ξ : xi [zai] $\quad$ ψ : psi
ϕ : phi $\quad$ η : Eta
χ : chi [kai] $\quad$ ζ : zeta

Lecture 8. Stresses around ~~cavities~~ & Excavations.

- 2D elasticity problems can be solved using the complex variable method.
- Stresses & displacements are represented in terms of two analytic functions of a complex variable.

- complex number $z = x + iy$.

$$x = \mathrm{Re}(z)$$
$$y = \mathrm{Im}(z)$$
$$\bar{z} = x - iy \quad : \text{complex conjugate of } z$$
$$\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z}), \quad \mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$$

- A complex valued function of a complex variable, $\zeta(z)$,

$$\zeta = \xi + i\eta = \xi(x,y) + i\eta(x,y).$$

- The function, $\zeta(z)$, is said to be analytic in a domain D if $\zeta(z)$ is defined and differentiable at all points of D. Analytic is often called 'holomorphic'

\# Cauchy-Riemann Equation.

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y}, \quad \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x}.$$

① … $\zeta'(z) = \frac{d\zeta}{dz} \overset{?}{=} \frac{\partial \xi}{\partial x} + i\frac{\partial \eta}{\partial x} = \frac{\partial \xi}{\partial x} - i\frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial y} + i\frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial y} - i\frac{\partial \xi}{\partial y}$.

if $\zeta(z)$ is analytic, both real and imaginary part of ζ satisfies Laplace's equation.

$$\nabla^2 \xi(x,y) = 0, \quad \nabla^2 \eta(x,y) = 0$$

harmonic function: function that satisfies Laplace's Eq.

Two harmonic functions related through C-R eq. are called conjugate harmonic function.

— don't confuse with complex conjugate

Lecture 8.

* Airy Stress Function

Date . . No.

in 2D compatibility Eq in terms of stress becomes,

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right)(\tau_{xx}+\tau_{yy}) = \frac{-4}{(\chi+1)}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right) \cdots ①$$

$$\chi = \begin{cases} 3-4\nu & \text{for plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{for plane stress.} \end{cases}$$

Body force can be expressed as the gradient of a potential function, V, that satisfies Laplace's Equation. (in terms of)

$$\nabla^2 V = \nabla\cdot(\nabla V) = 0 . \quad V = -\rho z .$$

$$\underline{F} = -\rho\cdot\nabla V .$$

$$F_x = -\rho\cdot\frac{\partial V}{\partial x}, \quad F_y = -\rho\frac{\partial V}{\partial y} .$$

→ note the difference from ~~the text book~~.

right hand side of ① vanishes. $\nabla^2 V = 0$.

$$\nabla^2(\tau_{xx}+\tau_{yy}) = 0 .$$

in stress-based formulation, in 2D, < 1 compatibility Eq, 2 stress Equilibrium Eq.

if we define the three independent stress component in terms of some function U,

$$\tau_{xx} = \frac{\partial^2 U}{\partial y^2}+\rho V, \quad \tau_{yy} = \frac{\partial^2 U}{\partial x^2}+\rho V, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x\partial y} .$$

① Equilibrium Equations automatically satisfied

② $\nabla^2\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+2\rho V\right) = \nabla^2(\nabla^2 U) = \nabla^4 U = 0 .$

U must satisfy the biharmonic equation.

$$\text{or } \frac{\partial^4 U}{\partial x^4}+2\cdot\frac{\partial^4 U}{\partial x^2\partial y^2}+\frac{\partial^4 U}{\partial y^4} = 0 .$$

U obtained from above automatically satisfy equilibrium Equation & compatibility Eq.

- ~~Method~~

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Date . . No.

- U : Airy stress function (or ϕ)
- Mathematical process of solving the elasticity Eq. has been reduced to the solution of a single 4th order pdf.

* Analyticity

A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be analytic at a point $z=z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0.

By an analytic function, we mean a function that is analytic in some domain.

