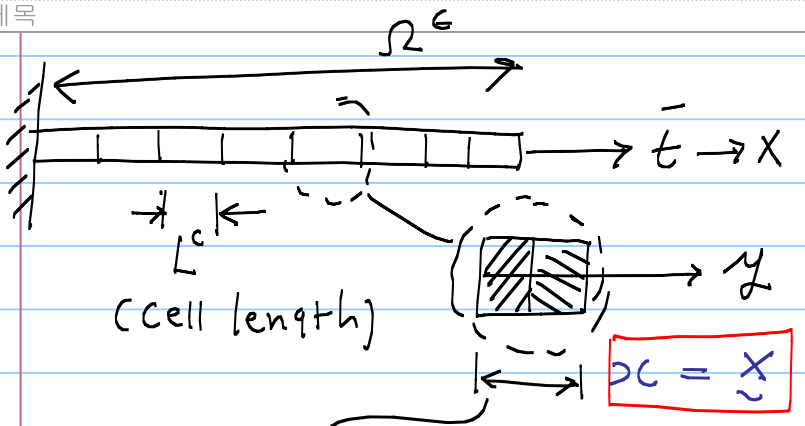
Homogenization: 1-D case



$\hat{\epsilon} = \dfrac{\text{Cell length}}{\text{Bar length}}$

$\boxed{x = \underset{\sim}{X}}$ (macroscopic coord.) (a)

$Y \in [Y_L, Y_R] = [0, l]$

$\boxed{y = X/\epsilon}$ (microscopic coord.) (b)

☑ Find Homogenized equations (as $\epsilon \to 0$) for periodic unit cells

⟨Field Equations⟩ ⟨Boundary Conditions⟩

distributed load

$$\begin{cases} \dfrac{\partial \sigma^\varepsilon}{\partial X} + b = 0 & (1) \\ \sigma_\varepsilon = E^\varepsilon \dfrac{\partial u^\varepsilon}{\partial X} & (2) \end{cases}$$

$$\begin{cases} u^\epsilon(X=0) = 0 & (d1) \\ \sigma^\varepsilon(X=L) = \bar{t} & (d2) \end{cases}$$

Easier to work with weak form:

$$\boxed{\int_{\Omega^\varepsilon} E \frac{\partial u^\varepsilon}{\partial X} \frac{\partial v}{\partial X} \, dX = \int_{\Omega} b v \, dx + \bar{t} v(L) \quad (e)}$$

(v : virtual disp with $v(x=0) = 0$)

To go from X to (x, y), we need

i) $dX = dx = \epsilon\, dy$ (f)

ii) Length (or volume) of unit cell

$$L_X^c = L_x^c = \epsilon L_y^c \qquad (g)$$

iii) $$\frac{d\phi}{dX} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial X}$$

(ϕ : some ftn) $$= \frac{\partial \phi}{\partial x} + \frac{1}{\epsilon}\frac{\partial \phi}{\partial y} \qquad (f)$$

iv) $$\lim_{\epsilon \to 0} \int_{\Omega^\epsilon} \phi(X)\, dX = \lim_{\epsilon \to 0} \int_{\Omega^\epsilon} \phi(x, y)\, dX$$

$$= \lim_{\epsilon \to 0} \int_{\Omega} \left[\text{volume average of } \phi(x,y) \text{ over } Y\right] dX$$

$$= \lim_{\epsilon \to 0} \int_{\Omega} \underbrace{\left[\frac{1}{L_y^c} \int_Y \phi(x, y)\, dy\right]}_{\text{ftn of } \underset{\sim}{x} \text{ only}} dx \qquad (h)$$

With (f–h), go back to the weak-form of the govern. eq:

Let
$$\begin{cases} u^\varepsilon(X) = u^\varepsilon(x,y) \\ \qquad\quad = u^0(x,y) + \epsilon u^1(x,y) + \epsilon^2 u^2(x,y) + \cdots \\ \sigma^\varepsilon(X) = \sigma^0(x,y) + \epsilon\cdot\sigma^1(x,y) + \cdots \end{cases} \qquad (1)$$

