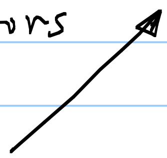
Lecture 1-1.

. Fundamentals on Vectors and Tensors

Will Study

① vectors



$$\begin{cases} \underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 \\ \Updownarrow \\ \underline{v} = v_1' \underline{e}_1' + v_2' \underline{e}_2' \end{cases}$$

$\Rightarrow \Rightarrow$ $v_i \Rightarrow$ related to v_i'
by a certain rule

② Tensor $\supset$ Vector
(Tensor $\sim$ Matrix ?)

③ Vector/Tensor operation
e.g. $\underset{\sim}{a} \times \underset{\sim}{b}$,

④ Vector/tensor calculus
Differentiation ($\underset{\sim}{\nabla}$, etc.)

Why study these??

1) Compact notation → "Convenient"

equilibrium equation:

$$
\begin{cases}
\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + \dfrac{\partial \sigma_{zx}}{\partial z} + f_x = 0 \\[2ex]
\dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + \dfrac{\partial \sigma_{zy}}{\partial z} + f_y = 0 \\[2ex]
\dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} + f_z = 0
\end{cases}
$$

⇒ In vector/Tensor form:

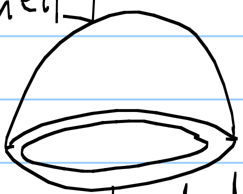
$$\sigma_{ji,j} + f_i = 0 \quad (i = 1, 2, 3)$$

2) Reduce the risk of "mistakes"

in describing physical principle

or obtaining governing equations

[Shell]



complicated geometry

→ Consider deriving governing field equations for shell

Famous people in shell theory:

(a) 「Reissner」 "Reissner theory"

→ Good, made a mistake in deriving shear strain ("twist) term

(why? He did not use vector/tensor approach)

(b) 「Sanders」

→ derived an elegant theory without defects

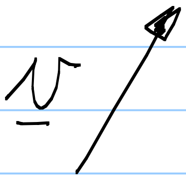
He used a tensor theory, so he did not make any mistake.

[large deformation theory]

→ Tensor-form of equations: better to describe the physics

3) To read journal papers for your research, you should know these.

「Vector?」



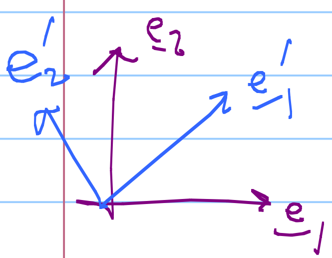
$$\boxed{\begin{aligned}\underline{v} &= v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 = \sum_{i=1}^{3} v_i \underline{e}_i \\ &\equiv v_i \underline{e}_i \text{ (summation convention)}\end{aligned}}$$

① quantities having magnitude and direction

② quantities introduced to represent <u>physical</u> quantities

③ Definition ① is okay, but we need more rigorous definition to precisely define it.

★ Vectors: Some quantities that are invariant under coordinate transformation



$$\boxed{\underline{v} = v_i \underline{e}_i = v_i' \underline{e}_i'} \quad (A)$$

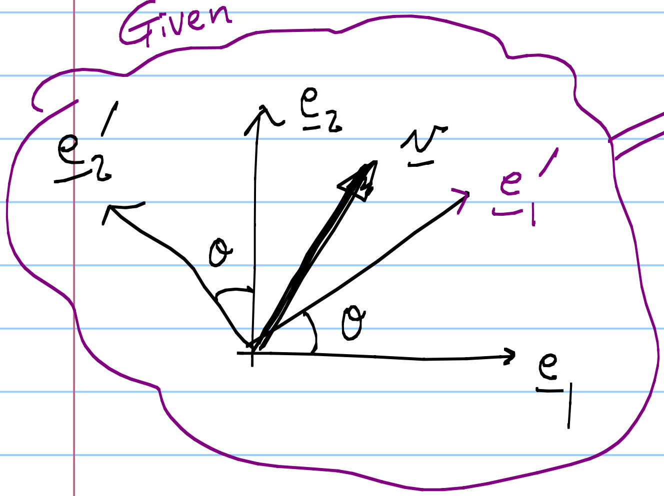
Because of A,

if components of $\underline{v}$ (i.e, v_i) in the $\underline{e}_i$ coordinate system ("OLD") are known,

then the components of $\underline{v}$ (i.e, $v_{i'}$) in the $\underline{e}_{i'}$ coordinate systems can be determed by a certain transformation rule.

