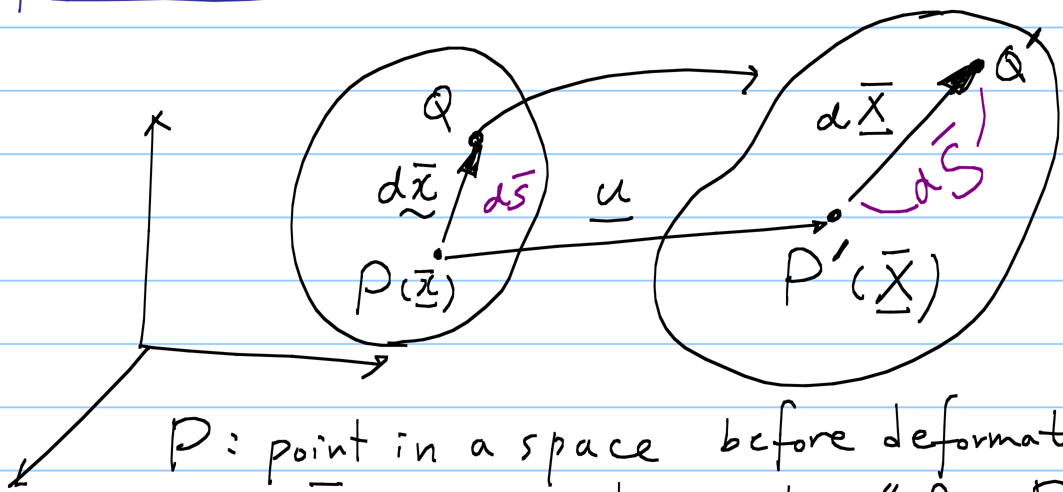
# Lecture 1-3 : Deformation






P : point in a space before deformation
( $\bar{x}$ : coordinate "values" for P )

P' : point in a space after deformation
( $\bar{X}$ : " " for P' )

## < length change & strain >

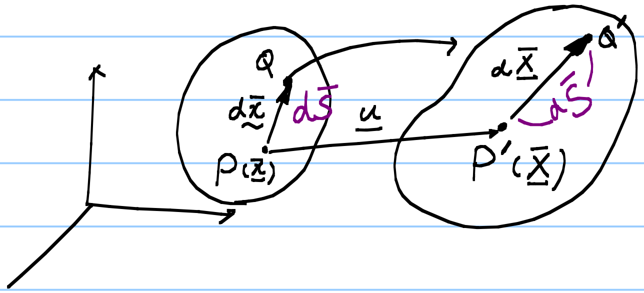
$$d\bar{S}^2 - d\bar{s}^2 \triangleq 2 E_{jk}\, d\bar{x}_j\, d\bar{x}_k = 2\, d\underset{\sim}{\bar{x}} \cdot \underline{E} \cdot d\underset{\sim}{\bar{x}}$$

← Lagrangian strain
"relative to undeformed coordinates"

$$\triangleq 2\, E^{*}_{jk}\, d\bar{X}_j\, dX_k = 2\, d\underset{\sim}{\bar{X}} \cdot \underline{E}^{*} \cdot d\underset{\sim}{X}$$

← Eulerian strain
"relative to deformed coord."

## < Analysis of $\underset{\sim}{E}$ >




Note

$$
\begin{cases}
d\bar{s}^2 = d\underset{\sim}{\bar{x}} \cdot d\underset{\sim}{\bar{x}} \\
d\bar{S}^2 = d\underset{\sim}{\bar{X}} \cdot d\underset{\sim}{\bar{X}}
\end{cases}
$$

