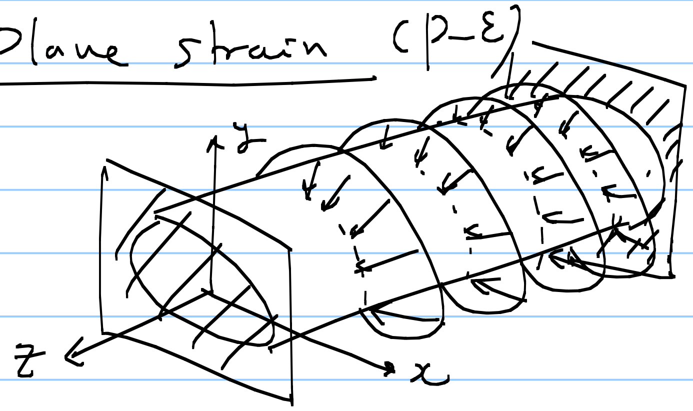
Plane Elasticity problems

<"plane strain" and "plane stress">

(notation)
$x_1 \to x, \; x_2 \to y, \; x_3 \to z$
$u_1 \to u_x, \; u_2 \to u_y, \; u_3 \to u_z$

1. Plane strain (P–ε)



☑ Consider a "long" cylinder under a uniform load

i) $u_z = 0$

ii) $\frac{\partial}{\partial z}(\cdot) = 0 \quad (\text{i.e,} \; \frac{\partial u_x}{\partial z} = \frac{\partial u_y}{\partial z} = 0)$

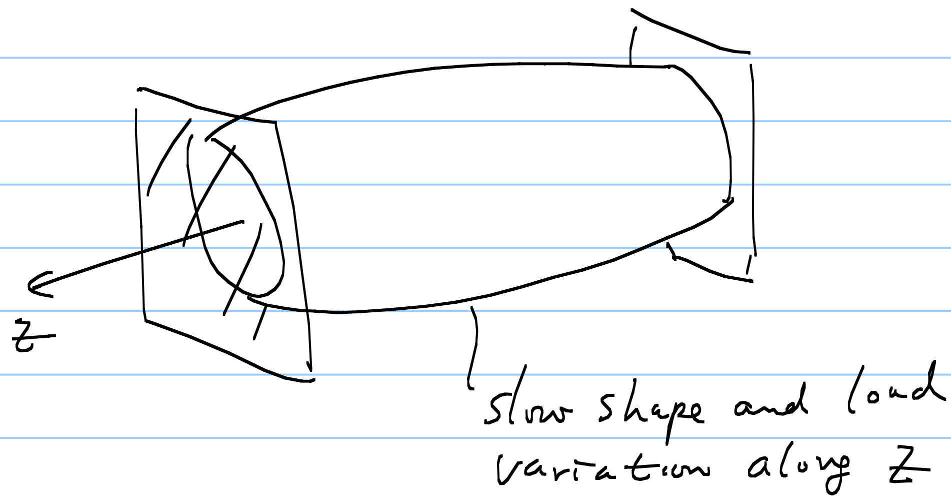
$$\rightarrow \quad \epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0 \qquad \text{— (1)}$$

↳ All strain components except the plane strain components ϵ_{xx}, ϵ_{xy}, ϵ_{yy} $\equiv 0$

Using $\quad \varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\,(\sigma_{xx} + \sigma_{yy})\right] \equiv 0$

$$\Longrightarrow \quad \left\{ \begin{aligned} \sigma_{zz} &= \nu\,(\sigma_{xx} + \sigma_{yy}) \\ \sigma_{zx} &= 2\mu\,\varepsilon_{zx} = 0 \\ \sigma_{zy} &= 2\mu\,\varepsilon_{zy} = 0 \end{aligned} \right\} \quad (2)$$

Note: to have the "exact" plane strain condition ($\epsilon_{zz} = \varepsilon_{zx} = \varepsilon_{zy} = 0$), non-vanishing $\sigma_{zz} = \nu\,(\sigma_{xx} + \sigma_{yy})$ must be present.



$\Longrightarrow$ plane-strain condition can be used.

Based on (1) and (2), we have the following form of field equations:

i) Eqm Eqs

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + b_x = 0 \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{cases}$$

ii) Strain – Disp relation

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} ; \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} ; \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

iii) Constitutive relation

$$\varepsilon_{xx} = \frac{1}{E^*}(\sigma_{xx} - \nu^* \sigma_{yy})$$

$$\varepsilon_{yy} = \frac{1}{E^*}(\sigma_{yy} - \nu^* \sigma_{xx})$$

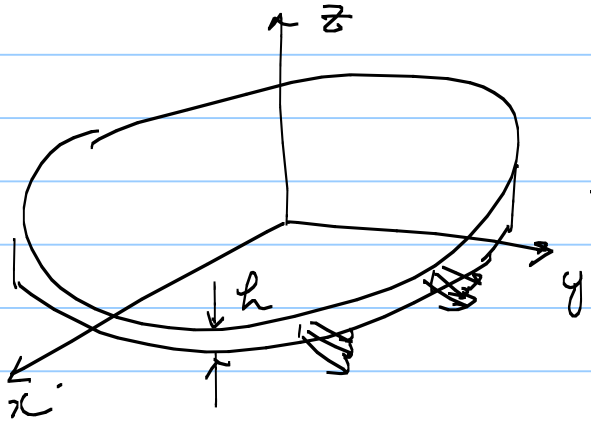
$$\varepsilon_{xy} = \frac{1+\nu^*}{E^*}\sigma_{xy} = \frac{1}{2\mu}\sigma_{xy}$$

where $E^* = \dfrac{E}{1-\nu^2}$; $\nu^* = \dfrac{\nu}{1-\nu}$

iv) Compatability (only 1 remains)