$\boxed{\text{Example}}$

Given



$$\underline{e}_{1'} = \overset{\cos\theta}{C_\theta}\, \underline{e}_1 + \overset{\sin\theta}{S_\theta}\, \underline{e}_2 \triangleq \beta_{j1'}\, \underline{e}_j \qquad \text{—}(1)$$

$$\underline{e}_{2'} = -S_\theta\, \underline{e}_1 + C_\theta\, \underline{e}_2 \triangleq \beta_{j2'}\, \underline{e}_j \qquad \text{—}(2)$$

i.e $$\boxed{\underline{e}_{i'} = \beta_{ji'}\, \underline{e}_j} \qquad \text{—}(B)$$

(6)

$$\boxed{\begin{aligned} \beta_{11'} &= (\underline{e}_1' \cdot \underline{e}_1) = C_\theta \, ; \quad \beta_{21'} = (\underline{e}_1' \cdot \underline{e}_2) = S_\theta \\ \beta_{12'} &= (\underline{e}_2' \cdot \underline{e}_1) = -S_\theta \, ; \quad \beta_{22'} = (\underline{e}_2' \cdot \underline{e}_2) = C_\theta \end{aligned}}$$

L(*)

Let us define:

$$v_{i'} \triangleq \overline{\beta}_{i'k} \, v_k \qquad \text{(C)}$$

$$(\, i' = 1, 2 \,)$$

(note: $\overline{\beta}_{i'k} \neq \beta_{ki'}$; they are different symbols)

i.e

$$\left(\begin{aligned} v_1' &= \overline{\beta}_{1'1} v_1 + \overline{\beta}_{1'2} v_2 \\ v_2' &= \overline{\beta}_{2'1} v_1 + \overline{\beta}_{2'2} v_2 \end{aligned} \right)$$

B, C $\Rightarrow$ A

$$v_i \underset{\sim}{e}_i = v_{j'} \underset{\sim}{e}_{j'}$$
$$= \left(\bar{\beta}_{j'k} v_k\right) \beta_{ij'} \underset{\sim}{e}_i$$
$$= \left(\bar{\beta}_{j'k} \beta_{ij'} v_k\right) \underset{\sim}{e}_i$$

$$\Rightarrow \quad v_i = \bar{\beta}_{j'k} \beta_{ij'} v_k$$

$$\underset{\text{must be}}{\equiv} \underbrace{\delta_{ki} v_k}_{\Updownarrow}$$ D

$$\boxed{\delta_{1i} v_1 + \delta_{2i} v_2 + \delta_{3i} v_3}$$

where

$$\delta_{ki} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{else} \end{cases}$$

"Kronecker Delta"

$(\delta_{ki} = \delta_{ik})$

and $\delta_{ki} = \underset{\sim}{e}_k \cdot \underset{\sim}{e}_i$

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From (D)

$$\boxed{\beta_{ij'}\,\bar{\beta}_{j'k} = \delta_{ik}}\,. \qquad (E)_1$$

$$\left(\delta_{ij} = \delta_i^j = \delta^j_i\right)$$

$$\underbrace{[\beta_{ij'}]}_{\substack{\text{Transformation}\\ \text{coeffs for } v_i'\\ \text{(to be computed)}}}\;\underbrace{[\bar{\beta}_{j'k}]}_{\substack{\text{Transformation}\\ \text{coeffs. for } \underline{e}_i'\\ \text{(Known)}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \leftarrow \text{Matrix Notation}$$

$$\rightarrow \quad \underline{\underline{\underset{(3\times3)}{[\bar{\beta}_{j'k}]} = \underset{(3\times3)}{[\beta_{ij'}]^{-1}}}} \qquad (E)_2$$

Messages: ①-1 vector components transform by a certain rule: $v_{i'} = \overline{\beta}_{i'k} v_k$

①-2 if coordinate relations $\underline{e}_{j'} = \beta_{mj'} v_m$ are given, $\overline{\beta}_{i'k}$ can be computed from $\beta_{ij'}$ by $Ⓔ_{①}$ or $Ⓔ_{②}$

* ② if certain quantities transform according to Ⓔ, then they are the components of a vector.

