4-1. NLP (Nonlinear Programming) Basics

"solving constrained nonlinear optimization involving multiple variables"

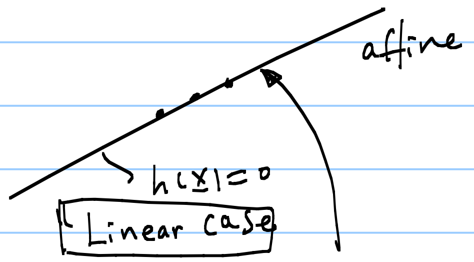
NLP: $\min f(\underset{\sim}{x}) \quad x \in R^n$

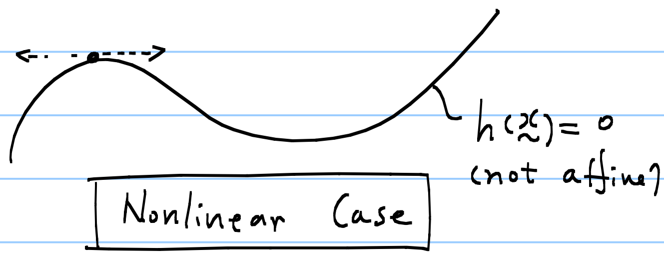
s.t $\quad \underset{\sim}{g}(\underset{\sim}{x}) \le 0 \quad (g_i \le 0,\ i = 1, \cdots, m)$

$\quad \underset{\sim}{h}(\underset{\sim}{x}) = 0 \quad (h_j(\underset{\sim}{x}) = 0,\ j = 1, \cdots, \ell)$

$\boxed{\text{NLP is difficult. WHY?}}$

① For Equality Constraint



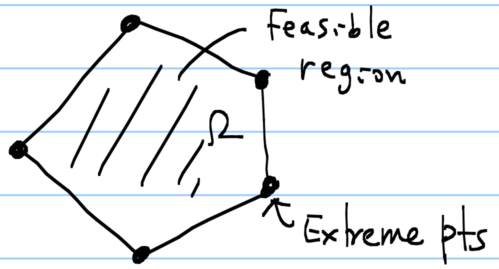


- Straight directions are feasible

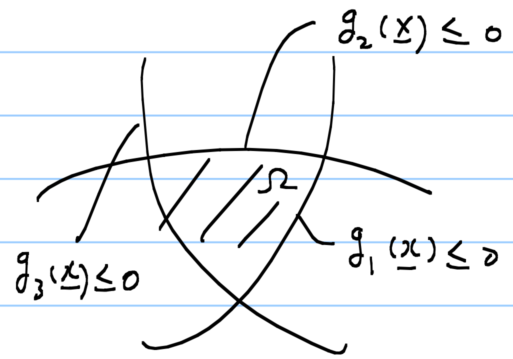
- No feasible straight direction → Difficult

(But for sufficient conditions Local Approx used)

② Inequality Constraint



Linear Case



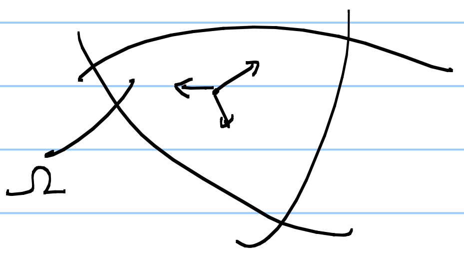
Nonlinear Case

- if $f(\underline{x})$ is linear, only extreme points need to be checked.
  (proof is possible for LP
  ↑ linear ↑ programming

- Need to check entire Ω, although optimal points usually appear where some of inequality constraints are active

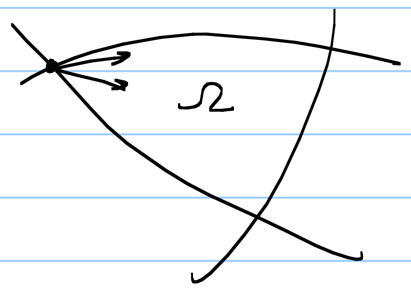
- When some (in) equality constraints are active, (i.e, $h(\underline{x}) = 0$, $\underline{g}(\underline{x}) = 0$) it is difficult to determine the feasible directions.