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0$$

2. Plane Stress (P-σ)



☑ Consider a "thin" plate

FACT: Traction-free top/bottom surfaces

$$\begin{cases} \sigma_{zz}(x, y, z = \pm h/2) = 0 \\ \sigma_{zx}(\quad " \quad) = 0 \\ \sigma_{zy}(\quad " \quad) = 0 \end{cases}$$

Because i) h is small compared other dimension
ii) the applied load does not vary along z
and acts only in the x-y plane,

one may approximate the stress state as

$$\sigma_{zz} \simeq 0, \quad \sigma_{zx} \simeq 0, \quad \sigma_{zy} \simeq 0$$

"everywhere" in the plate

"plane-stress condition"

$$\hookrightarrow \quad \sigma_{zz} = 0, \quad \sigma_{zx} = 0, \quad \sigma_{zy} = 0 \qquad (3)$$

$$\left(\text{note} \quad \varepsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}) \neq 0 \right)$$

Based on (3), we have

1) Eqm Eqs

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + b_x = 0 \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{cases}$$

ii) strain– Disp relation

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \; ; \; \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \; ; \; \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

iii) Constitutive relation

$$\boxed{\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\left(\sigma_{xx} - \nu\,\sigma_{yy}\right) \\ \varepsilon_{yy} &= \frac{1}{E}\left(\sigma_{yy} - \nu\cdot\sigma_{xx}\right) \\ \varepsilon_{xy} &= \frac{1+\nu}{E}\,\sigma_{xy} = \frac{1}{2\mu}\,\sigma_{xy} \end{aligned}}$$

iv) Compatability (only 1 remains)

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0$$

<Analogy Between p–σ and p–ε>

$$\begin{array}{ccc} \mu, \nu & \xrightarrow{\;\nu \to \frac{\nu}{1-\nu}\;} & \mu, \nu \\ & \xleftarrow[\;\nu \to \frac{\nu}{1+\nu}\;]{} & \\ p-\sigma & & p-\varepsilon \end{array}$$

Airy Stress Function ϕ

- Start from eqm equations

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} = 0 & \text{(a)} \\[2ex] \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} = 0 & \text{(b)} \end{cases}$$

① If $\sigma_{xx} = \dfrac{\partial F}{\partial y}$ and $\sigma_{yx} = -\dfrac{\partial F}{\partial x}$, equation (a) is satisfied.

② If $\sigma_{xy} = -\dfrac{\partial G}{\partial y}$ and $\sigma_{yy} = \dfrac{\partial G}{\partial x}$, equation (b) is satisfied.

Requirement : $\sigma_{xy} = \sigma_{yx}$

$$\frac{\partial F}{\partial x} = \frac{\partial G}{\partial y}$$

Thus Introduce $\phi(x, y)$ such that

$$F = \frac{\partial \phi}{\partial y}, \quad G = \frac{\partial \phi}{\partial x}$$

Then eqm eqs (a, b) are satisfied

if

$$\begin{cases} \sigma_{xx} = \dfrac{\partial^2 \phi}{\partial y^2} \\ \sigma_{yy} = \dfrac{\partial^2 \phi}{\partial x^2} \\ \sigma_{xy} = -\dfrac{\partial^2 \phi}{\partial x \partial y} \end{cases}$$

ϕ : Airy stress function

Remark : When strains (or stresses) are given, they must satisfy compatibility equation.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0 \qquad (*)$$

where

(p−σ)

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}(\sigma_x - \nu \sigma_y) = \frac{1}{E}(\phi_{,yy} - \nu \phi_{,xx}) \\ \varepsilon_{yy} &= \frac{1}{E}(\sigma_y - \nu \sigma_y) = \frac{1}{E}(\phi_{,xx} - \nu \phi_{,yy}) \\ \varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} = \frac{1+\nu}{E}(-\phi_{,xy}) \end{aligned}\right\} (**)$$

(**) → (*) yields

$$\boxed{\begin{gathered}\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \\ \text{or} \quad \nabla^2 \nabla^2 \phi = \nabla^4 \phi = 0\end{gathered}}$$

"Biharmonic" equation

Remark 1: When 2-D static problems are solved analytically, the Airy stress function approach is most popular.

Remark 2: If 2-D problems are solved by standard FEM using "displacement" based formulation, there is no need to consider the compability condition explicitly. The condition is needed when stress (or strain) formulations, such as the Airy stress ftn approach, are used.