon_{ij} = \frac{1}{2}(\tau_{11}\varepsilon_{11} + \tau_{12}\varepsilon_{12} + \tau_{13}\varepsilon_{13} + \cdots + \tau_{32}\varepsilon_{32} + \tau_{33}\varepsilon_{33})$

$$\frac{\partial W}{\partial \varepsilon_{11}} = \frac{1}{2}\tau_{11} = \frac{1}{2}C_{11kl}\varepsilon_{kl} = \frac{1}{2}(C_{1111}\varepsilon_{11} + C_{1112}\varepsilon_{12} + \cdots)$$

$$\frac{\partial}{\partial \varepsilon_{22}}\left(\frac{\partial W}{\partial \varepsilon_{11}}\right) = \frac{1}{2}C_{1122} = \frac{\partial^2 W}{\partial \varepsilon_{22}\partial \varepsilon_{11}} = \frac{\partial^2 W}{\partial \varepsilon_{11}\partial \varepsilon_{22}} = \frac{1}{2}C_{2211}$$

two of the cross partial derivatives of the strain energy with respect to two different strains must be equal

$$\frac{\partial^2 W}{\partial \varepsilon_{yy}\partial \varepsilon_{xx}} = \frac{\partial^2 W}{\partial \varepsilon_{xx}\partial \varepsilon_{yy}}.$$

now $\frac{\partial W}{\partial \varepsilon_{22}} = \frac{1}{2} \tau_{22} = \frac{1}{2} C_{22kl} \varepsilon_{kl} = \frac{1}{2} (C_{2211}\varepsilon_{11} + \varepsilon_{2212}\varepsilon_{12} + \ldots)$

$\frac{\partial}{\partial \varepsilon_{11}} \left(\frac{\partial W}{\partial \varepsilon_{22}} \right) = \frac{1}{2} C_{2211} = \frac{\partial^2 W}{\partial \varepsilon_{11} \partial \varepsilon_{22}}$

$\therefore C_{1122} = C_{2211}$

if we repeat this procedure with different pairs of $\varepsilon_{ij} \rightarrow {}_6C_2$

$C_{ijkl} = C_{klij} \rightarrow$ reduce 15 more

$\therefore$ there are 21 independent parameters.
physical meaning of each components → ppt.

Generalized Hookes Law ; in matrix notation.
Each of the six components of stress at any point of a body is a linear fn of the six components of strain at the point.

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{pmatrix} \begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

elastic stiffness tensor

$\tau_{ij} = C_{ijkl} \varepsilon_{kl}$ ~~$\tau = C \varepsilon$~~

elastic compliance tensor

$\varepsilon_{ij} = S_{ijkl} \tau_{kl}$

Because S_{ijkl} is a fourth order tensor, it follows the tensor rule. → tensor in transformed axis can be obtained by multiplication of direction cosine.

$$\underline{\underline{\tau}}' = \underline{\underline{R}}\,\underline{\underline{\tau}}\,\underline{\underline{R}}^T$$

recall that $\tau'_{ij} = \beta_{im}\beta_{jn}\tau_{mn}$

$$\beta_{ij} = \begin{pmatrix} \cos(x',x) & \cos(x',y) & \cos(x',z) \\ \cos(y',x) & \cos(y',y) & \cos(y',z) \\ \cos(z',x) & \cos(z',y) & \cos(z',z) \end{pmatrix} \quad \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

↑ rotation.

$$S'_{ijkl} = \beta_{im}\beta_{jn}\beta_{kp}\beta_{lq}S_{mnpq}.$$

above is mathematically elegant, but not convenient for practical calculation. ~~Matrix~~

Lekhnitskii introduced following 6×6 matrix transformation.

$$S'_{ij} = S_{mn}\,q_{mi}\,q_{nj}$$

component of q_{ij} → ppt file.

note that the orders of the expression for strains & stress are slightly different.

T~~herefore matrix~~

Elastic Symmetry.

- If a ~~material exhibits any physical symmetry~~, the number of independent constants can be reduced.

① Monoclinic material – one plane of elastic symmetry.

