Lecture 1-2 : More on Vectors & Tensors

To Carry out Tensor operations

1) In Cartesian coordinates

$\Rightarrow$ use the Cartesian tensor component

example: $$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} = 0$$

$$\Rightarrow \boxed{f_{,ii} = 0}$$

2) In non-Cartesian coordinates

a) use general tensor component and operation rules

(Base vectors vary as ftn of coordinates)

Laplace Eq: $\boxed{f\big|^{i}_{i} = 0}$

$(\cdot)\big|^{i}_{i}$ covariant and contravariant derivatives)

can be easier

b) use symbolic notation and carry out directly

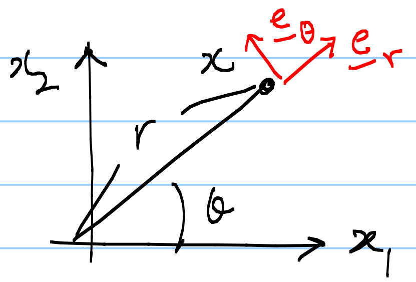
Laplace Eq: $\nabla^2 f = \underline{\nabla} \cdot \underline{\nabla} f = 0$

$$\underset{\sim}{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \underline{e}_z \frac{\partial}{\partial z}$$

(see Next Example)

⟨Example⟩ In Polar Coord. System:

$$\underline{\nabla}^2 f = ? \qquad \underline{\nabla}\,\underline{v} = ?$$



$$\underset{\sim}{\nabla} = \underline{e}_i \frac{\partial}{\partial x_i} = \underline{e}_1 \frac{\partial}{\partial x_1} + \underline{e}_2 \frac{\partial}{\partial x_2}$$

$$\Rightarrow \boxed{\underset{\sim}{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta}}$$

Note: $\dfrac{\partial \underline{e}_r}{\partial r} = 0, \qquad \dfrac{\partial \underline{e}_\theta}{\partial r} = 0$

$$\frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta, \qquad \frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r$$

① $$\underline{\nabla} f = \underline{e}_r \frac{\partial f}{\partial r} + \frac{1}{r} \underline{e}_\theta \frac{\partial f}{\partial \theta}$$

$$= \underline{e}_r \frac{\partial f}{\partial r} + \underline{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}$$

$$
\begin{aligned}
\nabla^2 f &= \underline{\nabla} \cdot \underline{\nabla} f \\
&= \left(\underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} \right) \cdot \\
&\quad \left(\underline{e}_r \frac{\partial f}{\partial r} + \underline{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} \right) \\
&= \underline{e}_r \cdot \frac{\partial}{\partial r} \left(\underline{e}_r \frac{\partial f}{\partial r} + \underline{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} \right) \\
&\quad + \underline{e}_\theta \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \left(\underline{e}_r \frac{\partial f}{\partial r} + \underline{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} \right) \\
&= \underline{e}_r \cdot \left[\underline{e}_r \frac{\partial^2 f}{\partial r^2} - \underline{e}_\theta \frac{1}{r^2} \frac{\partial f}{\partial \theta} \right] \\
&\quad + \underline{e}_\theta \cdot \frac{1}{r} \left[\underline{e}_\theta \frac{\partial f}{\partial r} + \underline{e}_r \frac{\partial^2 f}{\partial \theta \partial r} \right. \\
&\qquad \left. - \underline{e}_r \frac{1}{r} \frac{\partial f}{\partial \theta} + \underline{e}_\theta \frac{1}{r} \frac{\partial^2 f}{\partial \theta^2} \right] \\
&= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}
\end{aligned}
$$

② $$\underline{\nabla}\,\underline{V} = \left(\underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \otimes \left(v_r \underline{e}_r + v_\theta \underline{e}_\theta\right)$$

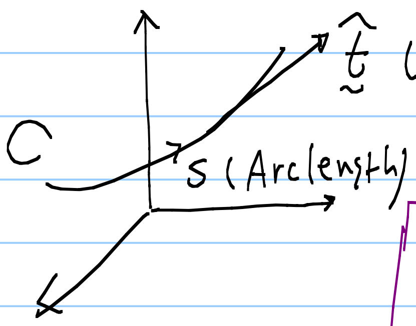
$$= \underline{e}_r \otimes \frac{\partial}{\partial r}(v_r \underline{e}_r) + \underline{e}_r \otimes \frac{\partial}{\partial r}(v_\theta \underline{e}_\theta)$$
$$+ \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \otimes (v_r \underline{e}_r) + \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \otimes (v_\theta \underline{e}_\theta)$$

$$\left(\frac{\partial \underline{e}_r}{\partial r} = \frac{\partial \underline{e}_\theta}{\partial \theta} = 0, \quad \frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta, \quad \frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r \right)$$

$$= \underline{e}_r \otimes \frac{\partial v_r}{\partial r} \underline{e}_r + \underline{e}_r \otimes \frac{\partial v_\theta}{\partial r} \underline{e}_\theta$$
$$+ \frac{\underline{e}_\theta}{r} \otimes \left(\frac{\partial v_r}{\partial \theta} \underline{e}_r + v_r \underline{e}_\theta \right)$$
$$+ \frac{\underline{e}_\theta}{r} \otimes \left(\frac{\partial v_\theta}{\partial \theta} \underline{e}_\theta - v_\theta \underline{e}_r \right)$$

$$= \frac{\partial v_r}{\partial r} \underline{e}_r \otimes \underline{e}_r + \left(\frac{1}{r} v_r + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} \right) \underline{e}_\theta \otimes \underline{e}_\theta$$
$$+ \frac{\partial v_\theta}{\partial r} \underline{e}_r \otimes \underline{e}_\theta + \left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \right) \underline{e}_\theta \otimes \underline{e}_r$$

⟨Some Useful Results⟩



$\hat{\underset{\sim}{t}}$ (unit tangent vector along a curve C)

$$\boxed{\frac{df}{ds} \text{ along } \hat{\underline{t}} = ?}$$

$$\Rightarrow \boxed{\frac{df}{ds} = \hat{\underline{t}} \cdot \underline{\nabla} f}$$

(Important, remember)

Derivation:

$$\frac{df}{ds} = \frac{d}{ds} f\left(x_1(s), x_2(s), x_3(s)\right)$$

$$= \frac{\partial f}{\partial x_1}\frac{ds}{dx_1} + \frac{\partial f}{\partial x_2}\frac{dx_2}{ds} + \frac{\partial f}{\partial x_3}\frac{dx_3}{ds}$$

$$= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) \cdot \underbrace{\left(\frac{dx_1}{ds}, \frac{dx_2}{ds}, \frac{dx_3}{ds}\right)}_{\text{Def. of } \hat{\underline{t}}};$$

$$= \underline{t} \cdot \underline{\nabla} f$$

Extension:

$$\boxed{\left.\frac{d\underset{\sim}{v}}{ds}\right|_{\text{Along } C} = \hat{\underset{\sim}{t}} \cdot \underline{\nabla}\, \underline{v}}$$

Remark:

$$\left.\frac{d}{ds}(\ )\right|_{\text{Along } C} \Rightarrow \text{"Directional Derivative"}$$