- $v(X) = v(\underline{x},\underline{y}) \leftarrow$ no expansion on ε needed

$$\begin{cases} v \in H^1(\Omega^\epsilon) \text{ with } v\big|_{X=0} = 0, & (\because v: \text{arbitrary}) \\ u \in H^1(\Omega^\epsilon) \text{ with } u\big|_{X=0} = 0 \end{cases}$$

$\left(H^1 = \{ w \mid w \in L_2,\ w_{,x} \in L_2 \},\ L_2 = \{ w \mid \int_0^1 w^2 dx < \infty \} \right)$

▫ (1), (f) $\Rightarrow$ (e) :

$$\int_{\Omega^\epsilon} E\left[\frac{\partial}{\partial x}(u^0 + \epsilon u^1 + \epsilon^2 u^2 + \cdots) + \frac{1}{\epsilon}\frac{\partial}{\partial y}(u^0 + \epsilon u^1 + \epsilon^2 u^2 + \cdots) \right] \left[\frac{\partial v}{\partial x} + \frac{1}{\epsilon}\frac{\partial v}{\partial y} \right] dX$$

$$= \int_{\Omega^\varepsilon} b(X)\, v\, dX + \bar{t}\, v(L) \qquad (2)$$

$$\rightarrow \int_{\Omega^\epsilon} E \Big\{ \frac{1}{\epsilon^2} \frac{\partial u^0}{\partial y}\frac{\partial v}{\partial y}$$

$$+ \frac{1}{\epsilon}\left[\left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y}\right)\frac{\partial v}{\partial y} + \frac{\partial u^0}{\partial y}\frac{\partial v}{\partial x}\right]$$

$$+ \left[\left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y}\right)\frac{\partial v}{\partial x} + \left(\frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y}\right)\frac{\partial v}{\partial y}\right]$$

$$+ \epsilon(\cdots)\Big\} dX$$

$$= \int b(X)\, v\, dX + \bar{t}\, v(L) \qquad (3)$$

Integrate term by term of (3)
by using (4) as $\epsilon \to 0$

$$\frac{1}{\epsilon^2}\int_\Omega \frac{1}{L_y^c}\left[\int_Y E \frac{\partial u^0}{\partial y}\frac{\partial v}{\partial y}\, dy\right] dx$$

$$+ \frac{1}{\epsilon}\int_\Omega \frac{1}{L_y^c}\left\{\int_Y E \left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y}\right)\frac{\partial v}{\partial y} + \frac{\partial u^0}{\partial y}\frac{\partial v}{\partial x}\Big]\, dy\right\} dx$$

$$+\int_{\Omega} \frac{1}{L_y^c} \left\{ \int_Y E \left[\left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y} \right) \frac{\partial v}{\partial x} + \left(\frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y} \right) \frac{\partial v}{\partial y} \right] dy \right\} dx$$

$$+ O(\epsilon)$$

$$= \int_{\Omega} \left(\frac{1}{L_y^c} \int_Y \underbrace{b}_{b(x,y)} \, v(x,y) \, dy \right) dx + \bar{t} v(L) \qquad (4)$$

Next Step: Equate terms of the same order of ϵ

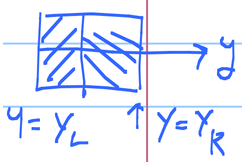
[1] $O\left(\frac{1}{\epsilon^2}\right)$ term

$$\int_{\Omega} \left[\frac{1}{L_y^c} \int_Y E \frac{\partial u^0}{\partial y} \frac{\partial v}{\partial y} dy \right] dx = 0 \qquad (5)$$

※ Because v is arbitrary, let us convert [ ] into something like $\int (\ ) v \, dy$

By Integration-by-Part (or Divergence)

$$\int_\Omega \frac{1}{L_{yc}} \left\{ \left[E\frac{\partial u^0}{\partial y} v \right]_{Y_L=0}^{Y_R=1} - \int_Y \frac{\partial}{\partial y}\left(E\frac{\partial u^0}{\partial y}\right) v \, dy \right\} dx = 0 \qquad (6)$$



Because $v(x,y)$ is arbitrary,

$$-\frac{\partial}{\partial y}\left(E\frac{\partial u^0}{\partial y}\right) = 0 \quad \text{in } Y$$

$$\sigma \longrightarrow E\frac{\partial u^0}{\partial y} \equiv 0 \quad \text{at} \begin{cases} y = y_L = 0 \\ y = y_R = 1 \end{cases}$$

meaning: $u^0(x,y)$ = indep of y

i.e., $u^0(x,y)$ does not vary as a ftn of y

$$\Longrightarrow \boxed{u^0(x,y) = u^0(x)} \qquad (8)$$

ftn of x only!