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & \frac{\eta_{x,xy}}{G_{xy}} & 0 & 0 \\ & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & \frac{\eta_{y,xy}}{G_{xy}} & 0 & 0 \\ & & \frac{1}{E_z} & \frac{\eta_{z,xy}}{G_{xy}} & 0 & 0 \\ & & & \frac{1}{G_{xy}} & 0 & 0 \\ & & & & \frac{1}{G_{yz}} & \frac{\mu_{yz,xz}}{G_{xz}} \\ & & & & * & \frac{1}{G_{zx}} \end{pmatrix} \begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix}$$

when xOy is symmetric plane

- if the elastic constants have the same values for every pair of coordinate systems that are the reflected images of one another with respect to the plane, 13 independent constants.

Elastic Symmetry

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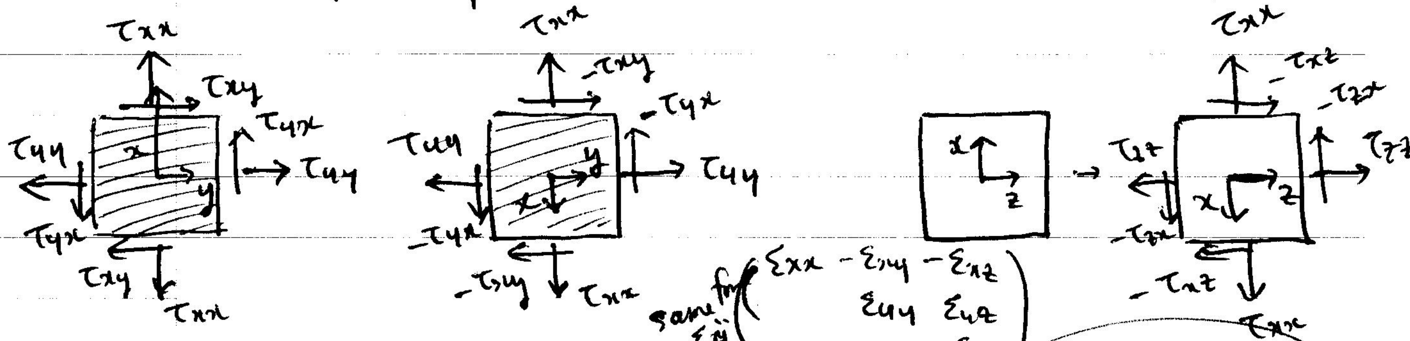
  - if the elastic constants have the same values for every pair of coordinate systems that are the reflected images of one another with respect to the plane.
- number of independent constant : 13.



same for ε_{ij} $\begin{pmatrix} \varepsilon_{xx} & -\varepsilon_{xy} & -\varepsilon_{xz} \\ & \varepsilon_{yy} & \varepsilon_{yz} \\ & & \varepsilon_{zz} \end{pmatrix}$

$$\begin{pmatrix} \tau'_{xx} & \tau'_{xy} & \tau'_{xz} \\ & \tau'_{yy} & \tau'_{yz} \\ & & \tau'_{zz} \end{pmatrix} = \begin{pmatrix} \tau_{xx} & -\tau_{xy} & -\tau_{xz} \\ & \tau_{yy} & \tau_{yz} \\ & & \tau_{zz} \end{pmatrix}$$

$C_{ij} = C'_{ij}$ because of symmetry.

From $\tau'_{xy} = C_{16}\varepsilon'_{xx} + C_{26}\varepsilon'_{yy} + C_{36}\varepsilon'_{zz} + C_{46}\varepsilon'_{yz} + C_{56}\varepsilon'_{xz} + C_{66}\varepsilon'_{xy}$

$$\frac{\partial \tau'_{xy}}{\partial \varepsilon_{xx'}} = C_{16} = \frac{-\partial \tau_{xy}}{\partial \varepsilon_{xx}} = -C_{16} \Rightarrow C_{16} = 0.$$

$$\frac{\partial \tau'_{xy}}{\partial \varepsilon'_{yy}} = C_{26} = -\frac{\partial \tau_{xy}}{\partial \varepsilon_{yy}} = -C_{26} \Rightarrow C_{26} = 0.$$

