



27. Power Series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \quad (1)$$

Thm

- (a) Every power series converges at the center z_0 .
- (b) If (1) converges at $z=z_1 \neq z_0$, it converges abs for every z closer to z_0 than z_1 .
- (c) If (1) diverges at $z=z_2$, it diverges for every z farther away from z_0 than z_2 .



- circle of convergence : the smallest circle with center z_0 that includes all the pts where (1) converges.

$$|z-z_0|=R$$

↗ radius of conv.

$$\begin{cases} R=\infty & \text{if (1) converges for all } z \\ R=0 & \text{if (1) converges only at } z=z_0 \end{cases}$$

Rmk (1) may converge on the circle of conv.
at some pts or none

Thm (Cauchy-Hadamard Formula)

Suppose that $\left| \frac{a_{n+1}}{a_n} \right| \rightarrow L^*$.

• If $L^* = 0$, then $R = \infty$.

• If $L^* > 0$, then

$$R = \frac{1}{L^*} = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

• If $\left| \frac{a_{n+1}}{a_n} \right| \rightarrow \infty$, then $R = 0$.

(4.3) Ftns given by Power Series

For simplicity, set $z_0 = 0$.

When the power series has a nonzero radius of convergence $R > 0$, its sum is a ftn of z . i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (|z| < R). \quad (2)$$

Q. Given $f(z)$ and a center, is the power series representation unique?

Thm $f(z)$ in (2) with $R > 0$ is conti at $z=0$.

Pf From (2), $f(0) = a_0$.

We need to show $\lim_{z \rightarrow 0} f(z) = f(0) = a_0$.

i.e. for a given $\varepsilon > 0$ there exists a $\delta > 0$

s.t. $|z| < \delta \Rightarrow |f(z) - a_0| < \varepsilon$.

} over

Thm in Sec 14.2 \Rightarrow (2) converges abs for
 $|z| \leq r$ with any $a < r < R$.

$$\Rightarrow \sum_{n=1}^{\infty} |a_n|r^n = \frac{1}{r} \sum_{n=1}^{\infty} (a_n r^n) = s$$

converges. $s \neq 0$
 $(s=0$ is trivial)

Then for $0 < |z| \leq r$,

$$\begin{aligned} |f(z) - z_0| &= \left| \sum_{n=1}^{\infty} a_n z^n \right| \leq |z| \sum_{n=1}^{\infty} |a_n| |z|^n \\ &\leq |z| \sum_{n=1}^{\infty} |a_n| r^n = |z| s. \end{aligned}$$

Given $\epsilon > 0$, choose δ s.t.

$$0 < \delta < r \text{ and } \delta < \frac{\epsilon}{s}.$$

Then

$$|f(z) - z_0| < \epsilon. \quad \#.$$

Thm² (Uniqueness of representation)

Suppose

$$\sum_{n=0}^{\infty} a_n z^n \text{ and } \sum_{n=0}^{\infty} b_n z^n$$

both converge for $|z| < R$ ($R > 0$) and have the same sum for all these z .

Then $a_n = b_n$ ($n=0, 1, \dots$).

Pf

$$a_0 + a_1 z + \dots = b_0 + b_1 z + \dots \quad (z) < R.$$

Both series are conti at $z=0$ (by Thm¹).

$$\Rightarrow \text{let } z \rightarrow 0 \Rightarrow a_0 = b_0.$$

$$\text{For } z \neq 0, a_0 + a_1 z + \dots = b_0 + b_1 z + \dots$$

$$\Rightarrow \text{let } z \rightarrow 0 \Rightarrow a_1 = b_1, \dots \quad \#$$

Operations on Power Series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad |z| < R_1$$

$$g(z) = \sum_{m=0}^{\infty} b_m z^m \quad |z| < R_2$$

- termwise addition/subtraction
 $\Rightarrow R = \min(R_1, R_2)$

- termwise multiplication (Cauchy product)

$$(f(z)g(z)) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0) z^n$$

Conv. abs. for each z within $R_1 \& R_2$.

- termwise differentiation/multiplication

$\left[\sum_{n=1}^{\infty} n a_n z^{n-1}$ has the same rad. of conv
as the original series.

$$\left[\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} \right] \quad " \quad "$$

Thm (Power Series represent analytic fns)

A power series with $R > 0$ represents an analytic fn at every point interior to its circle of convergence. The derivatives are obtained by differentiating term by term.

All the series thus obtained have the same R , as the original series. (Thus each of them represents an analytic fn).