[2] $O\left(\frac{1}{\epsilon}\right)$ term:

$$\int_{\Omega} \frac{1}{L_y^c} \left[\int_Y E \left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y} \right) \frac{\partial v}{\partial y} \, dy \right] dx = 0 \quad (9)$$

if $v(x,y) = v(y)$ (and $v(y_L) = v(y_R)$)

i.e.:

$$(W) \quad \boxed{\int_Y E \left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y} \right) \frac{dv}{dy} \, dy = 0} \quad (10)$$

- For simplicity, assume

$$E(x) = E(x, y) \equiv E(y)$$

periodic ($E(y_L) = E(y_R)$)

- $\frac{\partial u^0}{\partial x} = \frac{du^0(x)}{dx}$,

$$\hookrightarrow \int_Y E(y) \left[\frac{du^0(x)}{dx} + \frac{\partial u^1(x,y)}{\partial y} \right] \frac{dv}{dy} \, dy = 0 \quad (10)'$$

Meaning: u^1 must be in $F(x)\,G(y)$ form!

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Thus

Let $\boxed{u'(x,y) = -\frac{du^0(x)}{dx}\chi(y)}$ (11)

Then (10)' becomes

$$\frac{du^0(x)}{dx}\int_Y E(y)\left[1 - \frac{d\chi(y)}{dy}\right]\frac{dv}{dy}\,dy = 0$$

i.e. :

$$\boxed{\overbrace{\boxed{\int_Y E\frac{d\chi}{dy}\frac{dv}{dy}\,dy = \int_Y E\frac{dv}{dy}\,dy}} \quad \oplus \quad \chi\big|_{y_L} = \chi\big|_{y_R}, \quad \frac{d\chi}{dy}\Big|_{y_L} = \frac{d\chi}{dy}\Big|_{y_R}, \; - - -} \quad (12)$$

Microscopic equilibrium

For given E, χ is determined except a rigid-Body displacement (may set $\chi(y_L) = \chi(y_R) = 0$)

Q : what does Eq. (12) imply?

→ Check its strong form

Integrate (12) by part (or Divergence)

$$\left[\left(E\frac{dx}{dy}-E\right)v\right]_{Y_L}^{Y_R}$$

$$-\int_Y\left[\frac{d}{dy}\left(E\frac{dx}{dy}\right)-\frac{dE}{dy}\right]v\,dy=0 \qquad (*)$$

For (*) to hold ; ($v(y)$ = arbitrary, $v(Y_R)=v(y_L)$)

$$\begin{cases}\left(E\frac{dx}{dy}-E\right)_{Y_R}v\Big|_{Y_R}-\left(E\frac{dx}{dy}-E\right)_{Y_L}v\Big|_{Y_L}=0 & \text{(a)}\\[2ex] \frac{d}{dy}\left(E\frac{dx}{dy}\right)=\frac{dE}{dy}\quad \text{in } Y & \text{(b)}\end{cases}$$

Since $v\big|_{Y_R}=v\big|_{Y_L}$,

$$\text{(a)}\rightarrow\left(E\frac{dx}{dy}-E\right)_{Y_R}=\left(E\frac{dx}{dy}-E\right)_{Y_L}\qquad \text{(a)}'$$

and

$$E(y=y_L) = E(y=y_R)$$

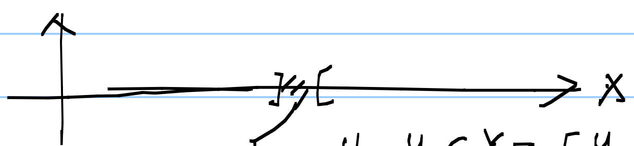
$$(a)' \rightarrow \quad \left(E\frac{dx}{dy}\right)_{y_R} = \left(E\frac{dx}{dy}\right)_{y_L} \qquad (a)''$$