⋮ ⋮

$C_{46} = 0.$

$$\frac{\partial \tau'_{xy}}{\partial \varepsilon'_{xz}} = C_{56} = \frac{-\partial \tau_{xy}}{-\partial \varepsilon_{xz}} = C_{56} \Rightarrow C_{56} \neq 0.$$

$C_{66} \neq 0.$

Similarly, $C_{15} = C_{25} = C_{35} = C_{45} = 0.$

e.g.) $\frac{\partial \tau'_{xz}}{\partial \varepsilon'_{xx}} = C_{15} = \frac{-\partial \tau_{xz}}{\partial \tau_{xx}} = -C_{15} \Rightarrow C_{15} = 0.$

$$\begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau \\ \\ \\ \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & 0 & 0 \\ S_{21} & S_{22} & S_{23} & S_{24} & 0 & 0 \\ S_{31} & S_{32} & S_{33} & S_{34} & 0 & 0 \\ * & * & * & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & S_{56} \\ 0 & 0 & 0 & 0 & * & S_{66} \end{pmatrix} \begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

② orthotropic – three orthogonal planes of elastic symmetry
– similar argument for x-y or xz planes.
$\rightarrow C_{14} = C_{24} = C_{34} = C_{56} = 0$.

$$\begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{pmatrix}.$$

or

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{pmatrix} \begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_{xx} \\ \\ \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ & & \frac{1}{E_z} & 0 & 0 & 0 \\ & & & \frac{1}{G_{yz}} & 0 & 0 \\ & & & & \frac{1}{G_{xz}} & 0 \\ & & & & & \frac{1}{G_{xy}} \end{pmatrix} \begin{pmatrix} \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

ν_{ij} : ratio of strain in j direction to i direction due to a stress acting in i direction.

From symmetry of compliance matrix. $\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}$

$\therefore$ ★ $\frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x}$, $\frac{\nu_{zx}}{E_z} = \frac{\nu_{xz}}{E_x}$, $\frac{\nu_{zy}}{E_z} = \frac{\nu_{yz}}{E_y}$.

9 independent constants.

transversely isotropic —

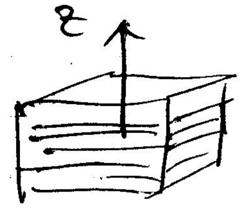
③ One axis of elastic symmetry of rotation ⑥

- if there are sets of equivalent directions in a body which can be superimposed by a rotation through an angle $2\pi/n$ about an axis, then this axis is an axis of symmetry of order n.
  - $n=2 \rightarrow$ Monoclinic
  - $n \rightarrow \infty \rightarrow$ Transversely Isotropic.
- Some say this name is misleading because this contains the word 'isotropic' but it is not. isotropic is a special case.

Anisotropic	Isotropic
general	special
normal	abnormal

- In Korean, 횡등방성, some say '평면이방성'. very misleading.

For z-axis as symmetric axis



$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu'}{E'} & 0 & 0 & 0 \\ & \frac{1}{E} & -\frac{\nu'}{E'} & 0 & 0 & 0 \\ & & \frac{1}{E'} & 0 & 0 & 0 \\ & & & \frac{1}{G'} & 0 & 0 \\ & & & & 1/G' & 0 \\ & & & & & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

Five independent constants.

By comparing with orthotropic case

$E = E_x = E_y$, $E' = E_z$, $G' = G_{yz} = G_{xz}$.

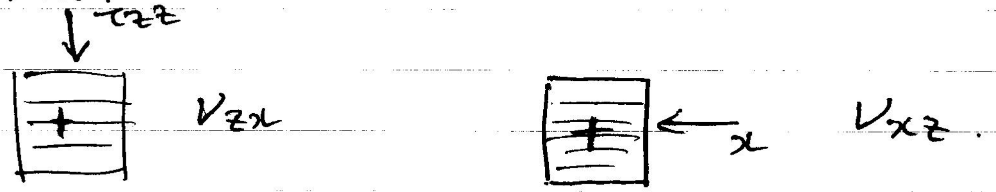
$\nu = \nu_{xy} = \nu_{yx}$, $\nu' = \nu_{zx} = \nu_{zy}$, $G = \dfrac{E}{2(1+\nu)}$.

Some people use different definition for this second poisson's ratio.