Displacement

with $\underset{\sim}{X} = \underset{\sim}{x} + \underset{\sim}{u}$

$$
\begin{aligned}
d\bar{S}^2 &= d\underset{\sim}{\bar{X}} \cdot d\underset{\sim}{\bar{X}} = d\bar{X}_i \, d\bar{X}_i \\
&= \left( \frac{\partial X_i}{\partial x_j} d\bar{x}_j \right) \left( \frac{\partial X_i}{\partial x_k} d\bar{x}_k \right) \\
&= \frac{\partial}{\partial x_j} (x_i + u_i) \frac{\partial}{\partial x_k} (x_i + u_i) \, d\bar{x}_j \, d\bar{x}_k \\
&= (\delta_{ij} + u_{i,j})(\delta_{ik} + u_{i,k}) \, d\bar{x}_j \, d\bar{x}_k
\end{aligned}
$$

$$= (\underbrace{\delta_{ij}\delta_{ik}}_{\delta_{jk}} + \underbrace{\delta_{ij} u_{i,k}}_{u_{j,k}} + \underbrace{u_{i,j}\delta_{ik}}_{u_{k,j}} + u_{i,j} u_{i,k})\, d\bar{x}_j\, d\bar{x}_k$$

$$= \underbrace{d\bar{x}_j\, d\bar{x}_j}_{d\bar{s}^2} + \underbrace{(u_{j,k} + u_{k,j} + u_{i,j} u_{i,k})}_{\triangleq 2E_{jk}}\, d\bar{x}_j\, d\bar{x}_k$$

From,

$$d\tilde{s}^2 - d\bar{s}^2 = 2 E_{jk}\, d\bar{x}_j\, d\bar{x}_k$$

$$= 2\, d\underset{\sim}{\bar{x}} \cdot \underset{\sim}{E} \cdot d\underset{\sim}{\bar{x}}$$

we can find

$$\boxed{\begin{aligned} &\underset{\sim}{E} = E_{ij}\, \underline{e}_i \otimes \underline{e}_j \\ &E_{ij} = (u_{i,j} + u_{j,i} + u_{i,k} u_{j,k})/2 \\ &\Rightarrow \text{Lagrangian strain tensor} \\ &\quad \text{Defined in "Undeformed" coordinate system} \end{aligned}}$$

Remark;

Eulerian strain tensor : $E^*_{ij}$

$$d\bar{S}^2 - d\bar{s}^2 = 2\, d\underset{\sim}{\bar{X}} \cdot \underset{\sim}{E}^* \cdot d\underset{\sim}{\bar{X}}$$

$\rightarrow$ defined in deformed coord. sys.

$$E^*_{ij} = \frac{1}{2}\left( u_{i,j} + u_{j,i} - u_{k,i}\, u_{k,j} \right)$$

$$E_{ij} = \frac{1}{2}\left( u_{i,j} + u_{j,i} + u_{k,i}\, u_{k,j} \right)$$

$( \ )_{,i} = \frac{\partial}{\partial x_i}$

$( \ )_{,i} = \frac{\partial}{\partial X_i}$

## For small deformation;

$$\frac{\partial}{\partial x_j}(\ ) \simeq \frac{\partial}{\partial X_j}(\ ) \qquad E_{ij} \simeq E^*_{ij} \Rightarrow \varepsilon_{ij}$$

linear strain

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

$$\underset{\sim}{\varepsilon} = \varepsilon_{ij}\, \underline{e}_i \otimes \underline{e}_j = \frac{1}{2}\left[\nabla \underline{u} + (\nabla \underline{u})^T\right]$$

$$\left[\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^T\right]$$

### Displacement gradient $\nabla \underline{u}$

$$\nabla \underline{u} = \underbrace{\frac{1}{2}\left[\nabla \underline{u} + (\nabla \underline{u})^T\right]}_{\underset{\sim}{\varepsilon}} + \underbrace{\frac{1}{2}\left[\nabla \underline{u} - (\nabla \underline{u})^T\right]}_{\underset{\sim}{\omega}}$$

symmetric part ($(\nabla \underline{u})_{sym}$)

