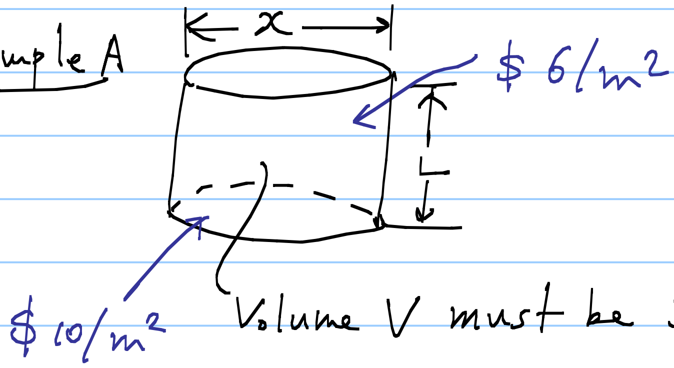
Lecture 2-1

One-Dimensional Unconstrained Minimization

- <u>Single-variable minimization</u>

<u>Example A</u>



Volume V must be $50 m^3$

⊕ \$80/m² to refregerate

Build a cylindrical tank of volume $= 50 m^3$ with min cost

- objective ftn $f = \frac{\pi}{4}x^2 \times 2 \times 10 + 6 \cdot (\pi x L)$
  $+ 80\left(\frac{\pi}{4}x^2 \times 2 + \pi x L\right)$ (1)
- constraint $V = \frac{\pi}{4}x^2 L = V_0 = 50$ (2)
  $x > 0, \quad L > 0$ (3)

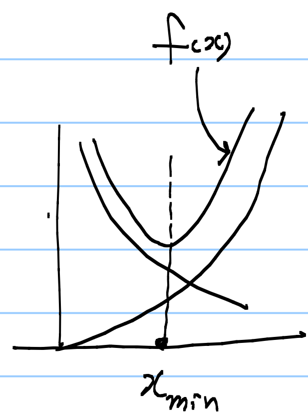
To solve for x_{min} (and L_{min})

i) eliminate L using (2)

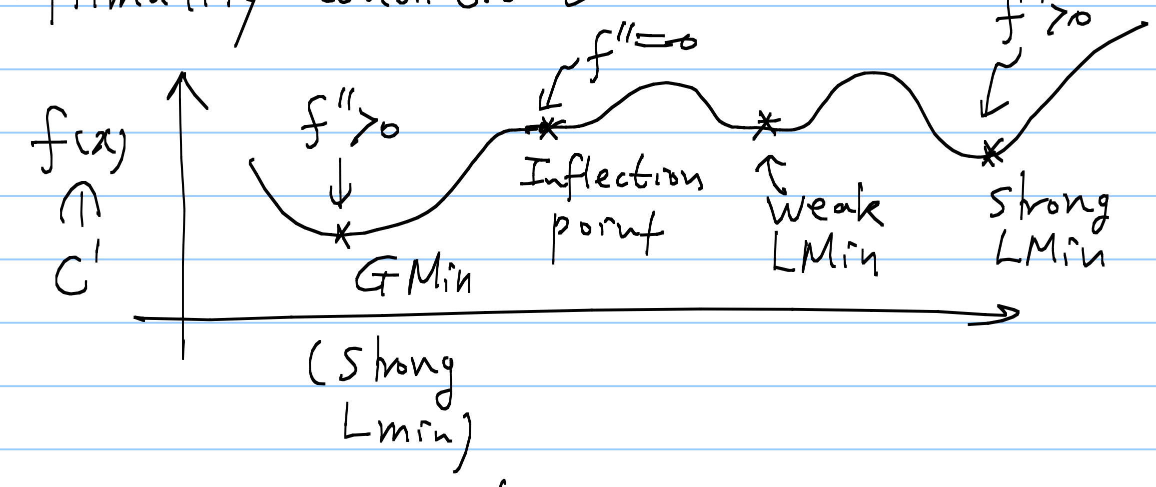
$$L = \frac{V_0 \cdot 4}{\pi x^2} = \frac{200}{\pi x^2}$$