~~$\nu' = \nu_{xz}$, $\nu_{xz} = \dfrac{E_x}{E_z} \cdot \nu_{zx}$~~ .

~~$S_{13} = \dfrac{\nu}{E}$~~

☆ important!



ν_{zx} & ν_{xz} are different but not independent!

$$\nu_{xz} = \frac{E_x}{E_z} \cdot \nu_{zx} = \frac{E}{E'} \cdot \nu'$$

if ν_{xz} is used as ν', $\quad \dfrac{\nu_{zx}}{E_z} = \left(\dfrac{E'}{E} \cdot \nu'\right) \cdot \dfrac{1}{E'} = \dfrac{\nu'}{E}$.

☆ Bounds of elastic constants.

Even if there are 21, 13, 9, 5 or 2 independent elastic constants, ~~the~~ constraints on their numerical value exist. ⇐ due to the requirement that elastic energy function must be positive definite. For W to be positive, we must have

$$C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl} > 0 \quad \text{or} \quad S_{ijkl}\,\tau_{ij}\,\tau_{kl} > 0 .$$

in quadratic form, $\tau^T S \tau > 0$.

matrix S_{ij} has to be positive definite.

Advanced Rock Mechanics
7. Anisotropic material

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Generalized Hooke's Law
Tensor & Matrix Form

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$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$$

Contracted form

$$\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16}\\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26}\\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36}\\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46}\\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56}\\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66}\end{pmatrix} \begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{pmatrix}$$

- Compliance matrix has 21 independent parameters
  (By the symmetry of stress tensor, strain tensor and consideration of strain energy)

More explicit expression - Lekhnitskii(1963), Hudson (1997)

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Coupling of normal in different directions

Coupling of normal in the same directions

Coupling of normal & Shear

Coupling of shear in different directions

Coupling of shear in the same directions

$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} =
\begin{bmatrix}
\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & \frac{\eta_{x,yz}}{G_{yz}} & \frac{\eta_{x,xz}}{G_{xz}} & \frac{\eta_{x,xy}}{G_{xy}} \\
-\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & \frac{\eta_{y,yz}}{G_{yz}} & \frac{\eta_{y,xz}}{G_{xz}} & \frac{\eta_{y,xy}}{G_{xy}} \\
-\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & \frac{\eta_{z,yz}}{G_{yz}} & \frac{\eta_{z,xz}}{G_{xz}} & \frac{\eta_{z,xy}}{G_{xy}} \\
\frac{\eta_{yz,x}}{E_x} & \frac{\eta_{yz,y}}{E_y} & \frac{\eta_{yz,z}}{E_z} & \frac{1}{G_{yz}} & \frac{\mu_{yz,xz}}{G_{xz}} & \frac{\mu_{yz,xy}}{G_{xy}} \\
\frac{\eta_{xz,x}}{E_x} & \frac{\eta_{xz,y}}{E_y} & \frac{\eta_{xz,z}}{E_z} & \frac{\mu_{xz,yz}}{G_{yz}} & \frac{1}{G_{xz}} & \frac{\mu_{xz,xy}}{G_{xy}} \\
\frac{\eta_{xy,x}}{E_x} & \frac{\eta_{xy,y}}{E_y} & \frac{\eta_{xy,z}}{E_z} & \frac{\mu_{xy,yz}}{G_{yz}} & \frac{\mu_{xy,xz}}{G_{xz}} & \frac{1}{G_{xy}}
\end{bmatrix}
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}
$$

Monoclinic

One plane of elastic symmetry

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$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} =
\begin{bmatrix}
\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & \frac{\eta_{x,xy}}{G_{xy}} \\
-\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & \frac{\eta_{y,xy}}{G_{xy}} \\
-\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & \frac{\eta_{z,xy}}{G_{xy}} \\
0 & 0 & 0 & \frac{1}{G_{yz}} & \frac{\mu_{yz,xz}}{G_{xz}} & 0 \\
0 & 0 & 0 & \frac{\mu_{xz,yz}}{G_{yz}} & \frac{1}{G_{xz}} & 0 \\
\frac{\eta_{xy,x}}{E_x} & \frac{\eta_{xy,y}}{E_y} & \frac{\eta_{xy,z}}{E_z} & 0 & 0 & \frac{1}{G_{xy}}
\end{bmatrix}
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}
$$

