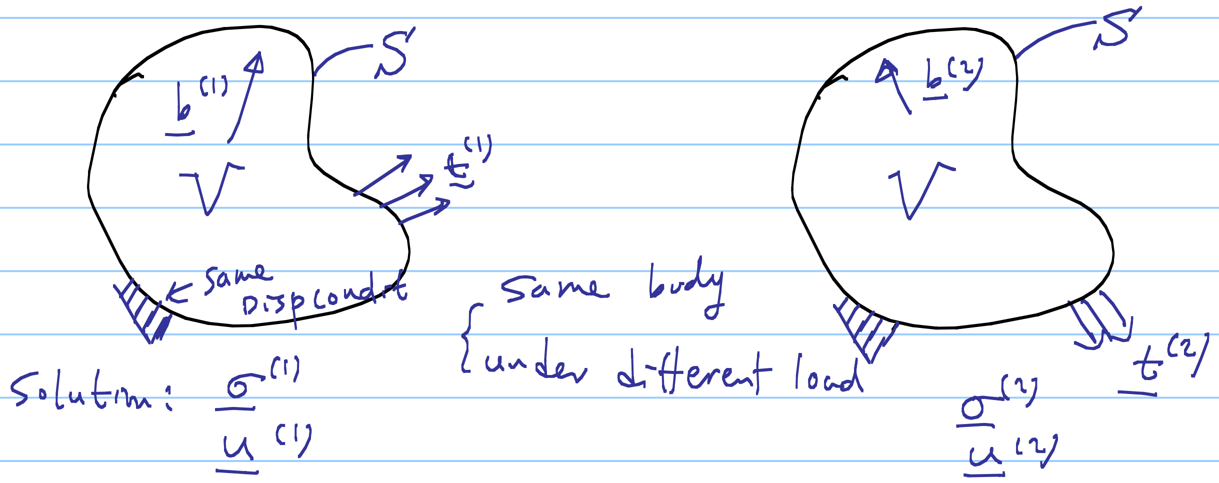
Principles Useful in Elasticity

① superposition principle ← related
② uniqueness
③ principle of virtual work
④ principle of minimum potential energy — related

< Superposition principle >



$\underline{\sigma}^{(\alpha)}, \underline{u}^{(\alpha)}$ $(\alpha = 1, 2)$ must satisfy

$$\underline{\nabla} \cdot \underline{\sigma}^{(\alpha)} + \underline{b}^{(\alpha)} = 0$$

$$\underline{\sigma}^{(\alpha)} = \underline{C} : \underline{\varepsilon}^{(\alpha)}$$

$$\underline{\varepsilon}^{\alpha} = [\, \underline{\nabla}\, \underline{u}^{(\alpha)} + (\underline{\nabla}\, \underline{u}^{(\alpha)})^T] / 2$$

$$\underline{t}^{(\alpha)} = \underline{n} \cdot \underline{\sigma}^{(\alpha)} \quad \text{on } S$$

Define $\underset{\sim}{\sigma}' = \underline{\sigma}^{(1)} + \underline{\sigma}^{(2)}$ (superposed solution)

$\underline{u}' = \underline{u}^{(1)} + \underline{u}^{(2)}$

Then $\underline{\sigma}'$ and $\underline{u}'$ must satisfy

$$\underline{\nabla} \cdot \underline{\sigma}' + (\underline{b}^{(1)} + \underline{b}^{(2)}) = 0$$

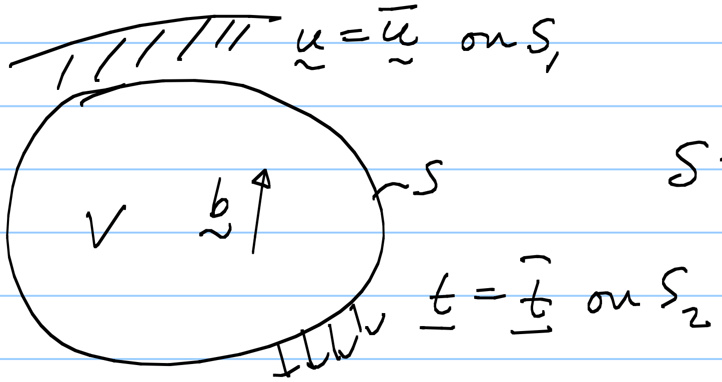
$$\underline{\sigma}' = \underline{C} : \underline{\varepsilon}' = \underline{C} : (\underline{\epsilon}^{(1)} + \underline{\epsilon}^{(2)})$$

$$\underline{t}' = \underline{n} \cdot (\underline{\sigma}^{(1)} + \underline{\sigma}^{(2)}) \quad \text{on } S$$

