# Physical variables for various physical problems

| Physical problem | Conservation Principle $\nabla \cdot q = 0$ | State Variable $u$ | Flux $\sigma$ | Material properties $k$ | Source f | Constitutive equation $\sigma = ku'$ |
|---|---|---|---|---|---|---|
| Elasticity | Conservation of linear momentum (equilibrium) | Displacement $u$ | Stress $\sigma$ | Young's modulus & Poisson's ratio | Body forces | Hooke's law |
| Heat conduction | Conservation of energy | Temperature T | Heat flux Q | Thermal conductivity k | Heat sources | Fourier's law |
| Porous media flow | Conservation of mass | Hydraulic head h | Flow rate Q | Permeability k | Fluid source | Darcy's law |
| Mass transport | Conservation of mass | Concentration C | Diffusive flux q | Diffusion coefficient D | Chemical source | Fick's law |

Structure of state variables and fluxes are mathematically similar – *a convenient truth!*

# Thermoelasticity

Date . . No.

Similar to poroelasticity. thermoelasticity.

$p \longrightarrow T$

- change in $p$ or $T$ induce equal normal strain in three orthogonal directions and no shear strain.
- $p$ and $T$ are governed by diffusion-type Eqn.
- TM simpler because $M \to T$ is ignored.

given temperature $T_0$, $\to$ change to $T$.

by assuming linearity,

compression positive.

$$\underline{\underline{\varepsilon}} = -\underline{\underline{\beta}}\,(T-T_0)$$

$\hookrightarrow$ thermal expansivity tensor.

if isotropic, $\underline{\underline{\beta}} = \beta \underline{\underline{I}}$

coefficient of linear thermal expansion

$$\underline{\underline{\varepsilon}} = -\beta\,(T-T_0)\,\underline{\underline{I}}.$$

$\alpha$ is more common, but here it may be confused with Biot alpha $\to$ use $\beta$

total strain, $\varepsilon_{total} = \varepsilon_E + \varepsilon_T$

$$\underline{\underline{\varepsilon}}_{total} = \frac{1}{2G}\underline{\underline{\tau}} - \frac{\nu}{2G(1+\nu)}\,\mathrm{trace}(\underline{\underline{\tau}})\,\underline{\underline{I}} - \beta(T-T_0)\,\underline{\underline{I}} \quad \cdots (1)$$

Taking traces of both sides,

$$tr(\underline{\underline{\varepsilon}}) = \varepsilon_b = \frac{\tau_m}{K} - 3\beta\,(T-T_0) \quad \cdots (2)$$

$\hookrightarrow$ volumetric thermal expansion coefficient.

by inverting (1) with the aid of (2)

$$\underline{\underline{\tau}} = 2G\,\underline{\underline{\varepsilon}} + \lambda\, tr(\underline{\underline{\varepsilon}})\,\underline{\underline{I}} + 3\beta K\,(T-T_0)\,\underline{\underline{I}}$$

$$\tau_{ij} = 2G\,\varepsilon_{ij} + \frac{\nu}{1+\nu}\cdot\left(3K\,tr(\varepsilon) + 9K\beta(T-T_0)\right) + 2G\beta(T-T_0)\,\underline{\underline{I}}$$

$$\frac{9\nu 2G}{3(1-2\nu)} + 2G\beta \qquad 2G\beta\cdot\left(\frac{9+3-6\nu}{3(1-2\nu)}\right) = 2G\beta.$$

compression (+)
term (−).

Date thermal stress.

$$\tau_{xx} = 2G\,\varepsilon_{xx} + \lambda(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}) + 3\beta K(T-T_0)$$
$$\tau_{yy} = 2G\,\varepsilon_{yy} + \lambda(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}) + 3\beta K(T-T_0)$$
$$\tau_{zz} = 2G\,\varepsilon_{zz} + \lambda(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}) + 3\beta K(T-T_0)$$
$$\tau_{xy} = 2G\,\varepsilon_{xy},\quad \tau_{xz} = 2G\,\varepsilon_{xz},\quad \tau_{yz} = 2G\,\varepsilon_{yz}.$$

$$\tau_{ij} = 2G\,\varepsilon_{ij} + \lambda\cdot\varepsilon_{\alpha\alpha}\cdot\delta_{ij} + 3\beta K(T-T_0)\,\delta_{ij}$$

T change per se does not cause stress

$\Delta T$ + mechanical restraint → thermal stress.

- free expansion → two terms cancel out.
- → normal stress.

ex) if free expansion,

$$\varepsilon_{zz} = \varepsilon_{yy} = \varepsilon_{xx} = \frac{1}{2G}\cancel{\tau_{xx}}^{\,0} - 0 - \beta(T-T_0).$$

$$\tau_{xx} = -2G\cdot\beta(T-T_0) - \lambda\cdot 3\beta(T-T_0) + 3\beta K(T-T_0)$$

roughly.

$$\sigma_T = \underline{3\beta K(T-T_0)} = \frac{E}{1-2\nu}\cdot\beta(T-T_0).$$

$\underline{3K = 2G + 3\lambda}$.