- With a plane of symmetry normal to z-axis
- 13 independent constants

Orthotropic
Three orthogonal planes of elastic symmetry

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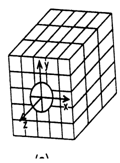
$$\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{pmatrix}=\begin{pmatrix}\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0\\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0\\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}}\end{pmatrix}\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{pmatrix}$$



- Three orthogonal planes elastic symmetry
- 9 independent constants

Transversely Isotropic
One axis of elastic symmetry of rotation

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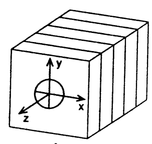
$$\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{pmatrix}=\begin{pmatrix}\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu'}{E'} & 0 & 0 & 0\\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu'}{E'} & 0 & 0 & 0\\ -\frac{\nu'}{E'} & -\frac{\nu'}{E'} & \frac{1}{E'} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G'} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G'} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G}\end{pmatrix}\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{pmatrix}$$



(d)

$E_x = E_y = E$

$E_z = E'$

$\nu_{xy} = \nu_{yx} = \nu$

$\nu_{zx} = \nu_{zy} = \nu'$

$G_{xz} = G_{yz} = G'$

$$\nu_{xz} = \nu_{yz} = \nu' \frac{E}{E'}$$

- 5 independent constants

Isotropic
Complete symmetry

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$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

$E' = E$

$\nu' = \nu$

$G' = G$

Bounds of elastic constants

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$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \qquad W = \frac{1}{2}\sigma^T S \sigma$$

- the 6×6 matrices of elastic constants must be positive definite (Ting, 1996)
- A necessary and sufficient condition for the quadratic form to be positive definite is that all principal minors of matrix (that is all minor determinants in the matrix having diagonal elements coincident with the principal diagonal of the matrix) are positive (Amadei et al 1987).

Bounds of elastic constants
Orthogonal

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$$E_x, E_y, E_z, G_x, G_y, G_z > 0$$

$$\begin{vmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} \end{vmatrix} = \frac{1}{E_xE_y} - \frac{\nu_{yx}\nu_{xy}}{E_xE_y} \rangle 0 \longrightarrow 1-\nu_{yx}\nu_{xy} \rangle 0$$

$$\begin{vmatrix} \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} \\ -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} \end{vmatrix} = \frac{1}{E_yE_z} - \frac{\nu_{zy}\nu_{yz}}{E_yE_z} \rangle 0 \longrightarrow 1-\nu_{zy}\nu_{yz} \rangle 0$$

$$\begin{vmatrix} \frac{1}{E_x} & -\frac{\nu_{zx}}{E_z} \\ -\frac{\nu_{xz}}{E_x} & \frac{1}{E_z} \end{vmatrix} = \frac{1}{E_xE_z} - \frac{\nu_{zx}\nu_{xz}}{E_xE_z} \rangle 0 \longrightarrow 1-\nu_{zx}\nu_{xz} \rangle 0$$

$$\begin{vmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} \end{vmatrix} = \frac{1}{E_xE_yE_z} - \frac{\nu_{zy}\nu_{yz}}{E_xE_yE_z} - \frac{\nu_{yx}\nu_{xy}}{E_xE_yE_z} - \frac{\nu_{xz}\nu_{zx}}{E_xE_yE_z} - \frac{\nu_{yx}\nu_{xz}\nu_{zy}}{E_xE_yE_z} - \frac{\nu_{zx}\nu_{xy}\nu_{yz}}{E_xE_yE_z} \rangle 0$$

$$\longrightarrow 1-\nu_{zy}\nu_{yz} - \nu_{yx}\nu_{xy} - \nu_{xz}\nu_{zx} - \nu_{yx}\nu_{xz}\nu_{zy} - \nu_{zx}\nu_{xy}\nu_{yz} \rangle 0$$