ii) $\min\limits_{x} f(x) = 45\pi x^2 + \frac{17200}{x}$

$(x \in R^+)$



Optimality Conditions



$* \Rightarrow f' = 0.$

- LMin (Local Min) : x^*

$$\begin{cases} \exists\, \delta > 0 \text{ such that } f(x^*) < f(x) \text{ for } |x - x^*| < \delta \ : \ \underline{\text{Strong LMin}} \\ \exists\, \delta > 0 \text{ such that } f(x^*) \le f(x) \text{ for } |x - x^*| < \delta \ : \ \text{weak LMin} \end{cases}$$

- GMin (Global Min) : x^*

$f(x^*) < f(x)$ for all feasible $x \in \Omega$

domain

Necessary Condition for LMin

$$\boxed{f'(x) = 0 \quad (\text{inside } \Omega)}$$

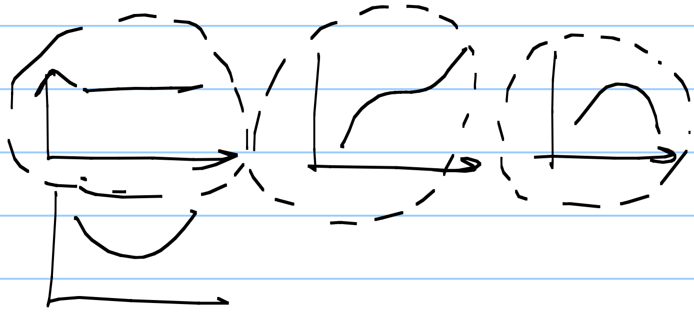
<proof>

- For small real number $h > 0$

$$f(x^* + h) = f(x^*) + h f'(x^*) + O(h^2) \geq f(x^*) \qquad (a)$$
$$f(x^* - h) = f(x^*) - h f'(x^*) + O(h^2) \geq f(x^*) \qquad (b)$$

- For (a), (b) to hold,

$$f'(x^*) \equiv 0 \text{ as } h \to 0$$



Sufficient Condition for LMin

$$\boxed{\begin{matrix} f'(x^*) = 0 \\ f''(x^*) > 0 \end{matrix} \Rightarrow \text{Strong LMin} \quad (\text{inside } \Omega)}$$

Proof: Use 2nd-order Taylor Expansion

$$f(x^* + h) = f(x^*) + \cancel{h f'(x^*)} + \frac{h^2}{2} f''(x^*) + O(h^3) > f(x^*)$$

$$\therefore f''(x^*) > 0$$

Back to Example A

① $f'(x) = 0 \rightarrow x^* = 3.93$

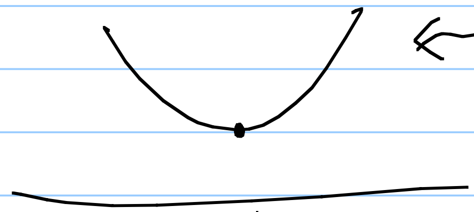
② check $f''(x^*) > 0$

$\therefore x^* = 3.93$ ($L^* = 4.12m$)

$\Rightarrow$ Strong LMin

Observation

If $f(x)$ looks like



← Convex ftn

LMin = GMin ⇒ very good News!!

(Remark: During actual numerical optimization, ftn can be approximated as a convex ftn at every iteration step)

∴ We will study

{ Convex function and convex set.

(* Convex ftn defined only on convex set)

Convex Set S

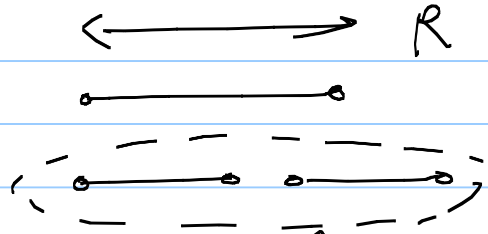
Def: A set S is convex

if $\alpha x_1 + (1-\alpha) x_1 \in S$

$(0 \leq \alpha \leq 1)$

for any $x_1, x_2 \in S$.