$K \approx 100\,GPa$, $\beta \approx 10^{-5}/°K$, $\Delta T = 10°C$.

$300\cdot 10^{9}\times 10^{-5}\times 10 = \underline{30\,MPa}$.

ex) underground nuclear repository
injection of cold water into geothermal wells.

Thermoelastic version of Navier's Eq. →
Duhamel-Neumann Eq.

$$G\nabla^2\underline{u} + (\lambda+G)\nabla(\nabla\cdot\underline{u}) = -\rho\underline{f} - \underbrace{3\beta K\nabla(T-T_0)}_{\text{mathematically similar to body force}}.$$

$u_x$, $u_y$, $u_z$ & $T-T_0$
fourth unknown

# ☆ Heat diffusion Eq.

Date . . No.

conductive heat flux vector $q_T$ ($J/m^2s$) or $W/m^2$.

Fourier's Law,

$$\underline{q}_T = -\underline{\underline{k}}_T \nabla \vartheta$$

often temperature dependent.

$\underline{\underline{k}}_T$ : thermal conductivity tensor $W/m^\circ K$

symmetric

pore structure has minor effect.

if isotropic, $\underline{\underline{k}}_T = k_T \underline{\underline{I}}$, $k_T$ : $1 \sim 10$ $W/m^\circ K$.

cf. k. change quite a lot.

Energy, $u$ (temperature dependent) $= u_0 + C_v \cdot \vartheta$

$u$ : energy per unit mass ($J/kg$)

$u_0$ : some arbitrary reference energy ($J/kg$)

$C_v$ : specific heat at constant strain ($J/kg^\circ K$).

From conservation of energy,

net flux of heat conducted into any region of rock = rate of change of internal energy.

$$\iiint_R \frac{\partial(\rho u)}{\partial t} dV = -\iint_{\partial R} (q_T \cdot n)\, dA.$$

$\rho u$ : energy per unit volume,

Using divergence theorem, (& constant density).

$$\rho \frac{\partial u}{\partial t} + \nabla \cdot q_T = 0.$$

$$\frac{\partial \vartheta}{\partial t} = \frac{k_T}{\rho C_v} \nabla^2 \vartheta.$$

$D_T$. $[m^2/s]$ thermal diffusivity.

$$\frac{\partial p_p}{\partial t} = \frac{k}{\mu \cdot \phi C_f} \nabla^2 p_p.$$

fluid flow through a rigid porous medium.

- Fully coupled TM

Date . . No.

if contribution of strain energy to overall energy balance is included,

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{k_T}{\rho C_v}\nabla^2 v + \dfrac{3K\beta T_0}{\rho C_v}\dfrac{\partial \varepsilon_b}{\partial t} \\ \text{or} \\ \dfrac{\partial v}{\partial t} = \dfrac{k_T}{(\rho C_v + 9K\beta^2 T_0)}\nabla^2 v + \dfrac{3\beta T_0}{(\rho C_v + 9K\beta^2 T_0)}\dfrac{\partial \tau_m}{\partial t} \end{cases}$$

$T_0$ → absolute T. (°K)

Fully coupled TM is rarely used in rock mechanics.

ex) $\Delta T : 10°K$, $\Delta \tau = 10 MPa$.

$k_T : 2\text{-}3\ W/m°K$. $\beta = 1.5\times10^{-6}$.

condition for uncoupled heat diffusion Eq.

$$\frac{9K\beta^2 T_0}{\rho C_v} \ll 1$$

Usually,

Heat diffusion Eqns can be tackled independently

classics → Carslaw & Jaeger (1959)

⇒ put into Duhamel-Neumann Eqns.

- nonuniform heating. $T\uparrow \to$ body expand $\to$ some portion has different temperature and cant expand.

method of strain suppression.

433 p. Timoshenko & Goodier

- a thin rectangular plate of uniform thickness in which the temperature T is an even function of y.
- $T \propto$ < even function of y. $T = T_0\left(1 - \frac{y^2}{c^2}\right)$
  < independent of x, z,

$y = -c \to T = 0$

$y = 0 \to T = T_0$

$y = c \to T = 0$

longitudinal thermal expansion $\alpha T$ will be entirely suppressed by applying each element of the plate the longitudinal stress

$$\sigma_x' = -\alpha T E \quad (a) \to \text{compressive for positive } T.$$

to get the thermal stress which is free from external forces, we have to superpose on (a) the stresses produced in the plate by tensile stress of intensity $\alpha T E$ distributed at the surface.

resultant $\to \int_{-c}^{+c} \alpha T E \, dy$.

at sufficient distance from the ends,

$$\frac{1}{2c}\int_{-c}^{+c} \alpha T E \, dy$$

thermal stress at a sufficient distance

$$\sigma_x = \frac{1}{2c}\int_{-c}^{+c} \alpha T E \, dy - \alpha T E$$

$$T = T_0\left(1 - \frac{y^2}{c^2}\right)$$

Date . . No.

$$\tau_{xx} = \frac{2}{3}\alpha T_0 E - \alpha T_0 E\left(1 - \frac{y^2}{c^2}\right)$$

there are many other applications & analytical solutions.