For Nonlinear Case,



if $\underline{x}^k$ inside Ω,

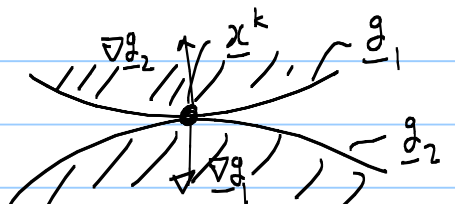
any direction is feasible ($\underline{d}$)



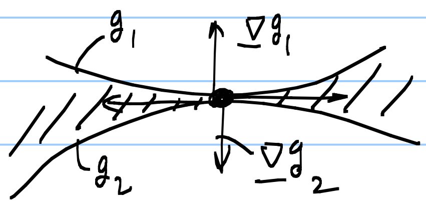
if $\underline{x}^k$ is on active boundary,

feasible $\underline{d}$ exists in this case

< Difficult Case >



no feasible direction



there is feasible direction

However, not possible to tell the existence of feasible directions only with ∇g_i ⇒ No theoretical analysis possible ⇒ will not consider

[Notation]

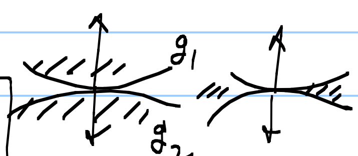
- $\Omega = \{ \underline{x} \in R^n \mid \underline{g}(\underline{x}) \le 0, \ \underline{h}(\underline{x}) = 0 \}$
- $\underset{\sim}{x}$ is feasible $\iff x \in \Omega$
  $\iff \underline{g}(\underline{x}) \le 0 \ \& \ \underline{h}(\underline{x}) = 0$
- $\begin{cases} h_i(\underset{\sim}{x}) = 0 \leftarrow \text{"active" constraint} \\ g_i(\underset{\sim}{x}) \equiv 0 \leftarrow \text{"active" "} \end{cases}$

$$\underset{\sim}{g} = \begin{cases} \underset{\sim}{g}^{A}(\underline{x}) = 0 : \text{active (tight, binding)} \\ \boxed{\underline{g}^{\bar{A}} < 0 : \text{inactive}} \end{cases}$$

↳ no need to consider for optimality condition

For all NLP analysis

$\boxed{\text{It is assumed that a constraint qualification (CQ) holds at } \underset{\sim}{x} \text{ for } \underline{g} \text{ and } \underline{h}.}$



will not consider these cases

CQ holds if i) $\underset{\sim}{x} \in \Omega$

ii) $\nabla \underset{\sim}{g}_i^A$ $(i=1, \cdots, n_A)$, and

∇h_j $(j=1, \cdots, m)$ are

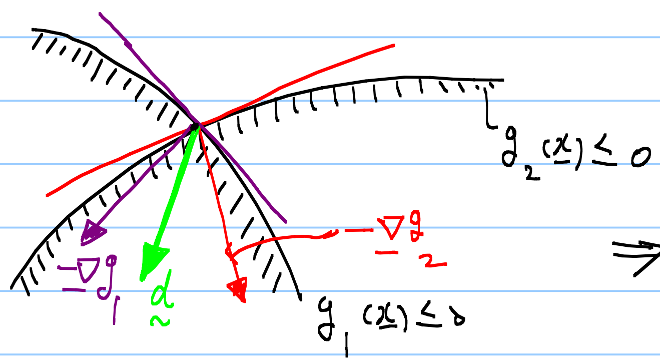
linearly Independent!

Why we need CQ?

↳ cannot determine the feasible direction $\underline{d}$ only with $\underline{\nabla g}^A$ and $\underline{\nabla h}$.

To understand why CQ is needed, consider the following cases:

CASE 1 : CQ holds ($\underline{\nabla g_1}$ not parrallel to ∇g_2 at $\underline{x}$)



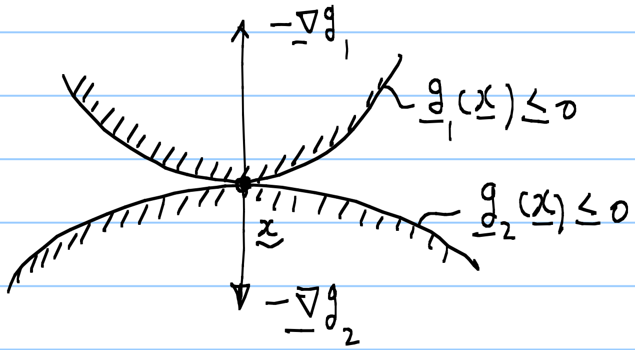
$\boxed{\underline{x}:\ \text{Regular point}}$