Therefore, $\underline{\sigma}'$ and $\underline{u}'$: solution satisfying the body force $\underline{b}^{(1)} + \underline{b}^{(2)}$ and traction boundary condition.

<Uniquness>

Question: For what types of boundary conditions, is the solution unique?



$S = S_1 \cup S_2$

Assume two solutions $(\underline{u}^{(1)}, \underline{\sigma}^{(1)})$ and $(\underline{u}^{(2)}, \underline{\sigma}^{(2)})$ satisfy the same load/boundary conditions:

Then,
$$\underline{\nabla} \cdot \underline{\sigma}^{(\alpha)} + \underline{b} = 0$$
$$\underline{\sigma}^{(\alpha)} = \underline{C} : \underline{\varepsilon}^{(\alpha)}$$
$$\underline{\varepsilon}^{(\alpha)} = [\underline{\nabla} \underline{u}^{(\alpha)} + (\underline{\nabla} \underline{u}^{(\alpha)})^T]/2$$
$$\underline{u}^{(\alpha)} = \bar{\underline{u}} \text{ on } S_1, \quad \underline{t}^{(\alpha)} = \bar{\underline{t}} \text{ on } S_2$$
$$(\alpha = 1, 2)$$

Form $\underline{u}' \triangleq \underline{u}^{(1)} - \underline{u}^{(2)}$

$\underline{\sigma}' \triangleq \underline{\sigma}^{(1)} - \underline{\sigma}^{(2)}$

Then $(\underline{u}', \underline{\sigma}')$ satisfies

$$\underline{\nabla} \cdot \underline{\sigma}' = 0$$

$$\underline{\sigma}' = \underline{C} : \underline{\varepsilon}'$$

$$\underline{\varepsilon}' = [\nabla \underline{u}' + (\nabla \underline{u}')^T]/2$$

$$\boxed{\underline{u}' = 0 \text{ on } S_1, \quad \underline{t}' = 0 \text{ on } S_2} \quad (*)$$

From $*$),

$$0 = \int_{S_1} \underline{t}' \cdot \underbrace{\underline{u}'}_{0} dS + \int_{S_2} \underbrace{\underline{t}'}_{0} \cdot \underline{u}' \, dS$$

$$= \int_S \underline{t}' \cdot \underline{u}' \, dS$$

$$= \int_S \underline{n} \cdot [\underline{\sigma}' \cdot \underline{u}'] \, dS$$

$$= \int_V \underline{\nabla} \cdot (\underline{\sigma}' \cdot \underline{u}')\, dV$$

$$= \int_V \left[(\underbrace{\underline{\nabla} \cdot \underline{\sigma}'}_{\equiv 0}) \cdot \underline{u}' + \underline{\sigma}' : \nabla \underline{u}' \right] dV$$

$$= \int_V \underline{\sigma}' : \underset{\sim}{\varepsilon}' \, dV = 2\, \underset{\text{Strain Energy}}{U}$$

$$= \int_V \underline{\varepsilon}' : \underline{C} : \underline{\varepsilon}' \, dV \qquad (**)$$

Because $\underline{\varepsilon} : C : \underline{\varepsilon} \geq 0$ ($= 0$ hold only for $\underline{\varepsilon} = 0$)

$(**)$ gives

$$\underset{\sim}{\varepsilon}' = 0 \iff \underset{\sim}{\varepsilon}_1 = \underline{\varepsilon}_2$$

If $S = S_2$ only, the solution is unique except rigid-body displacement.

Otherwise, everything including displacement $=$ unique!

