Lecture 4-4 : $\begin{cases} \text{Feasible-direction Method} \\ \text{Penalty Method} \end{cases}$

노트 제목

☒ Feasible-direction Method (Numerical Method for NLP with inequality constraints)

$\min f(\underset{\sim}{x})$

s.t $g_i(\underset{\sim}{x}) \le 0$, $i = 1, \cdots, m$

Approach:
- ∗ Start with a feasible point $\underset{\sim}{x}_k$ and determine which constraints are active
- ∗ look for feasible-descent direction $\underset{\sim}{d}_k$ ($\rightarrow$ becomes linear programming)
- ∗ 1-D search to find $\underset{\sim}{x}_{k+1}$
- ∗ Repeat until convergence

Simplied Version

i) feasible-descent direction finding

$$\min \nabla f^T(x_k)\, \underset{\sim}{d} \qquad \oplus \quad \nabla f^T d_k < \underset{\sim}{0}$$

$$\text{s.t} \quad \nabla g_i^A(\underset{\sim}{x}_k)\, \underset{\sim}{d} \le 0$$

$$-1 \le d_i \le 1$$

$\Rightarrow$ "LP" for $(d_1, d_2 \cdots d_n)$

$\Rightarrow$ OR

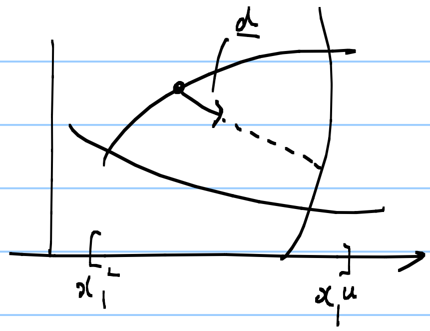
$$\max \beta$$

$$\text{s.t} \quad \begin{cases} \nabla f^T(x_k)\, \underset{\sim}{d} \le -\beta \\ \nabla g_i^A(x_k)\, \underset{\sim}{d} \le 0 \end{cases}$$

$$\beta > 0$$

$\Rightarrow$ LP for $\underset{\sim}{d}$ and β

ii) 1-D search



$$\min_{\alpha} f(x_k + \alpha d)$$

s.t. $g_i(x_k + \alpha \underline{d}) \le 0$

$\Rightarrow$ get α_k

Detailed algorithm, see Vanderplaat or Belegundu.

- Penalty and Barrier Method ← Interior Penalty Method

  ↳ or Exterior Penalty Method

Idea: Treat constrained problems as approximate unconstrained problems

Given
$$\boxed{\begin{aligned} &\min f(\underset{\sim}{x}) \quad \underset{\sim}{x} \in R^n \\ &\text{s.t } g_i(\underset{\sim}{x}) \le 0 \quad (i=1,\cdots,m) \\ &\quad\ h_j(\underset{\sim}{x}) = 0 \quad (j=1,\cdots,l) \end{aligned}} \; \text{---} \; (A)$$

$$\underset{\sim}{\Omega} = \{ \underset{\sim}{x} \mid \underset{\sim}{g}(x) \le 0,\ \underset{\sim}{h} = 0 \}$$

[1] Exterior Penalty Method

(Simply called Penalty Method)

$$(A) \Rightarrow \quad \min \phi(\underset{\sim}{x}; r) = f(\underset{\sim}{x}) + r P(\underset{\sim}{x})$$

↑ r: not variables but parameters

$$r = r_1, r_2, \cdots, r_i \to \infty \quad (r_i \ge 0)$$

(Typical value of $r_{i+1}/r \simeq 5$)

Typical choice of $P(x)$:

$$P(\underset{\sim}{x}) = \sum_{i=1}^{m} \left(\max[0, g_i(\underset{\sim}{x})]\right)^2 + \sum_{j=1}^{\ell} [h_j(\underset{\sim}{x})]^2$$

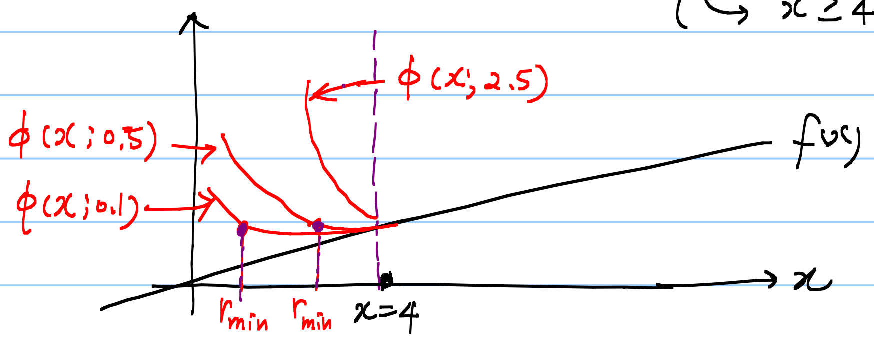
($P>0$ if any of constraints is violated!!)