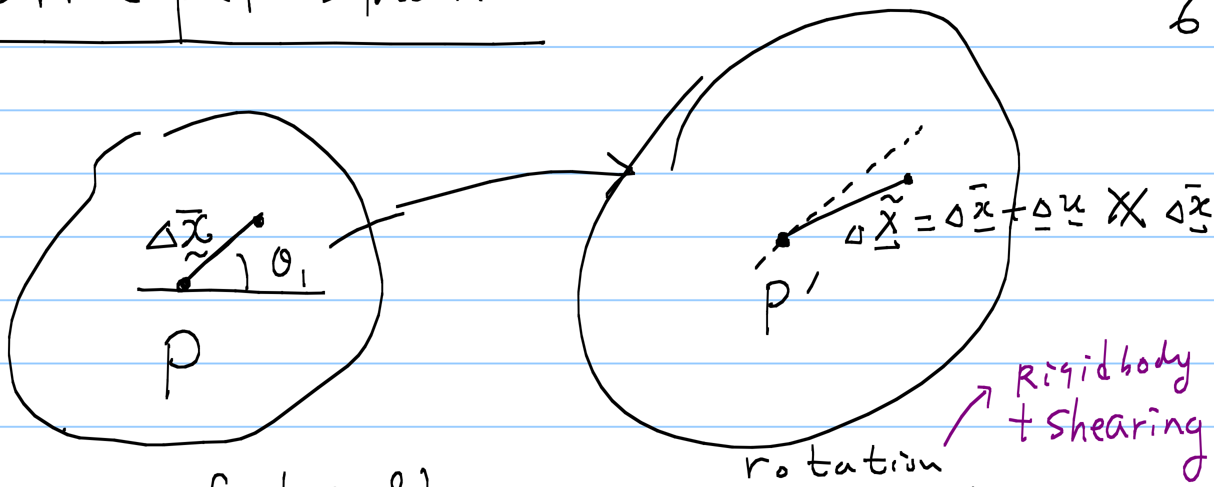
antisymmetric part

→ [illegible] [illegible] Rotation

(see page 49 – 50)

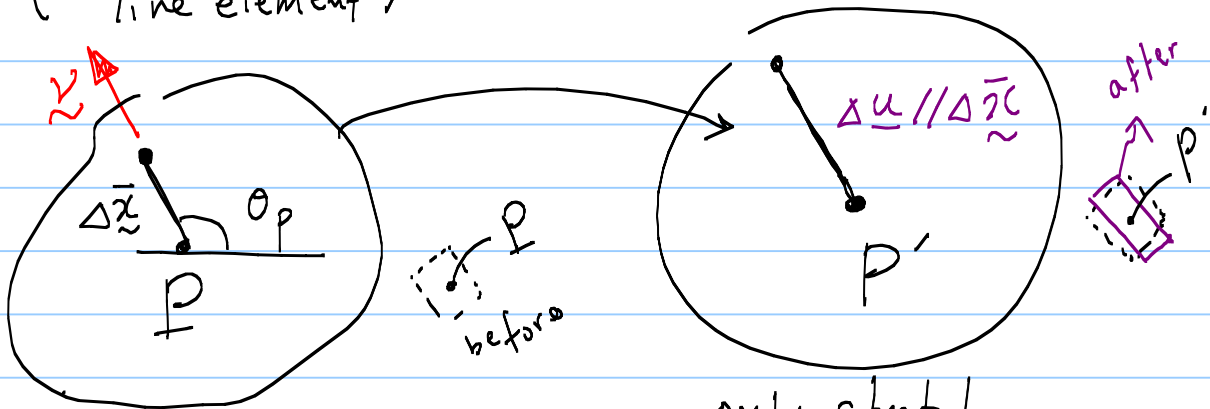
# Principal Strain






(all infinitesimal line element)

rotation + stretch




$\Delta \underset{\sim}{x} = \underset{\sim}{\nu}\, dx$

only stretch (i.e, no rotation)

For some orientation $\theta_p$, or unit tangent $\underline{\nu}$,

if $d\underset{\sim}{u} \,//\, d\underline{x}$ holds,

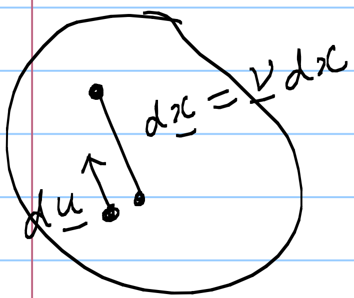
$\theta_p$ ; $\underline{\nu}$ $\Rightarrow$ "principal direction"

In this case,

one can set:

$$d\underline{u} \equiv \lambda \, d\underline{x} \equiv \lambda \, \underline{\nu} \, dx \qquad (a)$$

