2-3 Zeros of a Function

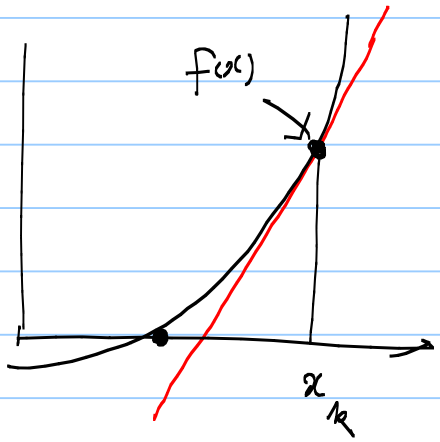
노트 제목 ①

Recall
- FONC : $h'(x^*) = 0$
- To find x^* $\Leftrightarrow$ to find sol. of $f(x^*) = 0$
  $(f(x) = h'(x))$

- Objective : to study numerical method to find solution of $f(x) = 0$.

i) Use an one point to start
  → ** Newton's method, Secant Method
ii) Use two points $\Leftrightarrow$ One pt + search Interval
  → Bisection method, etc

<< Newton's Method >>



i) Start with x_k
  (x_k : kth guess, $k = 1, 2 \ldots$)

ii) Use linear approximation of $f(x)$ $\boxed{\text{using } f(x_k) \ \& \ f'(x_k)}$

iii) Solve for an approx sol

IV) repeat until convergence

Analysis

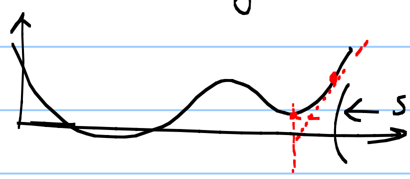
a) $\bar{f}(x) = f(x_k) + f'(x_k)(x - x_k)$ around x_k

b) $\bar{f}(x) \equiv 0 \rightarrow x = x_{k+1}$

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}} \qquad -(1)$$

Remark on Newton's Method

i) Newton's Method : Not guaranteed to work for arbitrary starting points



(← stopped because $f'(x_k) = 0$)

ii) Convergence Property ?

☑ Convergence Condition : $\left| \frac{f(x) f''(x)}{[f'(x)]^2} \right| < 1 \qquad (2)$

Rate of Convergence : 2nd-Order

i.e; $|e_{k+1}| = K\,|e_k|^2$ (3)

error at (k+1)th iteration

error at the kth iteration

Convergence Analysis

(to show Eqs (2) and (3))

- Trick : Rewrite $f(x) = 0$ as

$$\rightarrow \frac{f(x)}{f'(x)} = 0 \quad (\text{if } f'(x) \neq 0)$$

$$\rightarrow \boxed{x = x - \frac{f(x)}{f'(x)} \equiv g(x)} \qquad (4)$$

This form appeared in Eq.(1)

Observation when Eq.(4) is used

① if x^* satisfies $f(x^*) = 0$,

then $x^* = g(x^*)$ holds

② Newton's Algorithm can be viewed as

$$x_{k+1} = g_k(x_k) \qquad (k=1,2,\cdots)$$

③ Define error e_k

$$e_k = x_k - x^* \qquad \text{(if we happen to know } x^*\text{)}$$

A) To find the convergence condition (2):

$$e_{k+1} \underset{③}{=} \underset{\substack{\uparrow \\ g(x_k)}}{x_{k+1}} - \underset{\substack{\uparrow \\ g(x^*)}}{x^*}$$

$g(x_k)$ by Newton's Algorithm

$g(x^*)$ (Exact relation)

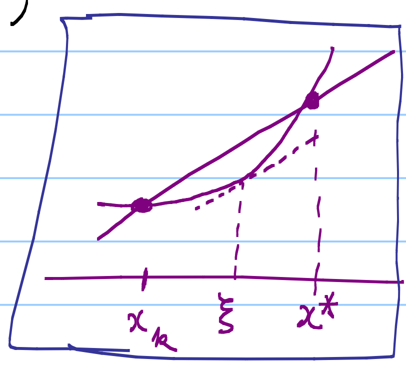
$$= g(x_k) - g(x^*)$$

$$= \frac{g(x_k) - g(x^*)}{x_k - x^*}(x_k - x^*)$$

$$= g'(\xi)(x_k - x^*)$$

By Mean Value Theorem

$$= g'(\xi)\, e_k \qquad (5)$$



(Mean value Theorem: when $g(x)$ is continuous on $[a,b]$, then at some point $\xi \in [a,b]$

$$g'(\xi) = \frac{g(b)-g(a)}{b-a}$$

provided that $g'(x)$ exists at all interior points.)