- A more modern term for 'analytic' in D is 'holomorphic' in D.

* Cauchy-Riemann Equations

$w = f(z) = u(x,y) + iv(x,y)$

f is analytic only if the first partial derivatives of u & v satisfies Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

ex) $f = z = x + yi$

$f = z^2 = (x+yi)(x+yi)$

$f = z^3 = (x+yi)(x+yi)(x+yi)$

$f = \bar{z} = x - yi$.

* Practical importance of complex analysis

Both real & imaginary part of of an analytic function satisfy the most important differential equation of physics, Laplace equation.

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right), \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right).$$

By adding $\nabla^2 u = 0$.

Similarly $\nabla^2 v = 0$.

- Airy Stress function can be expressed in terms of two analytic functions of a complex variable.

$$P = \tau_{xx} + \tau_{yy} = \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial x^2} = \nabla^2 U \quad \cdots ②\text{-}①$$

P is a harmonic function → its conjugate harmonic can be found,

$$f(z) = P + i\underset{\text{conjugate harmonic}}{\underline{Q}} \quad \cdots\cdots ②\text{-}2$$

We can define another analytic function, $\phi(z)$, by

$$\phi(z) = \frac{1}{4}\int f(z)\,dz = p + iq \quad \cdots ③$$

From ①, ②, ③.

$$\phi'(z) = \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x} = \frac{1}{4}\cdot f(z) = \frac{1}{4}(P + iQ)$$

$$\frac{1}{4}P = \frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}, \quad \frac{1}{4}Q = \frac{\partial q}{\partial x} = -\frac{\partial p}{\partial y} \quad \cdots ③\text{-}2$$

a function, $p_1 = U - px - qy$ is harmonic, since

$$\nabla^2(U - px - qy) = \nabla^2 U - x\nabla^2 p - 2\frac{\partial p}{\partial x} - y\nabla^2 q - 2\frac{\partial q}{\partial y} = 0$$

$$\left(\frac{\partial U}{\partial x} - \frac{\partial p}{\partial x}x - p - \frac{\partial q}{\partial x}y + \frac{\partial U}{\partial y} - \frac{\partial p}{\partial y}x - \frac{\partial q}{\partial y}y - q\right)$$

$\nabla^2 p_1 \rightarrow$ Therefore ~~By inspection~~, p_1 is the real part of an unknown function, $\chi(z)$.

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 p}{\partial x^2}x - \frac{\partial p}{\partial x} - \frac{\partial^2 q}{\partial x^2}y + \frac{\partial^2 U}{\partial x\partial y} - \frac{\partial^2 p}{\partial x\partial y}x - \frac{\partial p}{\partial y} - \frac{\partial^2 q}{\partial x\partial y}y - \frac{\partial q}{\partial x}$$

$\Rightarrow P = \nabla^2 U$

Also, $px + qy$ is real part of $\bar{z}\phi(z)$.

$$(x - iy)\cdot(p + iq) = px + qy + (qx - py)i.$$

$$\begin{cases} px + qy = Re(\bar{z}\phi(z)) \\ p_1 = Re(\chi(z)) \end{cases}$$

importantly U can be expressed in terms of two analytic functions, $\phi(z)$ & $\chi(z)$.

$$U = p_1 + px + qy = Re\{\chi(z)\} + Re\{\bar{z}\phi(z)\}$$

$$= \frac{1}{2}\left\{\chi(z) + \overline{\chi(z)} + \bar{z}\phi(z) + z\overline{\phi(z)}\right\} \quad \cdots ④$$

$$\frac{\partial (x+yi)}{\partial x}$$

$$\frac{\partial}{\partial x}\phi(z) = \frac{\partial}{\partial z}\phi(z)\cdot\frac{\partial z}{\partial x} = \phi'(z)\cdot 1$$

$$\frac{\partial}{\partial y}\phi(z) = \frac{\partial}{\partial z}\phi(z)\cdot\frac{\partial z}{\partial y} = \phi'(z)\cdot i$$

Date . . No.