Excluding Infinitesimal Rigid-body Rotation




Along $\underline{\nu}\, dx$ direction,

$$d\underline{u} = \underline{\nu} \cdot \underline{\nabla}\, \underline{u} \, dx$$

(Recall $\left.\dfrac{df}{ds}\right|_{\text{along } C} = \underset{\sim}{\hat{t}} \cdot \nabla f$, Eq (1.1.99)

choose: $f \Rightarrow \underline{u}$ ; $\underset{\sim}{\hat{t}} \Rightarrow \underline{\nu}$

$ds = dx$ )

Since $\underset{\sim}{\nabla}\, \underline{u} = \underset{\sim}{\varepsilon} + \underset{\sim}{\omega}$

$\underset{\sim}{\varepsilon}$: sym part

$\underset{\sim}{\omega}$: anti-sym Infinitesimal Rigid-Body Rotation

$$d\underline{u} = \underline{\nu} \cdot \underline{\nabla}\underline{u} \quad dx$$
$$= (\underline{\nu} \cdot \underline{\varepsilon} + \underline{\nu} \cdot \underset{\sim}{\omega})\, dx$$

$$\Rightarrow (\underline{\nu} \cdot \underline{\varepsilon})\, dx \qquad (b)$$

(not concerned with an infinitesimal rigid-body rotation)

$(a, b) \rightarrow$

$$\underline{\nu} \cdot \underline{\varepsilon} = \lambda \underline{\nu}$$

OR

$$\underline{\varepsilon}^T \cdot \underline{\nu} = \lambda \underline{\nu} = \lambda \underline{\mathbb{1}} \cdot \underline{\nu}$$

using

$$\underline{\varepsilon}^T = \underline{\varepsilon}$$

we have

$$\underset{\sim}{\varepsilon} \cdot \underline{\nu} = \lambda \mathbb{1} \cdot \underline{\nu} \qquad (C1)$$

$$(\underline{\varepsilon} - \lambda \underline{\mathbb{1}}) \cdot \underline{\nu} = 0 \qquad (C2)$$

In component

$$(\varepsilon_{ij} - \lambda \delta_{ij})\, \nu_j = 0 \qquad (C3)$$

"Eigenvalue problem"

For non trivial solution of (c);

$$|\varepsilon_{ij} - \lambda\,\delta_{ij}| = 0 \quad (\lambda_1, \lambda_2, \lambda_3) \quad (d)$$

FACT FOR SYMMETRIC TENSORS $\underline{\varepsilon}$

i) $\lambda$ (eigenvalue) = Real

Here, principal strain

ii) $\underline{\nu} = \nu_i \underline{e}_i$ (Eigenvector) = Real

Here, principal strain direction

* iii) Three eigenvectors = Mutually orthonormal

(However, eigenvalues may be not all distinct, i.e, $\lambda_1 \neq \lambda_2 = \lambda_3$ )

To expand (d), use Eq. (1.1.13) of book

$$\det A = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = e_{ijk} A_{1i} A_{2j} A_{3k} \quad (*)$$

Book (1.1.93)

Inserting in (1.1.93):

$$A_{ij} = \varepsilon_{ij} - \lambda \delta_{ij}$$

$$0 = | \varepsilon_{ij} - \lambda \delta_{ij} | \longleftarrow \text{characteristic eq}$$

$$\underset{(*)}{=} e_{ijk} (\varepsilon_{1i} - \lambda \delta_{1i}) (\epsilon_{2j} - \lambda \delta_{2j}) (\epsilon_{3k} - \lambda \delta_{3k})$$

$$= -\lambda^3 e_{ijk} \delta_{1i} \delta_{2j} \delta_{3k}$$

$$+ \lambda^2 e_{ijk} ( \epsilon_{1i} \delta_{2j} \delta_{3k} + \delta_{1i} \epsilon_{2j} \delta_{3k} + \delta_{1i} \delta_{2j} \epsilon_{3k} )$$

$$- \lambda^1 e_{ijk} ( \epsilon_{1i} \epsilon_{2j} \delta_{3k} + \delta_{1i} \epsilon_{2j} \epsilon_{3k} + \epsilon_{1i} \delta_{2j} \varepsilon_{3k} )$$