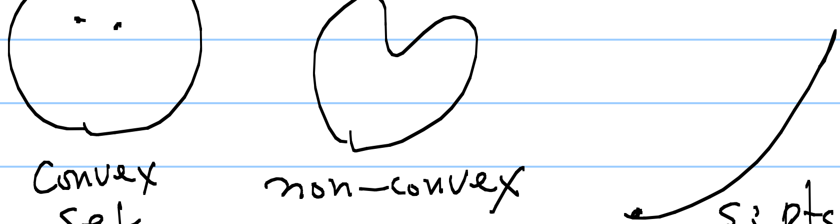
Example:



R

Not a convex set

S: point in



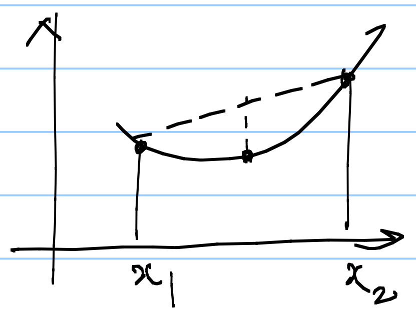
Convex set

non-convex

S: pts on this line

→ Non-convex

$\boxed{\text{Convex ftn } f \text{ (defined on a convex set)}}$



If for any $x_1, x_2 \in S$

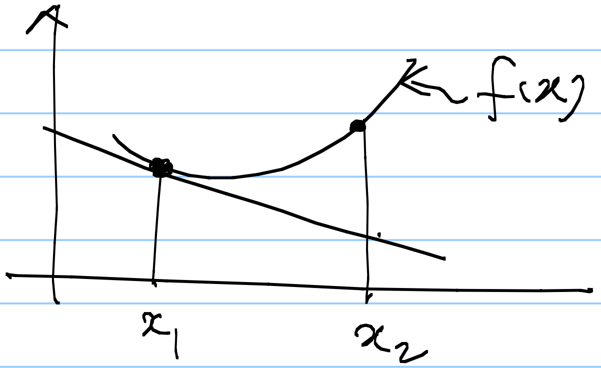
$$\alpha f(x_1) + (1-\alpha) f(x_2) \geq f(\alpha x_1 + (1-\alpha) x_2)$$

for $0 \leq \alpha \leq 1$

GOOD NEWS ABOUT CONVEX FUNCTION

① if $f \in C^1$ and $f =$ convex

$$\Leftrightarrow f(x_2) \geq f(x_1) + f'(x_1)(x_2 - x_1) \quad \text{Ⓐ}$$



☆ ② if $f \in C^2$ and $f =$ convex

$$\Leftrightarrow f''(x) \geq 0 \quad \text{for } x \in S$$

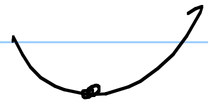
proof: using Ⓐ and Taylor expansion

$$f(x_2) \underset{\text{Taylor}}{=} f(x_1) + f'(x_1)(x_2 - x_1) + \frac{1}{2} f''(x_1)(x_2 - x_1)^2 + \cdots$$

$$\underset{\text{Ⓐ}}{\geq} f(x_1) + f'(x_1)(x_2 - x_1)$$

$$\therefore \; f''(x_1) \geq 0$$

3 For convex f



x^* is LMin $\rightarrow$ x^* is GMin

★ 4 $x^* =$ GMin of convex ftn f
(on a convex set)

$\Longleftrightarrow \frac{df}{dx}(x^*) = 0$ (inside Ω)

$\therefore$ FoNC $=$ SC

FoNC: First-order Necessary Condition

SC: Sufficient Condition