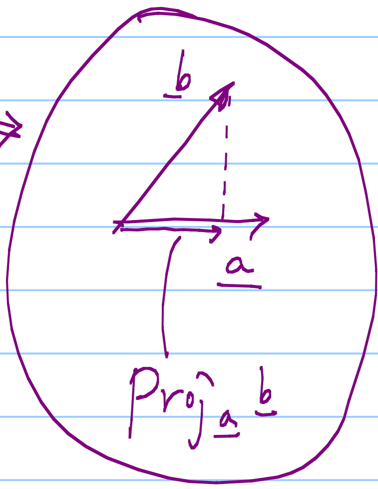
(Remark: we will simply write
$$\overline{\beta}_{i'k} = \beta^{i'k} .$$
)

Tensors?

1) also used to represent physical quatities
2) a linear transform mapping a vector to another vector
(① linear, ② works on vectors)

Tensor $\underset{\sim}{T}$: $\underline{v} \rightarrow \underline{u}$

Ex: Projection tensor $\underset{\sim}{T} = Proj_{\underline{a}} \Rightarrow$



$\rightarrow$ Projection of a vector onto $\underset{\sim}{a}$

Let $\hat{\underset{\sim}{a}} \triangleq \underset{\sim}{a} / |\hat{\underset{\sim}{a}}|$

$\underset{\sim}{T}\,\underline{b} = Proj_{\underline{a}}\, \underset{\sim}{b} = \underline{\hat{a}}\ (\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b})$

$\Rightarrow \underline{\hat{a}}\ \hat{\underset{\sim}{a}} \cdot \underline{b} = (\hat{\underset{\sim}{a}}\ \hat{\underset{\sim}{a}}) \cdot \underset{\sim}{b}$

$\Rightarrow (\hat{\underset{\sim}{a}} \otimes \hat{\underset{\sim}{a}}) \cdot \underset{\sim}{b}$

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In this case, $\underset{\sim}{I}$ is written as

$$\underset{\sim}{I} = \underbrace{\underset{\sim}{a} \otimes \underset{\sim}{a}}_{\text{Dyad}}$$

Operation of dyad onto a vector?

if $\underset{\sim}{I} = \underline{a} \otimes \underline{b}$,

then $\underline{I}\,\underset{\sim}{c} = \underset{\sim}{I} \cdot \underline{c}$

we will use this way

$$= (\underline{a} \otimes \underline{b}) \cdot \underline{c}$$

$$= \underline{a}\,(\underline{b} \cdot \underline{c}) = (\underline{b} \cdot \underline{c})\,\underline{a}$$

Remark : $\underline{a} \otimes \underline{b} \neq \underline{b} \otimes \underline{a}$

$\left(\because \ (\underline{a} \otimes \underline{b}) \cdot \underline{c} \neq (\underline{b} \otimes \underline{a}) \cdot \underline{c} \right)$

Claim: Any Tensor Can be Written as*

$$\underset{\sim}{T} = T_{ij}\, \underset{-}{e}_i \otimes \underset{-}{e}_j \quad (9 \text{ component})$$

("Dyadic Notation")

i.e

$$\begin{aligned}\underset{\sim}{T} &= T_{11}\, \underset{-}{e}_1 \otimes \underset{-}{e}_1 + T_{12}\, \underset{-}{e}_1 \otimes \underset{-}{e}_2 + T_{13}\, \underset{-}{e}_1 \otimes \underset{-}{e}_3 \\ &= T_{21}\, \underset{-}{e}_2 \otimes \underset{-}{e}_1 + T_{22}\, \underset{-}{e}_2 \otimes \underset{-}{e}_2 + T_{23}\, \underset{-}{e}_2 \otimes \underset{-}{e}_3 \\ &= T_{31}\, \underset{-}{e}_3 \otimes \underset{-}{e}_1 + T_{32}\, \underset{-}{e}_3 \otimes \underset{-}{e}_2 + T_{33}\, \underset{-}{e}_3 \otimes \underset{-}{e}_3\end{aligned}$$

In terms of its component,

$$\underset{\sim}{T} \sim \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

↑ matrix form.