$$+ \lambda^0 ( e_{ijk} \varepsilon_{1i} \varepsilon_{2j} \varepsilon_{3j} )$$

$$= -\lambda^3 \, e_{123} \rightarrow 1$$

$$+ \lambda^2 \underbrace{\left( e_{i23}\,\epsilon_{1i} + e_{1j3}\,\epsilon_{2j} + e_{12k}\,\epsilon_{3k} \right)}_{\underbrace{\epsilon_{11} + \epsilon_{22} + \epsilon_{33}}_{\mathrm{tr}\,\underline{\epsilon} \,\triangleq\, I_1}}$$

$$- \lambda \underbrace{\left( e_{ij3}\,\epsilon_{1i}\,\epsilon_{2j} + e_{1jk}\,\epsilon_{2j}\,\epsilon_{3k} + e_{i2k}\,\epsilon_{1i}\,\epsilon_{3k} \right)}$$

$$\Rightarrow \left[ \epsilon_{11}\epsilon_{22} - \epsilon_{12}^2 \right] + \left[ \epsilon_{22}\epsilon_{33} - \epsilon_{23}^2 \right] + \left[ \epsilon_{11}\epsilon_{33} - \epsilon_{13}^2 \right]$$

$$\Rightarrow \frac{1}{2} \left\{ \left( \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \right)^2 - \left( \epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2 + 2\epsilon_{12}^2 + 2\epsilon_{23}^2 + 2\epsilon_{31}^2 \right) \right\}$$

$$\Rightarrow \underbrace{\frac{1}{2} \left\{ (\mathrm{tr}\,\underline{\epsilon})^2 - (\epsilon_{ij}\,\epsilon_{ij})^2 \right\}}_{\triangleq I_2}$$

$$+ \lambda^0 \underbrace{\det \underline{\epsilon}}_{\triangleq I_3}$$

$$= -\left[\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3\right]$$

$$\begin{cases} I_1 = \operatorname{tr}\underline{\varepsilon} \quad (= \varepsilon_{ii}) \\ I_2 = \frac{1}{2}\left(I_1^2 - \underline{\varepsilon} : \underline{\varepsilon}\right) = \frac{1}{2}\left[(\varepsilon_{ii})^2 - \varepsilon_{ij}\varepsilon_{ij}\right] \\ I_3 = \det \underset{\sim}{\varepsilon} \end{cases}$$

$\Uparrow$ three $\underbrace{\text{"Invariants"}}_{\text{same values in any coordinate system}}$ $\rightsquigarrow$ Indep of coordinates

## ⟨Principal Strain and Direction⟩

$$(\underset{\sim}{\varepsilon} - \lambda \underline{I}) \cdot \underline{\nu} = 0$$

① need to solve

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

② After finding three $\lambda$'s, then find $\underline{\nu}$.

Ex 1.2.2

$$\underset{\sim}{\varepsilon}\Big|_{\text{at } P} = \left[ \underset{-}{e}_1 \otimes \underset{-}{e}_1 + \underset{-}{e}_3 \otimes \underset{-}{e}_3 + \sqrt{3}\,( \underset{-}{e}_1 \otimes \underset{-}{e}_2 + \underset{-}{e}_2 \otimes \underset{-}{e}_1 ) \right] \times 10^{-3}$$

$$\sim 10^{-3} \begin{bmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the principal strains and directions.

Sol.

$$\underline{\varepsilon} \cdot \underline{\nu} = \lambda \underline{\mathbb{1}} \cdot \underline{\nu}$$

