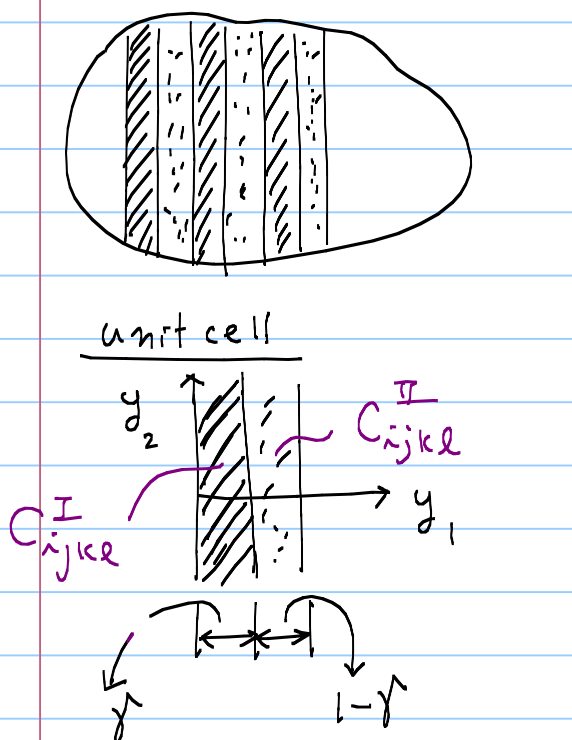
노트 제목

「Analytic Solution of the Homogenized Eqs for Laminated Composites – 2-D Case」

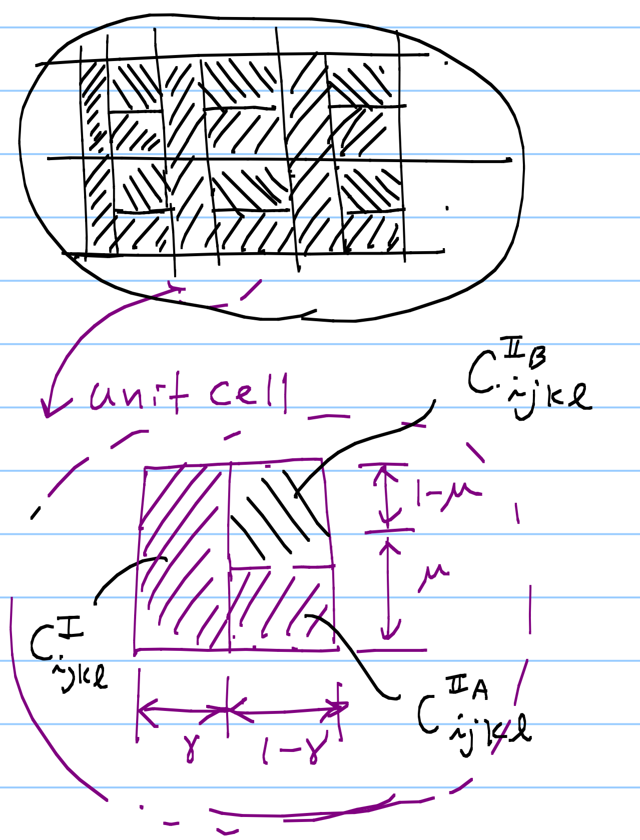
< Rank-1 Material >



$$Y = \bar{Y} = [0,1] \times \mathbb{R}$$

$$\frac{\partial}{\partial y_2}(\cdot) = 0$$

< Rank-2 Material >



$$Y = \bar{Y} = [0,1] \times [0,1]$$

(mainly work with isotropic C_{ijkl})

To find D_{ijkl} (Homogenized elastic constants), recall

$$\boxed{D_{ijkl}(\underline{x}) \triangleq \frac{1}{V_y^c}\int_{\bar{Y}}\left(C_{ijkl} - C_{ijpm}\frac{\partial \chi_p^{kl}}{\partial y_m}\right)dV_y} \quad (A)$$

$$\boxed{\int_{\bar{Y}} C_{ijkl}\frac{\partial \chi_k^{qr}}{\partial y_l}\frac{\partial v_i(\underline{y})}{\partial y_j}\,dV_y = \int_{\bar{Y}} C_{ijqr}\frac{\partial v_i(\underline{y})}{\partial y_j}\,dV_y} \quad (B)$$