By differentiating ④, derivatives of U can be expressed in terms of ϕ, χ.

⑤-1 $$2\frac{\partial U}{\partial x} = \phi(z) + \bar{z}\phi'(z) + \overline{\phi(z)} + z\overline{\phi'(z)} + \chi'(z) + \overline{\chi'(z)}$$

⑤-2 $$2\frac{\partial U}{\partial y} = -i\phi(z) + i\bar{z}\phi'(z) + i\overline{\phi(z)} - iz\overline{\phi'(z)} + i\chi'(z) - i\overline{\chi'(z)}$$

⑤-3 $$2\frac{\partial^2 U}{\partial x^2} = 2\phi'(z) + \bar{z}\phi''(z) + 2\overline{\phi'(z)} + z\overline{\phi''(z)} + \chi''(z) + \overline{\chi''(z)}$$

⑤-4 $$2\frac{\partial^2 U}{\partial y^2} = 2\phi'(z) - \bar{z}\phi''(z) + 2\overline{\phi'(z)} - z\overline{\phi''(z)} - \chi''(z) - \overline{\chi''(z)}$$

⑤-5 $$2\frac{\partial^2 U}{\partial x\partial y} = i\bar{z}\phi''(z) - iz\overline{\phi''(z)} + i\chi''(z) - i\overline{\chi''(z)}$$

also) ⑤-1 + ⑤-2 $\times i$ = $\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \phi(z) + z\overline{\phi'(z)} + \overline{\chi'(z)}$.

⑤-3 $\Rightarrow \tau_{yy}$, ⑤-4 $\Rightarrow \tau_{xx}$, ⑤-5 $\Rightarrow \tau_{xy}$.

* Stress components can be calculated!

Now let's look if we can calulate strain & displacement.

$$8G\varepsilon_{xx} = (K+1)\tau_{xx} + (K-3)\tau_{yy}$$

$$8G\varepsilon_{yy} = (K+1)\tau_{yy} + (K-3)\tau_{xx}$$

$K = 3-4\nu$; plane strain

$K = (3-\nu)/(1+\nu)$: plane stress

this is not a good form for integration.

After some manipulation,

$$2G\cdot\frac{\partial u}{\partial x} = 2G\varepsilon_{xx} = -\tau_{yy} + \frac{1}{4}(K+1)(\tau_{xx}+\tau_{yy})$$

$$2G\cdot\frac{\partial v}{\partial y} = 2G\varepsilon_{yy} = -\tau_{xx} + \frac{1}{4}(K+1)(\tau_{xx}+\tau_{yy})$$

from ③-2.

$$2G\cdot\frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x} + (K+1)\frac{\partial p}{\partial x}$$

$$2G\frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y} + (K+1)\frac{\partial q}{\partial y}.$$

Date . . No.

By integrating,   unknown fn of y.

$$\left\{\begin{aligned} 2Gu &= -\frac{\partial U}{\partial x} + (K+1)p + g(y) \\ 2Gv &= -\frac{\partial U}{\partial y} + (K+1)q + h(x) \end{aligned}\right) \quad ⑥-0$$

From $2G\varepsilon_{xy} = \tau_{xy}$, & Airy Stress Function.

$$2G\cdot\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -2\frac{\partial^2 U}{\partial x\partial y} \quad \cdots ⑥-1$$

Differentiate with x, & y and add.

$$2G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -2\frac{\partial^2 U}{\partial x\partial y} + (K+1)\frac{\partial p}{\partial y} + g'(y) + (K+1)\frac{\partial q}{\partial x} + h'(x)$$

$-\frac{\partial q}{\partial x}$

$$= -2\frac{\partial^2 U}{\partial x\partial y} + g'(y) + h'(x) \quad \cdots ⑥-2$$

comparison of ⑥-1 & ⑥-2.