① for $\lambda$, solve

- $| \varepsilon_{ij} - \lambda \delta_{ij} | = 0$
- $\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$

$$I_1 = \mathrm{tr}\, \underline{\varepsilon} = 2 \times 10^{-3}$$

$$I_2 = ( I_1^2 - \underline{\varepsilon} : \underline{\varepsilon} ) / 2$$

$$\left( \begin{aligned} \underline{\varepsilon} : \underline{\varepsilon} &= \mathrm{tr}\,( \underline{\epsilon}^T \underline{\epsilon} ) \\ &= \mathrm{tr} \begin{bmatrix} 4 & \sqrt{3} & 0 \\ \sqrt{3} & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times 10^{-6} \\ &= 8 \times 10^{-6} \end{aligned} \right)$$

$$= -2\times10^{-6}$$

$$I_3 = \det \underset{\sim}{\varepsilon} = -3\times10^{-9}$$

Solving

$$\lambda^3 - 2\times10^{-3}\lambda^2 - 2\times10^{-6}\lambda + 3\times10^{-9} = 0$$

$$\lambda = \underbrace{1\times10^{-3}}_{\varepsilon^{(1)}},\ \underbrace{\frac{1+\sqrt{13}}{2}\times10^{-3}}_{\varepsilon^{(2)}},\ \underbrace{\frac{1-\sqrt{13}}{2}\times10^{-3}}_{\varepsilon^{(3)}}$$

principal direction

② find $\underline{\nu}^{(i)}$ for each $\varepsilon^{(i)}$

i) $\varepsilon^{(1)} = 1\times10^{-3}$

$$\left(\underset{\sim}{\varepsilon} - \varepsilon^{(1)}\,\mathbb{1}\right)\cdot\underline{\nu}^{(1)} = 0$$

$$10^{-3}\begin{bmatrix} 0 & \sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \nu_1^{(1)} \\ \nu_2^{(1)} \\ \nu_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} \sqrt{3}\,\nu_2^{(1)} = 0 & \rightarrow \quad \nu_2^{(1)} = 0 \\ \sqrt{3}\,\nu_1^{(1)} - \nu_2^{(1)} = 0 & \qquad \nu_1^{(1)} = 0 \\ 0\,\nu_3^{(1)} = 0 & \qquad \nu_3^{(1)} = \text{arbitrary} \neq 0 \\ & \qquad\quad \equiv 1 \end{cases}$$

$\|\underline{\nu}\| = 1$ (normalization)

$$\underline{\nu}^{(1)} \sim \{0,\ 0,\ 1\}^T$$

ii) repeat for $\underline{\varepsilon}^{(2)}$ and $\underline{\varepsilon}^{(3)}$

$$\underline{\nu}^{(2)} = \{0.8,\ 0.6,\ 0\}^T$$

$$\underline{\nu}^{(3)} = \{0.6,\ -0.8,\ 0\}^T$$

Check:

$\underline{\nu}^{(i)}$ ; orthogonal to each other

## < Spectral Decomposition >

Base Vector at P was $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$.

$$\underline{\varepsilon} = \varepsilon_{ij}\, \underline{e}_i \otimes \underline{e}_j$$

How about using $\underline{\nu}^{(1)}, \underline{\nu}^{(2)}, \underline{\nu}^{(3)}$ as basis vector?

$$\underset{\sim}{\varepsilon} \overset{\Delta}{=} \sum_{i,j} \hat{\varepsilon}_{ij}\, \underline{\nu}^{(i)} \otimes \underline{\nu}^{(j)}$$

ith principal strain

$$\Rightarrow \boxed{\underline{\varepsilon} \doteq \sum_{i=1}^{3} \varepsilon^{(i)}\, \underline{\nu}^{(i)} \otimes \underline{\nu}^{(i)}} \quad (*)$$

"spectral decomposition"

proof of (*)

① $\underline{\varepsilon} \cdot \underline{\nu}^{(i)} = \varepsilon^{(i)}\, \underline{\nu}^{(i)}$, or $\varepsilon_{kl}\nu_l^{(i)} = \varepsilon^{(i)} \nu_k^{(i)}$

(no sum on $(i)$)

② orthogonality : $\nu^{(i)} \cdot \underline{\nu}^{(j)} = \delta^{(ij)}$

$$\nu_k^{(i)} \nu_k^{(j)} = \delta^{(ij)}$$

③ coordinate transform

$$\underline{e}_i \longleftrightarrow \underline{e}_{i'}$$

$$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} \qquad \{\underline{e}_{1'}, \underline{e}_{2'}, \underline{e}_{3'}\}$$

$$\qquad\qquad\qquad\qquad \underline{\nu}^{(1')}, \underline{\nu}^{(2')}, \underline{\nu}^{(3')}$$

$$\left.\begin{aligned} \underline{e}_{i'} &\stackrel{\Delta}{=} \beta_{ji'}\,\underline{e}_j \\ \underline{\nu}^{(i')} &= \nu^{(i')}_j\,\underline{e}_j \end{aligned}\right\} \quad \underline{\beta_{ji'} = \nu^{(i')}_j} \quad (*)$$