$\oplus$ periodicity on χ_k^{qr}

CASE 1 : $(qr) = (11)$

($Y = \bar{Y}$ in this case)

$$(B) \rightarrow \int_Y \Big[\left(C_{1111}\frac{\partial \chi_1^{11}}{\partial y_1} + C_{1122}\frac{\partial \chi_2^{11}}{\partial y_2}\right)\frac{\partial v_1}{\partial y_1} + C_{1212}\left(\frac{\partial \chi_1^{11}}{\partial y_2} + \frac{\partial \chi_2^{11}}{\partial y_1}\right)\left(\frac{\partial v_2}{\partial y_1} + \frac{\partial v_1}{\partial y_2}\right)$$

$(C_{2211} = C_{1122})$

$$+ \left(C_{2211}\frac{\partial \chi_1^{11}}{\partial y_1} + C_{2222}\frac{\partial \chi_2^{11}}{\partial y_2}\right)\frac{\partial v_2}{\partial y_2}\Big]\,dV_y$$

$$= \int_Y \left(C_{1111}\frac{\partial v_1}{\partial y_1} + C_{2211}\frac{\partial v_2}{\partial y_2}\right)dV_y \quad (1)$$

$\oplus$ χ_i^{11} : periodic

(A) →

$$D_{1111} = \frac{1}{V_y^c}\int_Y \left(C_{1111} - C_{1111}\frac{\partial \chi_1^{11}}{\partial y_1} - C_{1122}\frac{\partial \chi_2^{11}}{\partial y_2}\right) dV_y \quad (2)$$

$$D_{2211} = \frac{1}{V_y^c}\int_Y \left(C_{2211} - C_{2211}\frac{\partial \chi_1^{11}}{\partial y_1} - C_{2222}\frac{\partial \chi_2^{11}}{\partial y_2}\right) dV_y \quad (3)$$

⟨ Case 2 : $(qr = 22)$ ⟩

(B) →

$$\int_Y \Bigg[\left(C_{1111}\frac{\partial \chi_1^{22}}{\partial y_1} + C_{1122}\frac{\partial \chi_2^{22}}{\partial y_2}\right)\frac{\partial v_1}{\partial y_1} + C_{1212}\left(\frac{\partial \chi_1^{22}}{\partial y_2} + \frac{\partial \chi_2^{22}}{\partial y_1}\right)\left(\frac{\partial v_2}{\partial y_1} + \frac{\partial v_1}{\partial y_2}\right) + \left(C_{2211}\frac{\partial \chi_1^{22}}{\partial y_1} + C_{2222}\frac{\partial \chi_2^{22}}{\partial y_2}\right)\frac{\partial v_2}{\partial y_2}\Bigg] dV_y$$

$$= \int_Y \left(C_{1122}\frac{\partial v_1}{\partial y_1} + C_{2222}\frac{\partial v_2}{\partial y_2}\right) dV_y \quad (4)$$

$\oplus$ χ_i^{22} = periodic

(A) →

$$\begin{cases} D_{1122} = \dfrac{1}{V_y^c} \displaystyle\int_Y \left(C_{1122} - C_{1111} \dfrac{\partial \chi_1^{22}}{\partial y_1} - C_{1122} \dfrac{\partial \chi_2^{22}}{\partial y_2} \right) dV_y & (5) \\ D_{2222} = \dfrac{1}{V_y^c} \displaystyle\int_Y C_{2222} - C_{2211} \dfrac{\partial \chi_1^{22}}{\partial y_1} - C_{2222} \dfrac{\partial \chi_2^{22}}{\partial y_2}) \, dV_y & (6) \end{cases}$$

(Remark ; $D_{1122} \equiv D_{2211}$ ← used as a check)

< Case 3 : ($qr = 12$) (same as $qr = 21$) >

(B) →

$$\int_Y \Big[\left(C_{1111} \frac{\partial \chi_1^{12}}{\partial y_1} + C_{1122} \frac{\partial \chi_2^{12}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1}$$

$$+ C_{1212} \left(\frac{\partial \chi_1^{12}}{\partial y_2} + \frac{\partial \chi_2^{12}}{\partial y_1} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right)$$

$$+ \left(C_{1122} \frac{\partial \chi_1^{12}}{\partial y_1} + C_{2222} \frac{\partial \chi_2^{12}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \Big] \, dV_y$$