** $T_{ij} = T_{ij}\, \underset{-}{e}_i \otimes \underset{-}{e}_j$

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* Read page 11~12

By using, $T_{i'k'} = \underline{e}_{i'} \cdot (\underset{\sim}{T}) \cdot \underline{e}_{k'}$

$$= \underline{e}_{i'} \cdot \left(T_{jm} \underline{e}_j \otimes \underline{e}_m \right) \cdot \underline{e}_{k'}$$

One can show:

<u>Tensor Transformation Rule</u>

$$\underline{e}_j \longleftrightarrow \underline{e}_{i'} \quad \left(\beta_{i'j} \text{ or } \beta_{ji'} \right)$$

$$T_{ij} \longleftrightarrow T_{i'j'}$$

$$\begin{cases} T_{i'k'} = \beta_{i'j} \, \beta_{k'm} \, T_{jm} \\ T_{ij} = \beta_{ik'} \, \beta_{jm'} \, T_{k'm'} \end{cases}$$

(see page 13-14 for proof)

* Tensor Vs Matrix ?

- Tensor can be written in matrix form

$$\underset{\sim}{T} \sim [T_{ij}]$$

- Tensor : $\boxed{\text{physical}}$ quantities

Matrix : (two-) dimensional array.

- Thus, tensor component T_{ij} satisfies a certain coordinate transformation rule!

Vector / Tensor Operation

(Should not write it as $a_i \underline{e}_i \cdot b_i \underline{e}_i$)

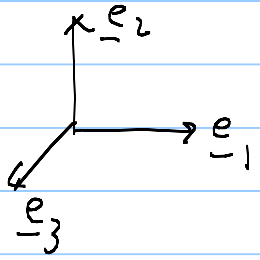
$$\underline{a} \cdot \underline{b} = a_i \underline{e}_i \cdot b_j \underline{e}_j$$

$$= a_i b_j (\delta_{ij}) = a_i b_i$$

$$\underline{a} \times \underline{b} = a_i \underline{e}_i \times b_j \underline{e}_j$$

$$= a_i b_j \, \underline{e}_i \times \underline{e}_j$$

$\underline{e}_i \times \underline{e}_j = ?$



$\underline{e}_1 \times \underline{e}_2 = +\underline{e}_3$

$e_2 \times \underline{e}_1 = -e_3$

$(\underline{e}_1 \times \underline{e}_1 = 0)$ etc

$\rightarrow$ introduce $\epsilon_{ijk} = \begin{cases} 0 & \epsilon_{112} \text{ etc} \\ +1 & \text{even permutation} \\ -1 & \text{odd} \quad \text{"} \end{cases}$

$$e_i \times \underline{e}_j = \epsilon_{ijk}\, \underline{e}_k$$

(Thus, $\underline{e}_1 \times e_2 = \epsilon_{12k}\, \underline{e}_k$

$= \cancel{\epsilon_{121}}\, \underline{e}_1 + \cancel{\epsilon_{122}}\, \underline{e}_2 + \epsilon_{123}\, \underline{e}_3$

$= \underline{e}_3$)

Remark: $\underset{\approx}{\epsilon} = \epsilon_{ijk}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \rightarrow$ Tensor (permutation tensor)

$\mathbb{1} = \delta_{ij}\, \underline{e}_i \otimes \underline{e}_j$ (metric tensor)

- $\operatorname{trace}(\underline{a} \otimes \underline{b}) \stackrel{\wedge}{=} \underline{a} \cdot \underset{\sim}{b}$

$$\text{Thus } \operatorname{trace} \underset{\sim}{T} = T_{ij} \operatorname{trace}(\underline{e}_i \cdot \underline{e}_j)$$
$$= T_{ij}\,\delta_{ij}$$
$$= T_{ii}$$
$$= T_{11} + T_{22} + T_{33}$$