Thus, unique solution is obtained if

$$\int_S \underline{t}' \cdot \underline{u}' \, ds = 0$$

i.e

on S_1 ; $u_i = \bar{u}_i$ $(i = 1, 2, 3)$

on S_2 ; $t_i = \bar{t}_i$ $(i = 1, 2, 3)$

Generalization:

- Mixed Boundary Condition

$(t_1 = \bar{t}_1, \; t_2 = \bar{t}_2, \; u_3 = \bar{u}_3)$

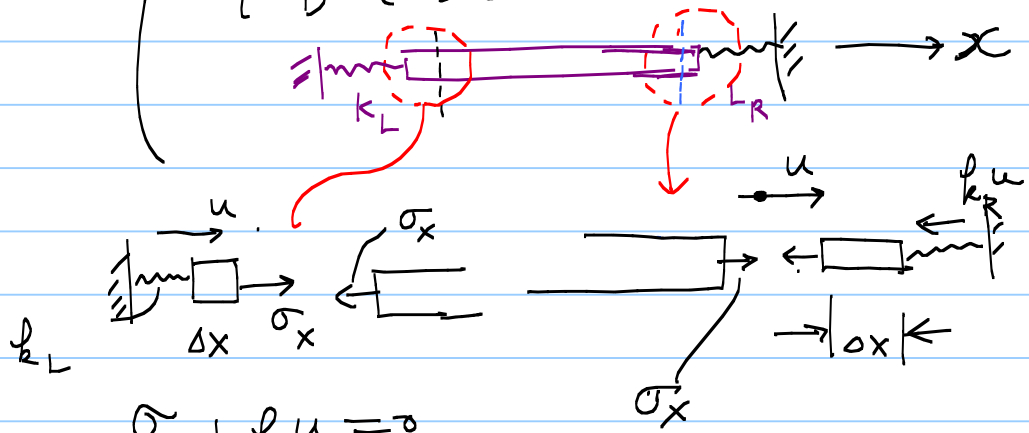
or $(t_1 = \bar{t}_1, \; u_2 = \bar{u}_2, \; u_3 = \bar{u}_3)$

or $(t_1 = \bar{t}_1, \; u_2 = \bar{u}_2, \; t_3 = \bar{t}_3)$ etc

- Elastically supported Boundary Condition

$$t_i + k_{ij} u_j = \bar{t}_i \quad (i = 1, 2, 3)$$

(1-D case



$$\sigma_x + k_L u = 0$$

$$\sigma_x + k_R u = 0$$)

Remark: energy (or work) conjugate cannot be prescribed simultaneously.

* unpermissible BC's

$$t_1 = \bar{t}_1, \quad u_1 = \bar{u}_1, \quad u_3 = \bar{u}_3$$

<Principle of Virtual Work>

$\subset$ "Variational principle"

"Strong form"

$$(S)\left[\begin{array}{ll}\nabla\cdot\underline{\sigma}+b=0 \text{ in } V & \text{— (a)}\\ u=\bar{u} \text{ on } S_1, \quad t=\bar{t} \text{ on } S_2 & \text{— (b)}\end{array}\right.$$

$$\Updownarrow$$

"Weak form"

$$(W) \quad \int_V \underline{\epsilon} : C : \epsilon_v \, dV = \int_V \underline{b}\cdot\underline{v}\, dV + \int_{S_2} \bar{t}\cdot v\, dS$$

where
$$\begin{cases}\epsilon = \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + (\underline{\nabla}\,\underline{u})^T\right) & \text{— (c)}\\ \underline{\sigma} = C : \underline{\varepsilon} & \text{— (d)}\end{cases}$$

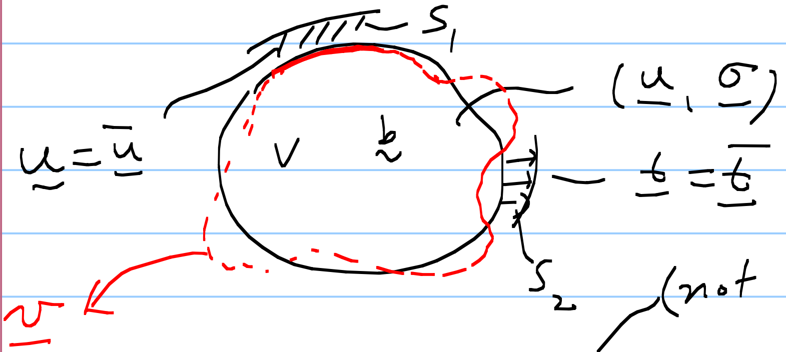
and
$$\epsilon_v = \frac{1}{2}\left(\underline{\nabla}\,\underline{v} + (\nabla\underline{v})^T\right)$$