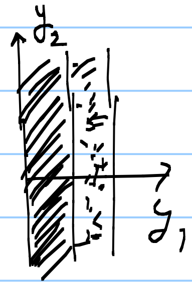
⊕ χ_i^{12} : periodic

$$= \int_Y C_{1212} \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) dV_y \qquad (7)$$

(A) →

$$D_{1212} = \frac{1}{V_y^c} \int_Y \left[C_{1212} - C_{1212} \left(\frac{\partial \chi_1^{12}}{\partial y_2} + \frac{\partial \chi_2^{12}}{\partial y_1} \right) \right] dV_y \qquad (8)$$

Explicit Calculations of Eqs (1-8) for Rank-1 Material



- Use the following conditions

  - $\frac{\partial}{\partial y_2}(\cdot) = 0,$
  - $v_i = v_i(y_1)$ (9)
  - $\chi_i = \chi_i(y_1)$
  - Let $V_y^c = L_y^c = 1$
  - Replace $\int_{V_y} (\cdot)\, dV_y = \int_{y_1^L = 0}^{y_1^u = 1} (\cdot)\, dy_1$

≪ For $(qr) = (11)$: $N_2 = 0$ ≫

$$(1) \rightarrow \int_Y C_{1111} \frac{\partial \chi_1''}{\partial y_1} \frac{\partial v_1}{\partial y_1} dy_1 = \int_{y_1} C_{1111} \frac{\partial v_1}{\partial y_1} dy_1 \quad (a)$$

$$\begin{cases} (2) \rightarrow & D_{1111} = \frac{1}{L_{y_1}^c} \int_Y \left(C_{1111} - C_{1111} \frac{\partial \chi_1''}{\partial y_1} \right) dy_1 \quad (b) \\ (3) & D_{2211} = \frac{1}{L_{y_1}^c} \int_Y \left(C_{2211} - C_{2211} \frac{\partial \chi_1''}{\partial y_1} \right) dy_1 \quad (c) \end{cases}$$

(※ Only $\partial \chi_1'' / \partial y_1$ is needed.)

Before integrating (a), convert (a) to

$$\int_0^1 \left(C_{1111} \frac{\partial \chi_1''}{\partial y_1} - C_{1111} \right) \frac{\partial v_1}{\partial y_1} dy_1 = 0 \quad (d)$$

Integrate the above result:

$$\underbrace{\left[\left(C_{1111} \frac{\partial \chi_1''}{\partial y_1} - C_{1111} \right) v_1 \right]_{y_1=0}^{y_1=1}}_{=0} - \int_Y \underbrace{\frac{d}{\partial y_1} \left[C_{1111} \frac{\partial \chi_1''}{\partial y_1} - C_{1111} \right]}_{=0} v_1 \, dy_1 = 0$$

$(\because v(0) = v(1),\ \chi = \text{periodic})$

$\Rightarrow$

$$C_{1111}\frac{\partial x_1''}{\partial y_1} - C_{1111} = a$$

$\Leftarrow$ (SAME procedure used in 1-D Homo. case)

$$\therefore \quad \frac{\partial x_1''}{\partial y_1} = 1 + \frac{a}{C_{1111}} \qquad \text{——— (e)}$$

$$x_1''(y_1) = y + a\int^{y_1}\frac{dy}{C_{1111}} + b \qquad (f)$$

Imposing $x_1''(0) = x_1''(1)$ yields

$$a = -\frac{1}{\underbrace{L_{y_1}^c}_{1}\int_Y \frac{dy_1}{C_{1111}}} = -\left\langle \frac{1}{C_{1111}} \right\rangle^{-1} \qquad (g)$$

Convenient to use Introduce the following notation:

$$\boxed{\langle f \rangle = \begin{cases} \dfrac{1}{L_{y_1}^c}\displaystyle\int_{y_1} f(y_1)\,dy_1 & \text{for 1-D} \\[2ex] \dfrac{1}{V_y^c}\displaystyle\int_Y f(\underset{\sim}{y})\,dV_y & \text{for 3-D} \end{cases}} \qquad (10)$$