Thus (12) is equivalent to solving

loading term

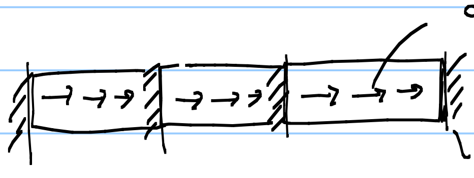
$$\boxed{\begin{array}{l} \dfrac{d}{dy}\left(E\dfrac{dx}{dy}\right) = \dfrac{dE}{dy} \quad \text{in } Y \qquad (12a)' \\[2ex] + \quad x|_{y_L} = x|_{y_R}, \quad \dfrac{dx}{dy}\Big|_{y_L} = \dfrac{dx}{dy}\Big|_{y_R} \qquad (12b)' \\[1ex] \text{(Thus } x \text{ has a rigid-body motion;} \\ \quad \text{may set } x(y_R) = x(y_L) = 0\text{)} \end{array}}$$

periodic condition



unit cell $y \in Y = [y_L, y_R]$

$dE/dy \leftarrow$ Body force



if $x(y_R) = x(y_L) = 0$

Integrating (12a)';

$$E \frac{dx}{dy} = E + a$$ ← Integration constant

$$\frac{dx(y)}{dy} = 1 + \frac{a}{E(y)}$$

Thus

$$x(y) = y + a \int_{y_L}^{y} \frac{dy}{E(y)} + b \qquad (C)$$

* use the periodicity condition

$$x(y_L) = x(y_R) \qquad \text{— } (*)$$

$$(**) \begin{cases} x(y = y_L) = y_L + b \\ x(y = y_R) = y_R + a \displaystyle\int_{y_L}^{y_R} \frac{dy}{E(y)} + b \end{cases}$$

$$\int_Y \frac{dy}{E(y)}$$

(**) → (*)

$$\rightarrow \quad -\underbrace{(y_R - y_L)}_{L_y^C} = a \int_Y \frac{dy}{E(y)}$$

$$\boxed{a = \frac{-L_y^c}{\int_Y \frac{dy}{E(y)}} = -\frac{1}{\left[\frac{1}{L_y^c}\int_Y \frac{dy}{E(y)}\right]}} \quad (d)$$

Summary

$$u'(x,y) = -\frac{du^0(x)}{dx}\, x(y) \quad (e)$$

$$x(y) = y + a\int_{y_L}^{y} \frac{dy}{E(y)} + b$$

$$a = -1\Big/\left[\frac{1}{L_y^c}\int_Y \frac{dy}{E(y)}\right] \quad (f)$$

※ Remark: if $u^0(x)$ is known, $u'(x,y)$ is determined (but not yet known!!)

For future use, we compute dx/dy explicitly:

$$\frac{dx}{dy} = 1 + \frac{a}{E(y)} \quad (g)$$

(yield governing eq. for $u^0(x)$!!)

[3] $O(1)$ term:

du^0/dx

$$\int_\Omega \frac{1}{L_y^c}\left[\int_Y E\left\{\left(\frac{\partial u^0}{\partial x}+\frac{\partial u^1}{\partial y}\right)\frac{\partial v}{\partial x} + \left(\frac{\partial u^1}{\partial x}+\frac{\partial u^2}{\partial y}\right)\frac{\partial v}{\partial y}\right\} dy\right] dx$$

$$= \int_\Omega \left[\frac{1}{L_y^c}\int_Y b\, v\, dy\right] dx + \bar{t}\, v(L) \qquad (13)$$

Choose $v(x,y) = v(x)$

Then (13) becomes

$$\boxed{\int_\Omega \frac{1}{L_y^c}\left[\int_Y E\left(\frac{du^0}{dx}+\frac{\partial u^1}{\partial y}\right) dy\right]\frac{\partial v}{\partial x}\, dx = \int_\Omega \left[\frac{1}{L_y^c}\int_Y \gamma(x,y)\, dy\right] v\, dx + \hat{t}\, v(L)} \qquad (14)$$

☐ Substitute $u^{1}(x,y)$ in page 12 into (14)

$$\int_{\Omega} \frac{1}{L_y{}^c}\int_{Y}\left(E - E\frac{d\chi}{dy}\right)dy\ \frac{du^{0}(x)}{dx}\frac{dv}{dx}\,dx$$

$$= \int_{\Omega}\left[\frac{1}{L_y{}^c}\int_{Y}\gamma\,dy\right] v\,dx + \hat{t}\,v(L)$$