$$\underset{\sim}{\nabla} P = 2 \sum_{i=1}^{m} \underset{\sim}{\nabla} g_i \underbrace{\max[0, g_i(\underset{\sim}{x})]}_{\text{ignore discontinuities at } g_i(\underset{\sim}{x})=0} + 2 \sum_{j=1}^{\ell} \underset{\sim}{\nabla} h_j(\underset{\sim}{x}) h_j(\underset{\sim}{x})$$

Case Study:

1) $f(x) = 0.5x, \quad g_1(x) = 4 - x \le 0$

($\hookrightarrow x \ge 4$)



$$\phi(x;r) = 0.5x + r\, P(x)$$

$$= 0.5x + r\{\max[0, \underset{g(x)}{(4-x)}]\}^2$$

FONC

$$\nabla\phi = \nabla f + r\nabla\phi = 0.5 + 2r\underset{g'(x)}{(-1)}\max[0, 4-x] = 0$$

if $x \le 4$

$$\nabla \phi = 0.5 - 2r(4-x) \equiv 0 \rightarrow 2rx = 8r - 0.5$$

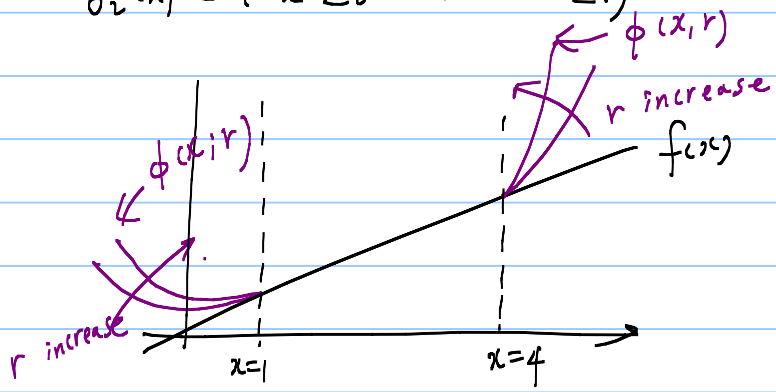
$$x_{min} = \frac{1}{4r}(16r-1) \rightarrow \begin{cases} 1.5 & \text{if } r = 0.1 \\ 3.5 & \text{if } r = 0.5 \\ \vdots & \\ 4 & \text{if } r = \infty \end{cases}$$

x_{min} of ϕ approaches x_{min} of f from the exterior of the feasible region → "Exterior penalty method"

2) $f(x) = 0.5x$

$g_1(x) = x - 4 \le 0 \quad (\leftarrow x \le 4)$

$g_2(x) = 1 - x \le 0 \quad (\leftarrow x \ge 1)$



$$\phi(x;r) = f(x) + r\left\{[\max(0, x-4)]^2 + [\max(0, 1-x)]^2\right\}$$

Example $\min f(x_1, x_2) = x_1^2 + 10x_2^2$

s.t $h = 4 - x_1 - x_2 \doteq 0$

[exact Min = (3.636, 0.3636)]

Solve By the exterior penalty method

Sol: $\phi(x; r) = f(x_1, x_2) + r\, h(x_1, x_2)^2$

$= x_1^2 + 10x_2^2 + r(4 - x_1 - x_2)^2$

FONC $\nabla\phi(x, r) = \begin{Bmatrix} 2x_1(1+r) + 2rx_2 - 8r \\ 2x_2(10+r) + 2rx_1 - 8r \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$x_1 = \dfrac{40r}{10+11r}$; $x_2 = \dfrac{4}{10+11r}$

r	(x_1, x_2)	f	ϕ
1	(1.905, 0.1905)	3.992	7.619
10	(3.333, 0.3333)	12.220	13.333
100	(3.604, 0.3604)	14.288	14.144
1000	(3.633, 0.3633)	14.518	14.532

Check $H_\phi = \nabla\nabla\phi$

$= \begin{bmatrix} 2(1+r) & 2r \\ 2r & 20+2r \end{bmatrix} \rightarrow$ becomes more and more ill-conditioned as $r \rightarrow \infty$