Volumetric average

Then

$$\frac{\partial x_1''}{\partial y_1}\Big|_{(e,g)} = 1 - \frac{1}{C_{1111}} \left\langle \frac{1}{C_{1111}} \right\rangle^{-1} \qquad (h)$$

Substituting (h) → (b, c) yields

$$\bullet \; D_{1111} = \frac{1}{L_{y_1}^c} \int_Y \left(C_{1111} - C_{1111} \frac{\partial x_1''}{\partial y_1} \right) dy_1$$

$$= \left\langle \frac{1}{C_{1111}} \right\rangle \qquad (11)$$

$$\bullet \; D_{2211} = \frac{1}{L_{y_1}^c} \int_Y \left(C_{2211} - C_{2211} \frac{\partial x_1''}{\partial y_1} \right) dy_1$$

$$= \frac{1}{L_{y_1}^c} \int_Y \frac{C_{2211}}{C_{1111}} \left\langle \frac{1}{C_{1111}} \right\rangle^{-1} dy_1$$

$$= \left(\frac{1}{L_{y_1}^c} \int_Y \frac{C_{2211}}{C_{1111}} dy_1 \right) \left\langle \frac{1}{C_{1111}} \right\rangle^{-1}$$

(12)

« For $(qr) = 22$: $v_2 = 0$ »

(4) → $$\int_Y C_{1111} \frac{\partial \chi_1^{22}}{\partial y_1} \frac{\partial v_1}{\partial y_1} dy_1 = \int C_{1122} \frac{\partial v_1}{\partial y_1} dy_1 \quad (a)'$$

(5) → $$D_{1122} = \frac{1}{L^c_{Y_1}} \int_Y \left(C_{1122} - C_{1111} \frac{\partial \chi_1^{22}}{\partial y_1} \right) dy_1 \quad (b)'$$

(6) $$D_{2222} = \frac{1}{L^c_{Y_1}} \int_Y \left(C_{2222} - C_{2211} \frac{\partial \chi_1^{22}}{\partial y_1} \right) dy_1 \quad (c)'$$

(∵ only $\partial \chi_1^{22} / \partial y_1$ is needed)

Before integrating (a)', convert (a)' into

$$\int_0^1 \left(C_{1111} \frac{\partial \chi_1^{22}}{\partial y_1} - C_{1122} \right) \frac{\partial v_1}{\partial y_1} dy_1 = 0 \quad (d)'$$

Integrate the above result:

$$\left[\left(C_{1111} \frac{\partial \chi_1^{22}}{\partial y_1} - C_{1122} \right) v_1 \right]_{y_1=0}^{y_1=1} - \int_Y \underbrace{\frac{\partial}{\partial y_1} \left[C_{1111} \frac{\partial \chi_1^{22}}{\partial y_1} - C_{1122} \right]}_{\underset{0}{\parallel}} v_1 dy_1 = 0$$

(∵ $v(0) = v(1)$, χ = periodic)

$$\Rightarrow \quad C_{1111}\frac{\partial \chi_1^{22}}{\partial y_1} - C_{1122} = a'$$

$$\therefore \quad \frac{\partial \chi_1^{22}}{\partial y_1} = \frac{C_{1122}}{C_{1111}} + \frac{a'}{C_{1111}} \qquad (e)'$$

$$\chi_1^{22}(y_1) = \int^{y_1} \frac{C_{1122}}{C_{1111}}\,dy_1 + a'\int^{y_1}\frac{1}{C_{1111}}\,dy_1 + b'$$

Imposing $\chi_1^{22}(0) = \chi_1^{22}(1)$

$$b' = \int_0^1 \frac{C_{1122}}{C_{1111}}\,dy_1 + a'\int_0^1 \frac{1}{C_{1111}}\,dy_1 + b'$$

Thus

$$\underline{a' = -\left\langle \frac{C_{1122}}{C_{1111}} \right\rangle \left\langle \frac{1}{C_{1111}} \right\rangle^{-1}} \qquad (g)'$$

Then $$\frac{\partial \chi_1^{22}}{\partial y_1} \underset{(e',g')}{=} \frac{C_{1122}}{C_{1111}} - \frac{1}{C_{1111}}\left\langle \frac{C_{1122}}{C_{1111}} \right\rangle \left\langle \frac{1}{C_{1111}} \right\rangle^{-1} \qquad (h)'$$