Then

$$\varepsilon_{ij}\ (\text{in } \underline{e}_i) \longrightarrow \varepsilon_{i'j'}\ (\text{in } \underline{e}_{i'} = \underline{\nu}^{(i')})$$

$$\varepsilon_{i'j'} = \beta_{i'k}\,\beta_{j'l}\,\varepsilon_{kl}$$

In orthonormal coord (e.g, Cartesian)

$$\beta_{i'k} = \beta_{ki'} \quad (\text{see prob 1.1.1})$$

Thus

$$\varepsilon_{i'j'} = \beta_{ki'}\,\beta_{lj'}\,\varepsilon_{kl} \underset{(*)}{=} \nu^{(i')}_k\,\nu^{(j')}_l\,\varepsilon_{kl}$$

$$= \nu_k^{(i')} \underbrace{\varepsilon_{kl}\, \nu_l^{(j')}}_{\xrightarrow{\;①\;} \varepsilon^{(j')} \nu_k^{(j')}}$$

(no sum on $j'$)

$$= \varepsilon^{(j')} \underbrace{\nu_k^{(j')} \nu_k^{(i')}}_{\xrightarrow{\;②\;} \delta^{(i'j')}} \quad \text{(no sum on } j'\text{)}$$

$$= \delta^{(i'j')} \varepsilon^{(j')} \quad \text{(no sum on } j'\text{)}$$

Thus

$$\underset{\sim}{\varepsilon} = \varepsilon_{i'j'}\, \underline{e}_{i'} \otimes \underline{e}_{j'}$$

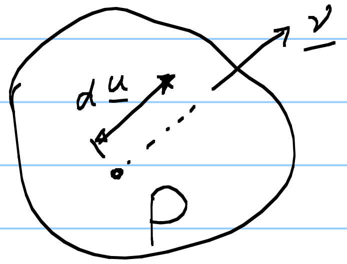
$$= \sum_{i',j'} \delta^{(i'j')} \varepsilon^{(j')}\, \underline{\nu}^{(i')} \otimes \underline{\nu}^{(j')}$$

$$= \sum_{j'} \varepsilon^{(j')}\, \underline{\nu}^{(j')} \otimes \underline{\nu}^{(j')}$$

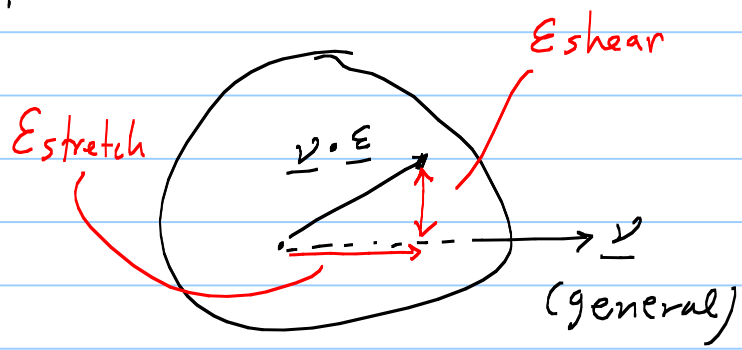
$$= \sum_{i} \varepsilon^{(i)}\, \underline{\nu}^{(i)} \otimes \underline{\nu}^{(j)}$$