$g'(y) + h'(x) = 0$.

general solution to this is,

$\frac{\partial g}{\partial x} = 0$, $\frac{\partial h}{\partial y} = 0$, $g(y) = wy + a$, $h(x) = -wx + b$.

$\frac{\partial g}{\partial y} + \frac{\partial h}{\partial x} = 0$, $\frac{1}{2}\left(\frac{\partial g}{\partial y} - \frac{\partial h}{\partial x}\right) = w_{xy}$

∴ the portion of u, v represented by g and h are rigid body motion that has no stress associated with it.

ignoring this rigid body motion, displacement can be written as a complex number,

$$2G(u+iv) = -\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right) + (K+1)\underbrace{(p+iq)}_{\phi(z)}$$

$\phi(z) + z\overline{\phi'(z)} + \overline{\chi'(z)}$

$$= K\phi(z) - z\overline{\phi'(z)} - \overline{\chi'(z)}$$

if we put $\psi(z) = \chi'(z)$

$$2G(u+iv) = K\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}$$

$(u+iv)(\cos\theta+i\sin\theta) = (u\cos\theta - v\sin\theta) + (u\sin\theta + v\cos\theta)\, i$

displacement & stress in transformed (rotated) axis.

$u'+iv' = (u+iv)(e^{-i\theta})$ $\qquad e^{-i\theta} = \cos\theta - i\sin\theta$

where $e^{+i\theta} = \cos\theta + i\sin\theta$,

$$\begin{cases} \tau_{y'y'} - \tau_{x'x'} + 2i\tau_{x'y'} = (\tau_{yy} - \tau_{xx} + 2i\tau_{xy})\, e^{2i\theta} & \cdots ② \\ \tau_{y'y'} + \tau_{x'x'} = \tau_{yy} + \tau_{xx} & \cdots ① \end{cases} \quad ⑦-$$

$① - ② = 2(\tau_{x'x'} - i\tau_{x'y'}) = \tau_{yy} + \tau_{xx} - (\tau_{yy} - \tau_{xx} + 2i\tau_{xy})\, e^{2i\theta}$

→ useful when you are dealing with Boundary condition.

in polar coordinates,

$\tau_{\theta\theta} - \tau_{rr} + 2i\tau_{r\theta} = (\tau_{yy} - \tau_{xx} + 2i\tau_{xy})\, e^{2i\theta}$

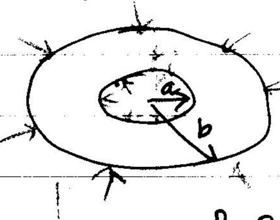
$\tau_{\theta\theta} + \tau_{rr} = \tau_{yy} + \tau_{xx}$.

⑤, Airy stress Fn, $\quad$ ⑦

$$\tau_{yy} + \tau_{xx} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 2\left[\phi'(z) + \overline{\phi'(z)}\right] = 4\,\mathrm{Re}(\phi'(z))$$

$$\tau_{yy} - \tau_{xx} + 2i\tau_{xy} = \frac{\partial^2 U}{\partial x} - \frac{\partial^2 U}{\partial y^2} - 2i\frac{\partial^2 U}{\partial x \partial y} = 2\left[\bar{z}\phi''(z) + \psi'(z)\right]$$

1) Pressurized hollow cylinder. (§8.4)



- hydrostatic stress.
- By putting $b \to \infty$ → similar to circular hole
- useful for laboratory test.

$$\text{B.C.} \begin{cases} \tau_{rr}(a) = P_i \\ \tau_{rr}(b) = P_o \end{cases}$$

We take the complex potential as

$$\phi(z) = cz, \quad \psi(z) = \frac{d}{z}$$

c, d are constants. Imaginary component → shear.

in this case → radial symmetry → c & d are real.