Substituting (h') into (b', c') yields

$$\bullet\ [\text{illegible}] = \frac{1}{L_y^c}\int_Y \left(C_{1122} - C_{1111}\frac{\partial \chi_1^{22}}{\partial y_1}\right) dy_1$$

$$\left\langle \frac{C_{1122}}{C_{1111}} \right\rangle \left\langle \frac{1}{C_{1111}} \right\rangle \qquad (13)$$

$\Big(= [\text{illegible}]$; note $C_{1122} = C_{2211}$ etc $\Big)$

$$\bullet\ [\text{illegible}] = \frac{1}{L_y^c}\int_Y \left(C_{2222} - C_{2211}\frac{\partial \chi_1^{22}}{\partial y_1}\right) dy_1$$

$$= \frac{1}{L_y^c}\int_Y \Bigg(C_{2222} - \frac{C_{1122}^2}{C_{1111}} - \frac{C_{1122}}{C_{1111}} \left\langle \frac{C_{1122}}{C_{1111}} \right\rangle \left\langle \frac{1}{C_{1111}} \right\rangle^{-1}\Bigg) dy_1$$

$$= \langle C_{2222} \rangle - \left\langle \frac{C_{1122}}{C_{1111}} \right\rangle \left\langle \frac{C_{1122}}{C_{1111}} \right\rangle^2 \left\langle \frac{1}{C_{1111}} \right\rangle^{-1} \qquad (14)$$

《 For $(qr) = (12)$; $v_1 = 0$ 》

$$(7) \rightarrow \int_Y C_{1212} \frac{\partial \chi_2^{12}}{\partial y_1} \frac{\partial v_2}{\partial y_1} dy_1 = \int_Y C_{1212} \frac{\partial v_2}{\partial y_1} dy_1 \quad (a)''$$

$$(8) \rightarrow D_{1212} = \frac{1}{|Y_1|} \int_Y \left(C_{1212} - C_{1212} \frac{\partial \chi_2^{12}}{\partial y_1} \right) dy_1 \quad (b)''$$

Before integrating (a''), convert $(a)''$ into

$$\int_0^1 \left(C_{1212} \frac{\partial \chi_2^{12}}{\partial y_1} - C_{1212} \right) \frac{\partial v_2}{\partial y_1} dy_1 = 0 \quad (c)''$$

Integrate the above equation:

$$\underbrace{\left(C_{1212} \frac{\partial \chi_2^{12}}{\partial y_1} - C_{1212} \right) v_1 \Big]_{y_1=0}^{y_1=1}}_{\overset{||}{0}} - \int_{y_1=0}^{y_2=1} \frac{\partial}{\partial y} \underbrace{\left[C_{1212} \frac{\partial \chi_2^{12}}{\partial y_1} - C_{1212} \right]}_{\overset{|||}{0}} v_2 \, dy = 0$$

(∵ $v(0) = v(1)$,
χ = periodic)

$$\Rightarrow C_{1212} \frac{\partial \chi_2^{12}}{\partial y_1} - C_{1212} = a'' \quad (d)''$$

↑ CONSTANT

$$\therefore \quad \frac{\partial \chi_2^{12}}{\partial y_1} = 1 + \frac{a''}{C_{1212}} \qquad (e)''$$

$$\chi_2^{12}(y_1) = y + a'' \int^{y_1} \frac{1}{C_{1212}} dy_1 + b \qquad (f'')$$

Imposing $\chi_2^{12}(0) = \chi_2^{12}(1)$ yields

$$a'' = - \frac{1}{\underbrace{L_Y^c}_{1=} \int_Y \frac{dy_1}{C_{1212}}} = - \left\langle \frac{1}{C_{1212}} \right\rangle^{-1} \qquad (g)''$$