⇒ In this case, we can find a feasible $\underset{\sim}{d}$ from: $(-\underline{\nabla g_1})^T \underline{d} > 0$ and $(-\underline{\nabla g_2})^T d > 0$

$\therefore$ $\boxed{\text{We can tell if a direction is feasible or not whenever } \nabla \underset{\sim}{g} \text{ are given at } \underline{x}.}$

Case 2 : CQ not holds $(\underline{\nabla g_1} \,//\, \underline{\nabla g_2})$

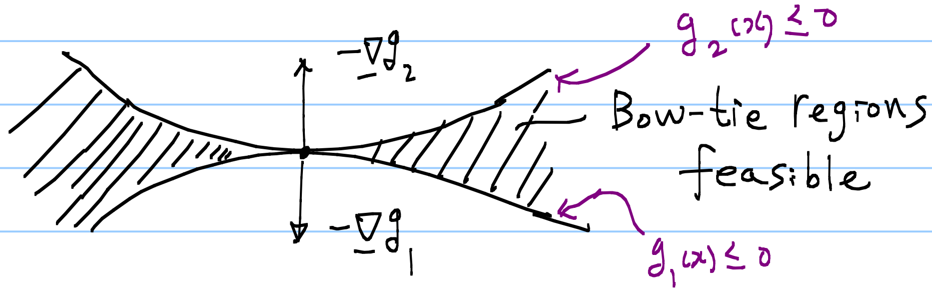
(A)



$\underline{x}$: "NOT Regular" point

- Ω consists of 1 point only
- No feasible direction exists

(B)



- feasible direction exists
- $\underline{\nabla g_1} \,//\, \underline{\nabla g_2}$

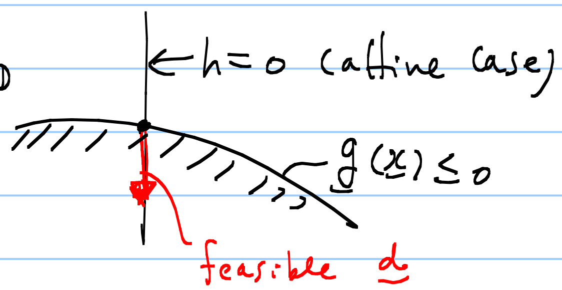
$\Rightarrow$ if CQ does not hold,

- ① it is difficult to distinguish (B) from (A)
- ② Furthermore, Case (B) is not easy to handle from the theoretical point of view

$\therefore$ We will mainly consider Case 1.

<Feasible Direction, Feasible-Enough Direction>

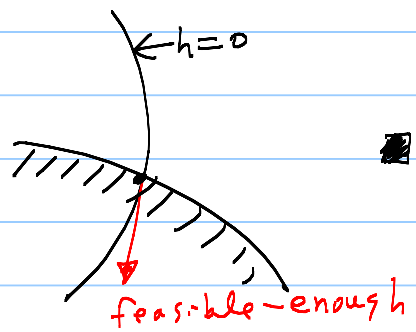
①



■ d is feasible if
$(\nabla g^A(x))^T d < 0$
and $\nabla h^T(x)\, d = 0$
with h = affine

• ⊕ CQ

②



■ d is feasible-enough if
$(\nabla g^A(x))^T d < 0$ and $\nabla h^T(x)\, d = 0$
⊕ CQ

※ The notion of the feasible direction will be used as a building block to derive the optimality condition of nonlinear constrained problems; → "KKT condition"
└ "Karushe-Kuhn-Tucher"

Approach to Study NLP

1) Equality Constraint

$$\min f(\underline{x})$$
$$\text{s.t.} \quad \underline{h}(\underline{x}) = 0$$

2) Inequality Constraint

$$\min f(\underline{x})$$
$$\text{s.t.} \quad \underline{g}(\underline{x}) \leq 0$$

3) Equality + Inequality Constraint

$$\min f(\underline{x})$$
$$\text{s.t.} \quad \underline{h}(\underline{x}) = 0 \text{ and } \underline{g}(\underline{x}) \leq 0$$