$$(\underline{\sigma}_v = \underline{C} : \underline{\epsilon}_v)$$

$$\oplus \quad v = 0 \text{ on } S_1$$

To convert (S) → (W),

i) Start with a body in equilibrium



(not real)

ii) Consider "virtual" displacement $\underset{\sim}{v}$ added to the real displacement $\underset{\sim}{u}$ such that

$$\underset{\sim}{u} + \underset{\sim}{v} = \bar{\underset{\sim}{u}} \quad \text{on } S_1$$

i.e. $\underset{\sim}{v} = 0$

iii) $\displaystyle\int_V \text{Eq (a)} \cdot \underset{\sim}{v}$

$$\Rightarrow \int_V \underset{\sim}{v} \cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma})\, dV + \int_V \underset{\sim}{v} \cdot \underset{\sim}{b}\, dV = 0 \quad (*)$$

Note $\int \underline{v} \cdot (\underline{\nabla} \cdot \underline{\sigma})\, dV$

$$= \int_V \underline{\nabla} \cdot (\underline{\sigma} \cdot \underline{v})\, dV - \int_V \underline{\sigma} : \nabla \underline{v}\, dV$$

$\int_{S_1} + \int_{S_2} \Longrightarrow$ ($\because \underline{v} = 0$ on S_1)

$$= \int_{S_2} \underline{n} \cdot \underline{\sigma} \cdot \underline{v}\, dS - \int_V \underline{\sigma} : \varepsilon_v\, dV$$

$\uparrow$ Sym part of $\underline{\nabla}\underline{v}$

$$= \int_{S_2} \underline{\bar{t}} \cdot \underline{v}\, dS - \int_V \underline{\sigma} : \varepsilon_v\, dV$$

Thus $(*)$ becomes,

$$\int_V \underline{\sigma} : \varepsilon_v\, dV = \int_V \underline{b} \cdot v\, dV + \int_{S_2} \underline{\bar{t}} \cdot \underline{v}\, dS$$

or

$$\boxed{\int_V \underbrace{\underline{\epsilon} : \underline{C}}_{\underline{\sigma}} : \varepsilon_v\, dV = \int_V \underline{b} \cdot \underline{v}\, dV + \int_{S_2} \underline{\bar{t}} \cdot \underline{v}\, dS}$$

principle of Virtual work

LHS = the virtual elastic energy
due to $\underset{\sim}{\sigma}$ by virtual strain $\underset{\sim}{\varepsilon}_v$
$\uparrow$
(True stress)

RHS = the virtual work by $\underline{b}$ and $\underline{\bar{t}}$
by virtual displacement $\underline{u}$

☑ Remark 1:

Strong form? $\uparrow$ 2nd-order differentiability required for $\underline{u}$

Weak form? $\uparrow$ 1st-order differentiability required for $\underline{u}$, $\underline{v}$

☑ Remark 2:

By Remark 2, weak form is better for solution approximation.

(S) $\to$ trial ftn $\underline{u}$ must be 2 times differentiable

(W) $\to$ only 1 time differentiable!

< Principle of Minimum Potential Energy >

Let $\underline{v} = \delta \underline{u}$

$\underbrace{\delta \underline{u}}$ "infinitesimal variation" of true displacement $\underline{u}$

Then the principle of virtual work becomes

$$\int_V \underline{\sigma} : \delta \underline{\varepsilon} \, dV = \int_V \underline{b} \cdot \delta \underline{u} \, dV + \int_{S_1} \underline{t} \cdot \delta \underline{u} \, dS$$

where

$$\delta \underset{\sim}{\epsilon} = \delta \cdot \frac{1}{2} \left[\underline{\nabla} \underline{u} + (\underline{\nabla} \underline{u})^T \right]$$

$$= \frac{1}{2} \left[\underline{\nabla} \delta \underline{u} + (\underline{\nabla} \delta \underline{u})^T \right]$$