Thus

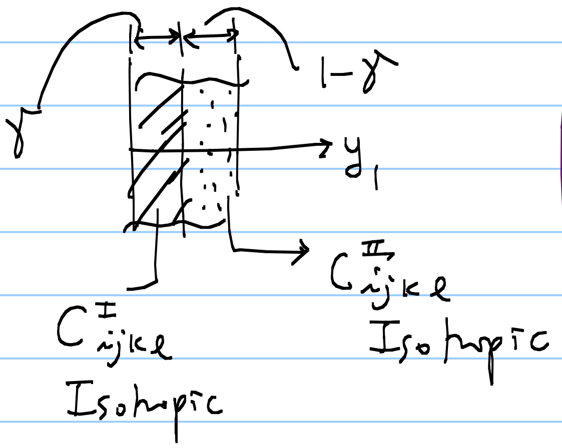
$$\frac{\partial \chi_2^{12}}{\partial y_1} = 1 - \frac{1}{C_{1212}} \left\langle \frac{1}{C_{1212}} \right\rangle^{-1} \qquad (h'')$$

Substituting (h'') into (b'') yield

$$= \frac{1}{L_{y_1}^c} \int_Y \left(C_{1212} - C_{1212} \frac{\partial \chi_2^{12}}{\partial y_1} \right) dy_1$$

$$(15)$$

Actual calculation of D_{ijkl} (under plane strain assumption)



$$\left[\begin{aligned} C_{ijkl} &= \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\ C_{1111} &= C_{2222} = \lambda + 2\mu \\ C_{1212} &= \mu \end{aligned}\right]$$

① $D_{1111} = \left\langle \dfrac{1}{C_{1111}} \right\rangle^{-1}$

$$\left\langle \frac{1}{C_{1111}} \right\rangle = \frac{1}{\underset{\underset{1}{\parallel}}{V_y^c}} \int_{y_1=0}^{1} \frac{dy_1}{C_{1111}} = \int_0^{\gamma} \frac{dy}{C^{I}_{1111}} + \int_{\gamma}^{1} \frac{dy}{C^{II}_{1111}}$$

$$= \frac{\gamma}{C^{I}_{1111}} + \frac{1-\gamma}{C^{II}}$$

$$D_{1111} = \frac{1}{\dfrac{\gamma}{C^{I}_{1111}} + \dfrac{1-\gamma}{C^{II}_{1111}}} \tag{16}$$

Useful Notation

- Arithmatic averge

$$A = \frac{a_1 + \cdots + a_n}{N}$$

- Harmonic averge

$$H = \frac{N}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

$\Rightarrow$ Generalization:

- Weighted Arithmatic average

$$A_\gamma(a, b) \triangleq \gamma a + (1-\gamma) b \qquad (0 \le \gamma \le 1)$$

- Weighted Harmonic average

$$H_\gamma(a, b) \triangleq \frac{1}{\frac{\gamma}{a} + \frac{1-\gamma}{b}} \tag{17}$$

Then, we write

- $$\underline{\underline{D_{1111} = H_\gamma\left(C^{I}_{1111}, C^{II}_{1111}\right)}} \tag{18}$$

② $$D_{2222} = \left\langle C_{2222} \right\rangle - \left\langle \frac{C_{1122}^{2}}{C_{1111}} \right\rangle + \left\langle \frac{C_{1122}}{C_{1111}} \right\rangle^{2} \left\langle \frac{1}{C_{1111}} \right\rangle^{-1}$$

• $$\left\langle C_{2222} \right\rangle = \frac{1}{V_y^c} \left[\int_{y_1=0}^{\gamma} C_{2222}^{I} dy_1 + \int_{\gamma}^{1} C_{2222}^{II} dy_2 \right]$$

$$= \gamma C_{2222}^{I} + (1-\gamma) C_{2222}^{II}$$

$$= A_\gamma \left(C_{2222}^{I}, C_{2222}^{II} \right)$$

$\left(\equiv A_\gamma \left(C_{1111}^{I}, C_{1111}^{II} \right) \text{ for Isotropy} \right)$

• $$\left\langle \frac{C_{1122}^{2}}{C_{1111}} \right\rangle = A_\gamma \left(\frac{\left(C_{1122}^{I}\right)^2}{C_{1111}^{I}}, \frac{\left(C_{1122}^{II}\right)^2}{C_{1111}^{I}} \right)$$