Let

Homogenized elasticity coefficient

- $$E^{H} \triangleq \frac{1}{L_y{}^c}\int_{Y}\left(E - E\frac{d\chi}{dy}\right)dy$$

$$\underset{\text{page 12, (9)}}{=} \frac{1}{L_y}\int_{Y}\left[E - E\left(1 - \frac{a}{E}\right)\right]dy$$

a

$$= a = 1\Big/\frac{1}{L_y{}^c}\int_{Y}\frac{dy}{E(y)} \tag{15}$$

- $$\overset{a}{b}(x) \triangleq \frac{1}{L_y{}^c}\int_{Y} b(x,y)\,dy \tag{16}$$

Then

$$(W)\quad \boxed{\int_\Omega E^H \frac{du^\circ}{dx}\frac{dv}{dx}\,dx = \int_\Omega b(x)\, v(x)\, dx + \hat{t}\, v(L)} \quad (17)$$

Macroscopic equation for $u^\circ(x)$ (with E^H, $b^a(x)$)

☒ Integrating (17) by part to obtain strong form

$$\int_\Omega \left[\frac{d}{dx}\left(E^H \frac{du^\circ}{dx}\right) + b^a(x)\right] v(x)\, dx$$

$$- \left(E^H(x)\frac{du^\circ}{dx} - \bar{t}\right) v(L) = 0$$

$$(\because v(0) = 0)$$

(stress term)

Thus

$$(S)\quad \begin{cases} \dfrac{d}{dx}\left(E^H \dfrac{du^\circ}{dx}\right) + b = 0 \\ E^H(x)\dfrac{du^\circ}{dx} = \bar{t} \quad \text{at } x = L \end{cases}$$

How about $\sigma^0(x,y)$?

① $\sigma(X) = \sigma^0(x,y) + \varepsilon\,\sigma^1(x,y) + \cdots$

② $\sigma^0(x,y) = E\,\varepsilon^0(x,y)$ etc

To find $\varepsilon^0(x,y)$, consider

③ $\epsilon(X) = \epsilon(x,y) \stackrel{\Delta}{=} \epsilon^0(x,y) + \epsilon\,\epsilon^1(x,y) + \cdots$

④ $$\frac{\partial u}{\partial X} = \left(\frac{\partial}{\partial x} + \frac{1}{\epsilon}\frac{\partial}{\partial y}\right)\left(u^0 + \epsilon u^1 + \cdots\right)$$

$$= \frac{1}{\epsilon}\cancel{\frac{\partial u^0}{\partial y}} + \underbrace{\left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y}\right)} + \epsilon(\cdots)$$

because $u^0 = u^0(x)$,

From ③ = ④

$$\epsilon^0(x,y) = \left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y}\right)$$

$$= \left(\frac{\partial u^0}{\partial x} - \frac{\partial \chi}{\partial y}\frac{du^0}{dx}\right)$$

use $\left(\frac{d\chi}{dy} = 1 - \frac{a}{E}\right)$

$$= \left(1 - \frac{\partial \chi}{\partial y}\right)\frac{du^0}{dx} = \frac{a}{E(y)}\frac{du^0}{dx}$$

$$= \frac{E^H}{E(y)}\frac{du^0}{dx}$$

Thus

$$\sigma^0(x,y) = E\,\varepsilon^0(x,y)$$

$$= \underbrace{E^H \frac{du^0(x)}{dx}}_{\text{ftn of } x \text{ only.}} \equiv \sigma^0(x)$$

Because $\sigma^h(x)$ is usually defined as

$$\sigma^h(x) = E^H \frac{du^0(x)}{dx}$$

E^H — homogenized material coeff

$\sigma^h(x)$ can be shown to be

$$\sigma^h(x) = \frac{1}{L_Y{}^c} \int_Y \sigma^0(x,y)\,dy$$

in general case.

In the present case,

$$\sigma^h(x) = \sigma^0(x,y)$$

$\sigma^0(x,y)$ — $\sigma^0(x)$