Bounds of elastic constants
Orthogonal

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$$E_x, E_y, E_z, G_x, G_y, G_z > 0$$

$$-\sqrt{\frac{E_x}{E_y}} \langle \nu_{xy} \langle \sqrt{\frac{E_x}{E_y}}$$

$$-\sqrt{\frac{E_y}{E_z}} \langle \nu_{yz} \langle \sqrt{\frac{E_y}{E_z}}$$

$$-\sqrt{\frac{E_x}{E_z}} \langle \nu_{xz} \langle \sqrt{\frac{E_x}{E_z}}$$

$$1 - \frac{E_z}{E_y}\nu_{yz}^2 - \frac{E_y}{E_x}\nu_{xy}^2 - \frac{E_z}{E_x}\nu_{xz}^2 - 2\frac{E_z}{E_x}\nu_{xy}\nu_{xz}\nu_{yz} \rangle 0$$

Bounds of elastic constants
Transversely Isotropic

$$E, E', G' > 0$$

$$-1 < \nu < 1$$

$$-\sqrt{\frac{E'}{E}\frac{(1-\nu)}{2}} < \nu' < \sqrt{\frac{E'}{E}\frac{(1-\nu)}{2}}$$

Bounds of elastic constants
Isotropic

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$$E > 0, G > 0$$

$$-1 < \nu < 1 \longrightarrow$$

$$-1 < \nu < \frac{1}{2}$$

$$E > 0$$

$$-\sqrt{\frac{(1-\nu)}{2}} < \nu < \sqrt{\frac{(1-\nu)}{2}}$$

Application to fractured rock masses - Amadei (1981)

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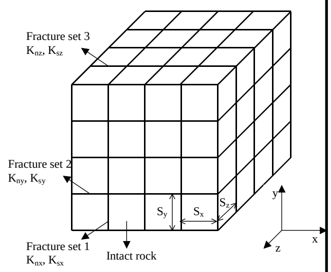
Rock masses with three perpendicular fracture sets can modelled as orthogonally isotropic rock

$$\begin{pmatrix} \frac{1}{E_x}+\frac{1}{K_{nx}S_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{yx}}{E_y} & \frac{1}{E_y}+\frac{1}{K_{ny}S_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{zx}}{E_z} & -\frac{\nu_{zy}}{E_z} & \frac{1}{E_z}+\frac{1}{K_{nz}S_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}}+\frac{1}{K_{sy}S_y}+\frac{1}{K_{sz}S_z} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}}+\frac{1}{K_{sx}S_x}+\frac{1}{K_{sz}S_z} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}}+\frac{1}{K_{sx}S_x}+\frac{1}{K_{sy}S_y} \end{pmatrix}$$



Transformation of compliance tensor under the transformation of axis

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- 0th order tensor (scalar) : no need to transform, independent of coordinate
- 1th order tensor (vector) :
- 2nd order tensor :
  - i.e. stress, strain, permeability
- 4th order tensor :
  - Compliance tensor

$$x_i' = \beta_{ij} x_j$$

$$\sigma_{ij}' = \beta_{im}\beta_{jn}\sigma_{mn}$$

$$S_{ijkl}' = \beta_{im}\beta_{jn}\beta_{kp}\beta_{lp}S_{mnpq}$$

$$\beta_{ij} = \begin{pmatrix} \cos(x',x) & \cos(x',y) & \cos(x',z) \\ \cos(y',x) & \cos(y',y) & \cos(y',z) \\ \cos(z',x) & \cos(z',y) & \cos(z',z) \end{pmatrix} \qquad \beta_{ij} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

General transformation Rotation

Compliance matrix
Transformation

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$$S'_{ijkl} = \beta_{im}\beta_{jn}\beta_{kp}\beta_{lp}S_{mnpq}$$

$$\downarrow$$

$$S'_{ij} = S_{mn}q_{mi}q_{nj}$$

$$\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16}\\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26}\\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36}\\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46}\\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56}\\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66}\end{pmatrix}\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{pmatrix}$$