(Cause numerical problems with too large r's)

Back to the Exterior Penalty Method:

Question: Solution Behavior $x^*(r)$ as a ftn of penalty parameter

i) $x^*(r) \overset{?}{\approx} \underline{a} + \underline{b}\, r$ $\quad$ X

ii) $x^*(r) \overset{?}{\approx} \underline{a} + \dfrac{\underline{b}}{r}$ ---(1) yes

$\uparrow$ converged

$\boxed{\text{Trick to improve the Solution Convergence!}}$

Use (1) with two values of r_{i-1} and r_i

$$\left.\begin{aligned}
&\left\{\begin{aligned} x^*(r_{i-1}) &= \underline{a} + \frac{\underline{b}}{r_{i-1}} \\ x^*(r_i) &= \underline{a} + \frac{\underline{b}}{r_i} \end{aligned}\right. \\
\Rightarrow \quad \underset{\sim}{a} &= \frac{\alpha\, x^*(r_{i-1}) - x^*(r_i)}{\alpha - 1} \\
\underset{\sim}{b} &= \left[x^*(r_{i-1}) - \underline{a} \right] r_{i-1}
\end{aligned}\right\} \text{Strategy A}$$

where $\quad \alpha = r_{i-1} / r_i$

If Strategy A is applied to Example;

$r_1 = 1, \quad r_2 = 10, \qquad \alpha = 0.1 \qquad$ [Exact $\underline{x}^* = (3.636, 0.3636)^T$]

$\left(x_i^* \triangleq x^*(r_i) \right)$

$x_1^* = (1.905, 0.1905)^T \qquad x_2^* = (3.333, 0.3333)^T$

$\underset{\sim}{a} = \{3.492, 0.3492\}$

↑ Better estimate

Convergence Criteria

i) $\| \underline{x}_i^* - a \| \le \epsilon_1 \qquad \epsilon_1 = \text{small value} \times \| \underline{x}_i^* \|$

OR

ii) $|(\phi - f)/f| \le \epsilon_2$

OR

iii) $|f_i^* - f_{i-1}^*| / |f_i^*| \le \epsilon_3$

Lagrange Multiplier estimate at optimal pt: $(\underline{\nabla}\phi = 0)$

$$\mu_i \simeq 2r \max [0, g_i]$$

$$\lambda_i \simeq 2r\, h_j$$

[2] Interior Penalty Method [Barrier Method]

$$\min \ \phi(x;r) = f(x) + \frac{1}{r} B_1(r) + r^{\frac{1}{2}} B_2(r)$$

$$r = r_1, r_2, \cdots, r_i \to \infty \quad (r_i > 0)$$

$$\text{s.t.} \quad x \in \text{interior of } \Omega$$

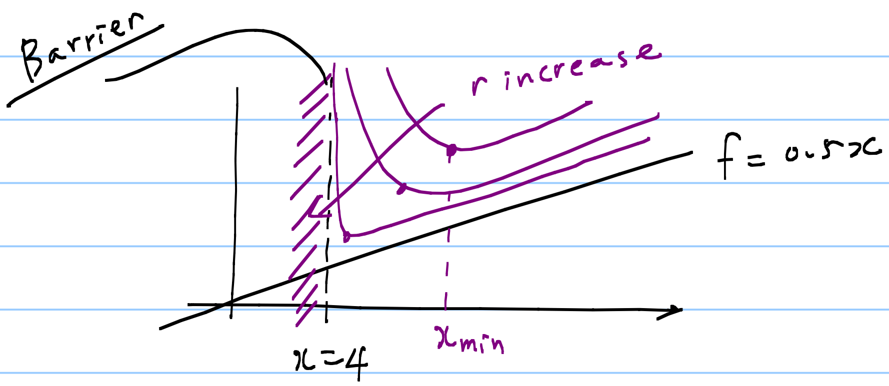
Typical functional forms of B_i

$$B_1(\underline{x}) = -\sum_{i=1}^{m} \frac{1}{g_i(\underline{x})} \quad \leftarrow \text{Inverse Barrier function}$$

$$B_2(x) = \sum_{i=1}^{\ell} h_i^2(\underline{x})$$

example: $f(x) = 0.5x \qquad g_1(x) = 4 - x \le 0$

$$\phi(x;r) = 0.5x - \frac{1}{r(4-x)}$$



① it produces a series of feasible solutions as r increases

② Thus, it requires a feasible starting point.