Using $\underline{\sigma} = \underline{C} : \underline{\varepsilon}$

$$\int \underset{\sim}{\varepsilon}(\underline{u}) : \underline{C} : \delta \underline{\epsilon}(\underline{u})\, dV = \int_V \underline{b} \cdot \delta \underline{u}\, dV + \int_{S_2} \underline{\bar{t}} \cdot \delta \underline{u}\, dS \quad (*)$$

* physical Interpretation :

as $\underline{u} \to \underline{u} + \delta \underline{u}$,

$$\underline{\epsilon}(\underline{u}) \to \underline{\epsilon}(\underline{u} + \delta \underline{u}) = \underline{\epsilon}(\underline{u}) + \delta \underline{\epsilon}(u) + \cdots$$

$$\underline{\sigma}(\underline{u}) \to \underline{\sigma}(\underline{u} + \delta \underline{u}) = \underline{\sigma}(\underline{u}) + \underbrace{\delta \underline{\sigma}(\underline{u})}_{\text{up to 1st order}} + \cdots$$

$\Rightarrow$ Meaning $\delta \underline{\epsilon}(\underline{u})$ in $*)$ is the first variation of $\underline{\epsilon}(\underline{u})$

Thus, $*)$ can be written as

$$\delta \left[\int_V \frac{1}{2} \underline{\varepsilon}(\underline{u}) : \underline{C} : \underline{\varepsilon}(\underline{u})\, dV - \int_V \underline{b} \cdot \underline{u}\, dV - \int_{S_2} \underline{\bar{t}} \cdot \underline{u}\, dS \right] = 0$$

Let $$U_{system} = \frac{1}{2}\int_V \underline{\varepsilon}(\underline{u}) : \underset{\sim}{C} : \underline{\epsilon}(\underline{u})\, dV$$

$$= \frac{1}{2}\int_V \underline{\sigma}(\underline{u}) : \underline{\varepsilon}(\underline{u})\, dV$$

(elastic energy stored in a system)

$$U_{external} = -W_{external}$$

Work done by external force

$$= -\int_V \underline{b}\cdot\underline{u}\, dV - \int_{S_2} \underline{\bar{t}}\cdot\underline{u}\, ds$$

(Potential energy due to external force)

Then

$$\delta U_{system} + \delta U_{external} = 0$$

Define $$\Pi = U_{system} + U_{external}$$

↑ Total potential energy

Then $$\boxed{\delta \Pi (u) = 0}$$

↑ First Variation of $\Pi = 0$

If we assume

$$u_q(\underline{x}) = a_q^0 h_q^0(\underline{x}) + \sum_{k=1}^{N} a_q^k h_q^k(x) \qquad (*)$$

$(q = 1, 2, 3)$
(no sum on q)

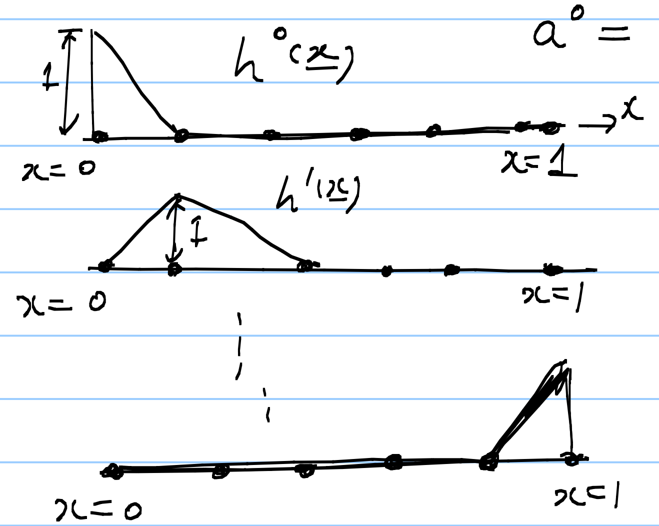
such that $\left\{ \begin{array}{l} \bar{\underset{\sim}{u}}^0(\underline{x}) = a_q^0 h_q^0(x) \\ \underset{\sim}{h}^k(x) = 0 \end{array} \right\}$ on $S = S_1$