	X	Y	z
X'	α_1	β_1	γ_1
Y'	α_2	β_2	γ_2
Z'	α_3	β_3	γ_3

	1	2	3	4	5	6
1	α_1^2	α_2^2	α_3^2	$2\alpha_2\alpha_3$	$2\alpha_3\alpha_1$	$2\alpha_1\alpha_2$
2	β_1^2	β_2^2	β_3^2	$2\beta_2\beta_3$	$2\beta_3\beta_1$	$2\beta_1\beta_2$
3	γ_1^2	γ_2^2	γ_3^2	$2\gamma_2\gamma_3$	$2\gamma_3\gamma_1$	$2\gamma_1\gamma_2$
4	$\beta_1\gamma_1$	$\beta_2\gamma_2$	$\beta_3\gamma_3$	$\beta_2\gamma_3+\beta_3\gamma_2$	$\beta_1\gamma_3+\beta_3\gamma_1$	$\beta_1\gamma_2+\beta_2\gamma_1$
5	$\gamma_1\alpha_1$	$\gamma_2\alpha_2$	$\gamma_3\alpha_3$	$\gamma_2\alpha_3+\gamma_3\alpha_2$	$\gamma_1\alpha_3+\gamma_3\alpha_1$	$\gamma_1\alpha_2+\gamma_2\alpha_1$
6	$\alpha_1\beta_1$	$\alpha_2\beta_2$	$\alpha_3\beta_3$	$\alpha_2\beta_3+\alpha_3\beta_2$	$\alpha_1\beta_3+\alpha_3\beta_1$	$\alpha_1\beta_2+\alpha_2\beta_1$

Transformation of compliance tensor
Elastic modulus and Poisson's ratio

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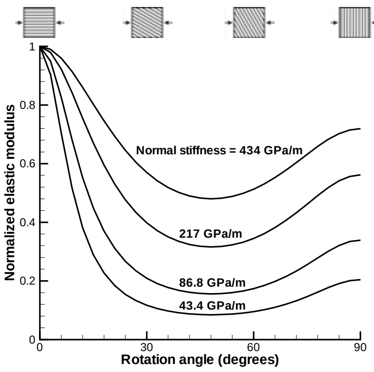
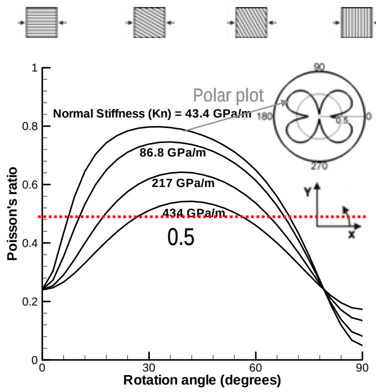


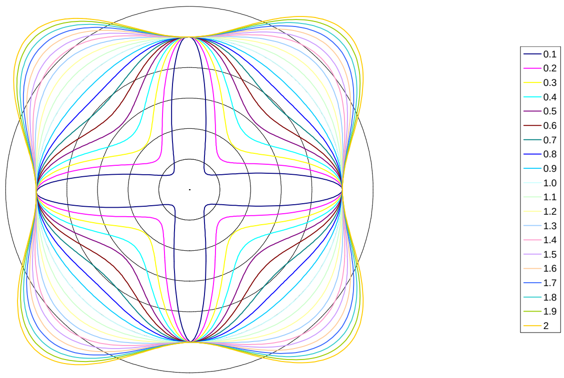


Transversely Isotropic rock

Transformation of compliance tensor
Elastic modulus

Ex variation with Gxy/G ratio

Ex=Ey
Nu = 0.25
Gxy/G varies



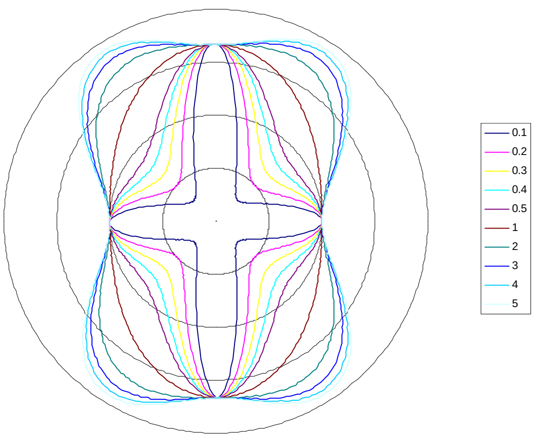
- Orthotropic rock

Transformation of compliance tensor
Elastic modulus

Ex variation with Ks/Kn ratio

K_n for set 1

= 2* K_n for set 2



Fractured Rock Masses