$\square$

# Deformation along general direction $\underline{\nu}$
(not the principal direction)




if $\underline{\nu}$: principal

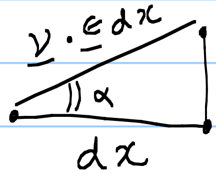



$$d\underline{u} = \underline{\nu}\cdot\nabla\underline{u}$$
$$= \underline{\nu}\cdot(\underline{\varepsilon} + \underbrace{\underline{\omega}}_{\text{Rigid-Body rotation}})\,dx$$

$$\Rightarrow \boxed{\underline{\nu}\cdot\underline{\varepsilon} : \text{deformation measure along } \nu \text{ excluding Rigid-Body Rotation}}$$

$$\varepsilon_{stretch} = \underline{\nu}\cdot\underline{\varepsilon}\cdot\underline{\nu}$$
$$\varepsilon_{shear}^2 = (\underline{\nu}\cdot\underline{\varepsilon})^2 - \varepsilon_{stretch}^2$$

( $\tan\alpha \approx \alpha$ (for small $\alpha$)




$\leftarrow \varepsilon_{shear}\,dx$ ; $\alpha \approx \dfrac{\varepsilon_{shear}\cdot \cancel{dx}}{\cancel{dx}}$ )

One can show:

$$\begin{cases} \varepsilon_{stretch}\big|_{max} = \text{Largest of } (\epsilon^{(1)}, \epsilon^{(2)}, \epsilon^{(3)}) \\ \varepsilon_{stretch}\big|_{min} = \text{smallest of } (\epsilon^{(1)}, \epsilon^{(2)}, \epsilon^{(3)}) \end{cases}$$

$$\varepsilon_{shear}\big|_{max} = \max \frac{1}{2}\left\{ \left|\varepsilon^{(1)} - \varepsilon^{(2)}\right|, \left|\varepsilon^{(2)} - \varepsilon^{(3)}\right|, \left|\varepsilon^{(3)} - \varepsilon^{(1)}\right| \right\}$$

<u>Back to Ex 1.2.2</u>

$$\underset{\sim}{\varepsilon} = \left[ \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3 + \sqrt{3}\left(\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2 + \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_3\right)\right] \times 10^{-3}$$

problem c). $\nu = \frac{1}{\sqrt{2}} (\underset{\sim}{e}_1 + \underset{\sim}{e}_2)$

$$\varepsilon_{stretch} = \underset{\sim}{\nu} \cdot \underset{\sim}{\epsilon} \cdot \underset{\sim}{\nu} = \frac{1}{2}(1 + 2\sqrt{3}) \times 10^{-3}$$

$$\varepsilon_{shear} = \left( (\underset{\sim}{\nu} \cdot \underset{\sim}{\epsilon})^2 - \epsilon_{strech}^2 \right)^{\frac{1}{2}}$$

$$= \frac{1}{2} \times 10^{-3}$$

(Rotation angle)

(problem c) $\varepsilon_{shear}\big|_{max} = \frac{1}{2}(\epsilon^{(2)} - \epsilon^{(3)}) = \frac{\sqrt{13}}{2} \times 10^{-3}$

# Compability of Strain

(Read p56–61 for derivation)

3 Displacement $\longrightarrow$ 6 Strains

$(u_1, u_2, u_3)$ $\qquad$ $(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{23}, \epsilon_{31})$

$$\begin{cases} \varepsilon_{33,22} + \varepsilon_{22,33} - 2\varepsilon_{23,23} = 0 \\ \varepsilon_{11,33} + \varepsilon_{33,11} - 2\varepsilon_{31,31} = 0 \\ \varepsilon_{22,11} + \varepsilon_{11,22} - 2\varepsilon_{12,12} = 0 \end{cases}$$

$$\begin{cases} -\epsilon_{11,23} + (-\varepsilon_{23,1} + \varepsilon_{31,2} + \varepsilon_{12,3})_{,1} = 0 \\ -\varepsilon_{22,31} + (-\varepsilon_{31,2} + \varepsilon_{12,3} + \epsilon_{23,1})_{,2} = 0 \\ -\epsilon_{33,12} + (-\varepsilon_{12,3} + \varepsilon_{23,1} + \epsilon_{31,2})_{,3} = 0 \end{cases}$$

May be written as

$$\begin{cases} \nabla \times (\underline{\nabla} \times \underline{\varepsilon})^T = 0 \\ \text{or} \quad e_{ilk}\, e_{jmp}\, \varepsilon_{lm,kp} = 0 \end{cases}$$