((h_q^0, $h_q^k(x)$: known ftns

example:

$a^0 = \bar{u}(x=0)$

$\underset{\sim}{x} = x \, \underline{e}_x$



))

Substituting (*) into $\delta\Pi = 0$ yields

known coeff.

$$\delta\Pi\left(a_q^0 h_q^0(x) + \sum_{k=1}^{N} a_q^k h_q^k(x) \right) = 0$$

$$\Rightarrow \quad \frac{\partial \Pi}{\partial a_q^k} = 0 \quad \text{for } q = 1, 2, 3, \quad k = 1, \cdots, N$$

(3N unknowns, 3N equations)

This approximate method using $\delta\Pi = 0$ is called the Rayleigh-Ritz method.

(*Requirement on the trial function is considerably more relaxed than in the direct solution of strong form)

< Summary of Field Equation for Isotropic elastic bodies >

① Eqm Eq (# = 3)

$$\nabla \cdot \underset{\sim}{\sigma} + \underset{\sim}{b} = 0$$

$$\sigma_{ji,j} + b_i = 0$$

② Constitutive equation

$$\underset{\sim}{\sigma} = 2\mu \underset{\sim}{\varepsilon} + \lambda \, \mathrm{tr}\underset{\sim}{\varepsilon} \, \mathbb{1}$$

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij}$$

③ Strain-Displacement relation

$$\underset{\sim}{\varepsilon} = \frac{1}{2}\left[\nabla \underset{\sim}{u} + (\nabla \underset{\sim}{u})^T\right]$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j})$$

③ → ② → ①,

Eqm in terms of $\underset{\sim}{u}$ (Displacement)

$$(\lambda+\mu)\, \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) + \mu \nabla^2 \underset{\sim}{u} + \underset{\sim}{b} = 0$$

$$\lambda+\mu\; u_{k,ik} + \mu\, u_{i,kk} + b_i = 0$$

called Navier equations

④ Compability equation on $\underline{\varepsilon}$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{\varepsilon})^T = 0$$

$$e_{ilk}\, e_{pmj}\, \varepsilon_{ml,kp} = 0$$

Because $\underline{\sigma} = C : \underline{\varepsilon}$, the compatability can be written in terms of $\underline{\sigma}$:

$$\nabla^2 \underset{\sim}{\sigma} + \frac{1}{1+\nu} \underline{\nabla}\, \underline{\nabla} (\operatorname{tr} \underline{\sigma})$$
$$= -\frac{\nu}{1-\nu} (\underline{\nabla} \cdot \underset{\sim}{b})\, \underset{\sim}{\mathbb{1}} - \underline{\nabla}\, \underline{b} - (\underline{\nabla}\, \underline{b})^T$$

$$\left(\sigma_{ij,kk} + \frac{1}{1+\nu} \sigma_{ll,ij} = -\frac{\nu}{1-\nu} \delta_{ij}\, b_{k,k} - b_{i,j} - b_{j,i} \right)$$

$\Rightarrow$ called Beltrami–Michell equation

* Energy expression

$$U_{system} = \frac{1}{2}\int_V \underset{\sim}{\epsilon} : \underset{\sim}{C} : \underset{\sim}{\epsilon}\, dV$$

$$= \frac{1}{2}\int_V \left[2\mu\, \underset{\sim}{\varepsilon} : \underset{\sim}{\varepsilon} + \lambda (tr\, \underset{\sim}{\varepsilon})^2\right] dV$$

$$= \frac{1}{2}\int_V \Big[2\mu\, (\epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2 + 2\epsilon_{12}^2 + 2\epsilon_{23}^2 + 2\epsilon_{31}^2) + \lambda (\epsilon_{11} + \epsilon_{22} + \epsilon_{33})^2\Big] dV$$

In terms of E, ν, μ

$$U_{system} = \frac{1}{2}\int_V \Big[\frac{E\nu}{(1+\nu)(1-\nu)}(\epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2) + 2\mu(\epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2 + \epsilon_{12}^2 + \epsilon_{23}^2 + \epsilon_{13}^2)\Big] dV$$