① Stresses. ∵ from ⑦

$\tau_{xx} + \tau_{yy} = \tau_{rr} + \tau_{\theta\theta} = 4\,\mathrm{Re}[\phi'(z)] = 4c$ $\quad$ real

$\tau_{yy} - \tau_{xx} + 2i\tau_{xy} = 2[\bar{z}\phi''(z) + \psi'(z)] = -2\frac{d}{z^2} = -2d$ $\quad$ real

if we put $z = re^{i\theta}$, $\quad = \dfrac{-2d \cdot e^{-2i\theta}}{r^2}$.

Date . . No.

more convenient to use polar coordinate.

$$\tau_{\theta\theta} - \tau_{rr} + 2i\tau_{r\theta} = (\tau_{yy} - \tau_{xx} + 2i\tau_{xy})e^{2i\theta}$$

$$= -2\frac{d \cdot e^{-2i\theta}}{r^2} \times e^{2i\theta} = -\frac{2d}{r^2}.$$

Separating real & imaginary part.

$$\begin{cases} \tau_{\theta\theta} - \tau_{rr} = -\frac{2d}{r^2}, \quad \tau_{r\theta} = 0. \\ \tau_{rr} + \tau_{\theta\theta} = 4c. \end{cases}$$

$$\tau_{rr} = 2c + \frac{d}{r^2}, \quad \tau_{\theta\theta} = 2c - \frac{d}{r^2}.$$

By imposing BC.

τ_{rr} at $r = a, \to P_i$

$r = b, \to P_o$.

$$\begin{pmatrix} 2c + \frac{d}{a^2} = P_i \\ 2c + \frac{d}{b^2} = P_o \end{pmatrix} \to c = \frac{b^2 P_o - a^2 P_i}{2(b^2 - a^2)}, \quad d = \frac{a^2 b^2 (P_i - P_o)}{(b^2 - a^2)}$$

$$\therefore \begin{cases} \tau_{rr} = \frac{(b^2 P_o - a^2 P_i)}{b^2 - a^2} + \frac{a^2 b^2 (P_i - P_o)}{(b^2 - a^2)} \frac{1}{r^2}. \\ \tau_{\theta\theta} = \frac{(b^2 P_o - a^2 P_i)}{b^2 - a^2} - \frac{a^2 b^2 (P_i - P_o)}{(b^2 - a^2)} \frac{1}{r^2}. \end{cases}$$

② Displacement. from ⑥ - ③ $\quad k \cdot cz - \bar{z} \cdot c - d/\bar{z}$

$$2G(u + iv) = (k-1)cz - d/\bar{z}$$

using polar coordinate, $z = re^{i\theta}$.

$$2G(u + iv) = (k-1)c \cdot re^{i\theta} - d \cdot e^{i\theta}/r.$$

in order to describe u_r, u_θ,

$$2G(u_r + iu_\theta) = 2G \cdot (u + iv)e^{-i\theta} = (k-1)cr - d/r.$$

since c & d are real,

$$u_r = \frac{1}{2G} \cdot \left((k-1)cr - d/r\right), \quad u_\theta = 0.$$

$K-1 = 2(1-2\nu)$

Date . . No.

$$u_r = \frac{(K-1)}{2G}\left(\frac{(b^2P_o - a^2P_i)r}{2(b^2-a^2)} - \frac{a^2b^2(P_i-P_o)}{(b^2-a^2)r}\right).$$

$(1-2\nu)\ P_o r - (P_i - P_o)\frac{a^2}{r}$

The solution for a circular hole in an infinite rock mass with a far-field hydrostatic stress P_o & internal pressure P_i can be found by letting $b \to \infty$. (for plane strain).

$$\tau_{rr} = P_o + (P_i - P_o)\left(\frac{a}{r}\right)^2$$

$$\tau_{\theta\theta} = P_o - (P_i - P_o)\left(\frac{a}{r}\right)^2$$

$$u_r = \frac{1}{2G}\left[(1-2\nu)P_o r - (P_i - P_o)\left(\frac{a^2}{r}\right)\right]$$