- <u>Tensor product $\underline{T} \cdot \underline{S}$</u>

Def: $(\underline{T} \cdot \underline{S}) \cdot \underset{\sim}{v} = \underline{T} \cdot (\underline{S} \cdot \underline{v})$ for any $\underline{v}$

works on $\underline{v}$

if $\underline{U} = \underline{T} \cdot \underline{S}$,

then $U_{ij} = T_{ik} S_{kj}$

$$\left(\text{why? } \underline{T} \cdot \underline{S} = (T_{im}\,\underline{e}_i \otimes \underline{e}_m) \cdot (S_{kj}\,\underline{e}_k \otimes \underline{e}_j)\right.$$
$$= T_{im} S_{kj}\,\delta_{mk}\,\underline{e}_i \otimes \underline{e}_j$$
$$= T_{ik} S_{kj}\,\underline{e}_i \otimes \underline{e}_j$$
$$= U_{ij}\,\underline{e}_i \otimes \underline{e}_j$$

(⊕ Read pp 15–16)

Differentiation

① $\underset{\sim}{\nabla}(\ ) \triangleq \underline{e}_i \dfrac{\partial}{\partial x_i}(\ )$

gradient

$$\underline{\nabla} f = \underline{e}_i \frac{\partial f}{\partial x_i}$$

$$\underline{\nabla}\, \underline{v} = \underline{e}_i \frac{\partial}{\partial x_i} \otimes v_j \underline{e}_j$$

$$= \underline{e}_i \otimes \left(\frac{\partial v_j}{\partial x_i} \underline{e}_j + v_j \cancel{\frac{\partial \underline{e}_j}{\partial x_i}} \right)$$

Cartesian

$$= v_{j,i}\, \underline{e}_i \otimes \underline{e}_j$$

Diff

② Div (div) = divergence

div = $\underline{\nabla} \cdot$

$$\text{div}\, \underline{v} = \underline{\nabla} \cdot \underline{v} = \underline{e}_i \frac{\partial}{\partial x_i} \cdot (v_j \underline{e}_j)$$

$$= \frac{\partial v_i}{\partial x_i} = v_{i,i}$$

③ curl $= \underline{\nabla} \times (\ )$

$$\underline{\nabla} \times \underline{v} = \underline{e}_i \frac{\partial}{\partial x_i} \times v_j \underline{e}_j$$

$$= (\underline{e}_i \times \underline{e}_j) \frac{\partial v_j}{\partial x_i}$$

$$= \epsilon_{ijk} \underline{e}_k \, v_{j,i}$$

$$= \epsilon_{ijk} \frac{\partial}{\partial x_i}(v_j) \underline{e}_k$$

④ proof of :

$$\underline{\nabla}(\underline{u} \cdot \underline{v}) = (\underline{\nabla} \underline{u}) \cdot \underline{v} + (\underline{\nabla} \underline{v}) \cdot \underline{u}$$

$$LHS = \underline{e}_i \frac{\partial}{\partial x_i}(u_j v_j)$$

$$= \underline{e}_i (u_{j,i} v_j + u_j v_{j,i})$$

Trick! →

$$= \underline{e}_i \Big(u_{j,i} \underbrace{\delta_{kj}}_{\underline{e}_j \cdot \underline{e}_k} v_k + v_{j,i} \underbrace{\delta_{jk}}_{\underline{e}_j \cdot \underline{e}_k} u_k \Big)$$

$$= (u_{j,i}\, \underline{e}_i \otimes \underline{e}_j) \cdot v_k \underline{e}_k$$

$$+ (v_{j,i}\, \underline{e}_i \otimes \underline{e}_j) \cdot u_k \underline{e}_k$$

$$= (\underline{\nabla}\, \underline{u}) \cdot \underline{v} + (\underline{\nabla}\, \underline{v}) \cdot \underline{u}$$

⑤ Divergence Theorem

$$\int_V \underline{\nabla} \cdot \underset{\uparrow \text{ or } \underset{\sim}{T}}{\underline{v}}\, dV = \int_S \underset{\sim}{n} \cdot \underset{\uparrow \text{ or } \underset{\sim}{T}}{\underline{v}}\, dS$$

+) Green's Theorm

⑥ Stoke's Theorem

$$\int_S \underline{n} \cdot (\underline{\nabla} \times \underline{v})\, dS = \int_C \underset{\sim}{\hat{t}} \cdot \underline{v}\, dl$$

unit tangent vector along a contour C