$$\therefore \ |e_{k+1}| = |g'(\xi)|\,|e_k| \qquad (6)$$

■ Observation: For $|e_{k+1}|$ to be less than $|e_k|$, the following eq must hold

$$|g'(\xi)| \leq K < 1 \qquad (7)$$

i.e.,

$$|g'(\xi)| \underset{(4)}{=} \left| \frac{d}{d\xi}\left(\xi - \frac{f(\xi)}{f'(\xi)}\right) \right|$$

$$= \left| 1 - \frac{f'(\xi)f'(\xi) - f(\xi)f''(\xi)}{[f'(\xi)]^2} \right|$$

$$= \underbrace{\left| \frac{f(\xi)\ f''(\xi)}{[f'(\xi)]^2} \right| < 1}_{\text{condition for convergence}}$$

$\Rightarrow$ Equation (2)

Furthermore,

$$g'(x^*) = \frac{f(x^*)\, f''(x^*)}{[f'(x^*)]^2} = 0 \qquad (8)$$

$$(\because f(x^*) = 0)$$

B) To find the relation between e_{k+1} and e_k;

$$e_{k+1} \overset{③}{=} x_{k+1} - x^*$$

$$= \underbrace{g(x_k)}_{\text{Newton's Algorithm (By ②)}} - \underbrace{g(x^*)}_{\text{By ①}}$$

By ⑧

$$\underset{\text{Taylor Expansion about } x = x^*}{=} \Big[g(x^*) + \cancel{g'(x^*)}(x_k - x^*) + \frac{1}{2} g''(x^*)(x_k - x^*)^2 + \cdots \Big] - g(x^*)$$

$$= \frac{1}{2} g''(x^*)(x_k - x^*)^2 + \cdots$$

$$= \frac{1}{2} g''(\xi)(x_k - x^*)^2$$

for $\xi \in (x^*, x_k)$

$$= \frac{1}{2} g''(\xi)\, e_k^{2}$$

$$\therefore \quad e_{k+1} = \beta\, e_k^{2}$$

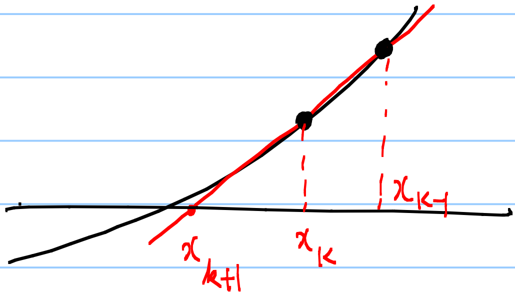
$$|e_{k+1}| = |\beta|\, |e_k|^{2} \leftarrow \text{2nd-order}$$

$\Rightarrow$ "The Rate of Convergence is of Order 2."

* Remark; For the case of double root, the convergence rate becomes quadratic if $x_{k+1} = x_k - 2\dfrac{f(x)}{f'(x)}$ is used.

《 Secant Method 》

A straight line that cuts a curve at one or more points (In this case, 2 points)



① Similar to Newton's method

② approximate $f'(x_k)$ as

$$f'(x_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

→ No derivative of $f(x)$ is need
(Useful when df/dx is difficult to calculated anaytically)

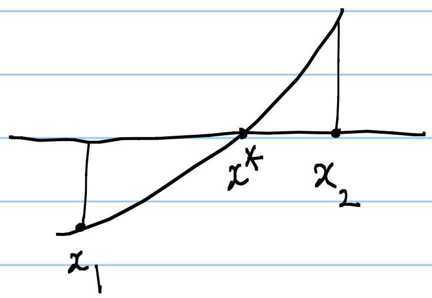
$$\therefore f(x) = f(x_k) + f'(x_k)(x - x_k) + \cdots$$

$$\Rightarrow \bar{f}(x) = \underset{\underset{f_k}{\triangleq}}{f(x_k)} + \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}(x - x_k) + \cdots$$

Imposing $\bar{f}(x) = 0$, x_{k+1} is obtained

$$\boxed{x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k}$$

« Bi-section Method »

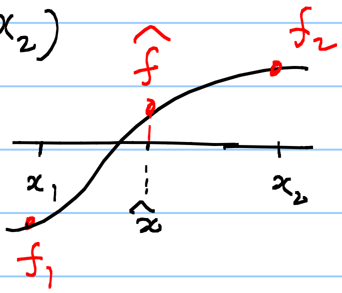


• start with $I = [x_1, x_2] \ni x^*$

i) $\hat{x} = \frac{1}{2}(x_1 + x_2)$

ii) if $f_1 \hat{f} < 0$

$I_{New} = [x_1, \hat{x}]$



else

$I_{NEW} = [\hat{x}, x_1]$

iii) repeat ii) until convergence

Remark: Bisection Methods can be combined with polynomial approaches.