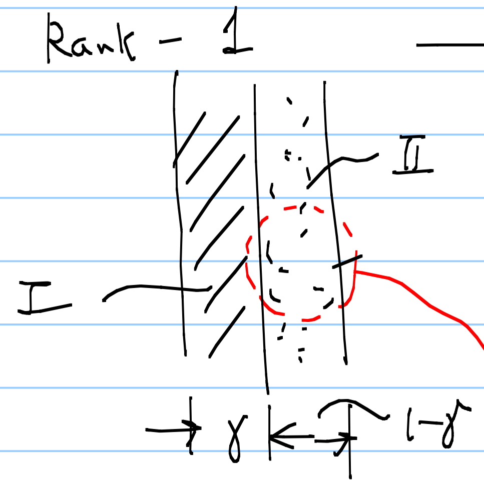
• $$\left\langle \frac{C_{1122}}{C_{1111}} \right\rangle = A_\gamma \left(\frac{C_{1122}^{I}}{C_{1111}^{I}}, \frac{C_{1122}^{II}}{C_{1111}^{I}} \right)$$

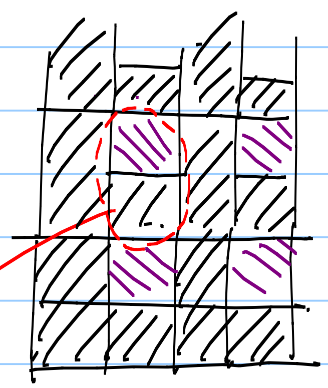
$$D_{2222} = A_\gamma\left(C^{\mathrm{I}}_{2222},\ C^{\mathrm{II}}_{2222}\right)$$
$$- A_\gamma\left(\frac{(C^{\mathrm{I}}_{1122})^2}{C^{\mathrm{I}}_{1111}},\ \frac{(C^{\mathrm{II}}_{1122})^2}{C^{\mathrm{II}}_{1111}}\right)$$
$$+ \left[A_\gamma\left(\frac{C^{\mathrm{I}}_{1122}}{C^{\mathrm{I}}_{1111}},\ \frac{C^{\mathrm{II}}_{1122}}{C^{\mathrm{II}}_{1111}}\right)\right]^2 H_\gamma\left(C^{\mathrm{I}}_{1111},\ C^{\mathrm{II}}_{1111}\right)$$

③ $$D_{1212} = \left\langle \frac{1}{C_{1212}} \right\rangle^{-1}$$
$$= H_\gamma\left(C^{\mathrm{I}}_{1212},\ C^{\mathrm{II}}_{1212}\right)$$

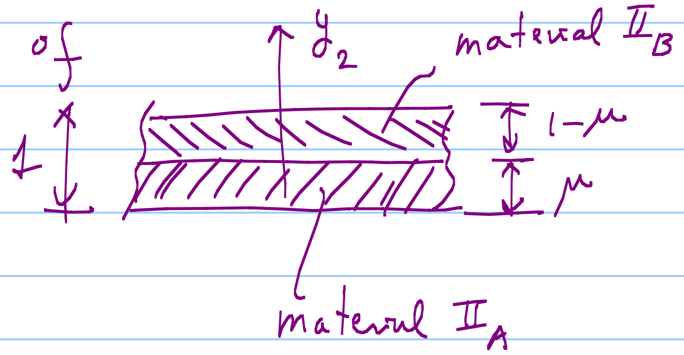
Explicit Calculations of Eqs (1-8) for Rank-2 Material

Rank - 1 ⟶ Rank - 2





May treat C^{II}_{ijkl} as the Homogenized unit cell of



Therefore

Replace C^{II}_{ijkl} in D_{ijkl} in the rank-1 material by

- $$C^{II}_{1111} \leftarrow A_\mu\left(C^{IA}_{1111},\ C^{IB}_{1111}\right) - A_\mu\left(\frac{(C^{IA}_{1122})^2}{C^{IA}_{1111}},\ \frac{(C^{IB}_{1122})^2}{C^{IB}_{1111}}\right) + \left[A_\mu\left(\frac{C^{IA}_{1122}}{C^{IA}_{2222}},\ \frac{C^{IB}_{1122}}{C^{IB}_{2222}}\right)\right]^2 H_\mu\left(C^{IA}_{2222},\ C^{IB}_{2222}\right)$$

- $$C^{II}_{2222} \leftarrow H_\mu\left(C^{IA}_{2222},\ C^{IB}_{2222}\right)$$

- $$C^{II}_{1212} \leftarrow H_\mu\left(C^{IA}_{1212},\ C^{IB}_{1212}\right)$$