Airy Stress Function ϕ

노트 제목

① In Cartesian coords,

$$\nabla^4\phi = \nabla^2\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \phi(x,y) = 0$$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x \partial y} \qquad (a)$$

Note : $\sigma_{xx} + \sigma_{yy} = \mathrm{tr}\,\underset{\sim}{\sigma} = \nabla^2\phi$

observation:

$$\begin{cases} \sigma_{xx} = \phi_{,yy} = \nabla^2\phi - \phi_{,xx} \\ \sigma_{yy} = \phi_{,xx} = \nabla^2\phi - \phi_{,yy} \\ \sigma_{xy} = -\phi_{,xy} = 0 - \phi_{,xy} \end{cases}$$

$$\Rightarrow \boxed{\sigma_{\alpha\beta} = \delta_{\alpha\beta}\nabla^2\phi - \phi_{,\alpha\beta}} \qquad (\alpha=1,2)$$

i.e. $$\boxed{\underset{\sim}{\sigma} = \nabla^2\phi\,\underset{\sim}{1} - \underset{\sim}{\nabla}\,\underset{\sim}{\nabla}\,\phi} \qquad (*)$$

symbolic form

now can be used in any coordinates!

② In polar coordinates,

use
- $(*)$
- $\nabla^2\phi(r,\theta) = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$
- $\underset{\sim}{\nabla}\phi = \underset{\sim}{e}_r\frac{\partial\phi}{\partial r} + \underset{\sim}{e}_\theta\frac{1}{r}\frac{\partial\phi}{\partial\theta}$

Some details

$$\underline{\nabla}\,\underline{\nabla}\phi = \left(\underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \otimes \left(\underline{e}_r \frac{\partial \phi}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$= \underline{e}_r \otimes \underline{e}_r \frac{\partial^2 \phi}{\partial r^2} + \underline{e}_r \otimes \underline{e}_\theta \left(-\frac{1}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}\right)$$

$$+ \boxed{\underline{e}_\theta \otimes \underline{e}_r \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}} + \boxed{\underline{e}_\theta \otimes \underline{e}_\theta \frac{1}{r}\frac{\partial \phi}{\partial r}}$$

$$\boxed{- \underline{e}_\theta \otimes \underline{e}_r \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}} + \boxed{\underline{e}_\theta \otimes \underline{e}_\theta \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}}$$

$$= \frac{\partial^2 \phi}{\partial r^2} \underline{e}_r \otimes \underline{e}_r + \left(-\frac{1}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}\right)\left(\underline{e}_r \otimes \underline{e}_\theta + \underline{e}_\theta \otimes \underline{e}_r\right) + \underline{e}_\theta \otimes \underline{e}_\theta \left(\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right)$$

$$\sigma_{rr} = \nabla^2 \phi - \underline{e}_r \cdot (\underline{\nabla}\,\underline{\nabla}\phi)\, \underline{e}_r$$

$$= \underline{\underline{\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}}}$$

$$\sigma_{\theta\theta} = \nabla^2 \phi - \underline{e}_\theta \cdot (\underline{\nabla}\,\underline{\nabla}\phi)\, \underline{e}_\theta = \underline{\underline{\frac{\partial^2 \phi}{\partial r^2}}}$$

$$\sigma_{r\theta} = - \underline{e}_r \cdot (\underline{\nabla}\,\underline{\nabla}\phi) \cdot \underline{e}_\theta \quad \left(= - \underline{e}_\theta \cdot \nabla\nabla\phi \cdot \underline{e}_r\right)$$

$$= \underline{\underline{\frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}}}$$

Some typical solutions of $\nabla^4 \phi(x, y) = 0$ in polynomial form

a) Choose $\phi = \frac{a_2}{2} x^2 + b_2 xy + \frac{c_2}{2} y^2 \equiv \phi_2$

$\rightarrow$ automatically satisfies $\nabla^4 \phi_2 = 0$ for any a_2, b_2, c_2

b) Choose $\phi = \frac{a_4}{12} x^4 + \frac{b_4}{6} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{6} x y^3 + \frac{e_4}{12} y^4 \equiv \phi_4$

To satisfy $\nabla^4 \phi_4 = 0$

$e_4 = -2(2c_4 + a_4)$,

a_4, b_4, c_4, d_4 arbitrary

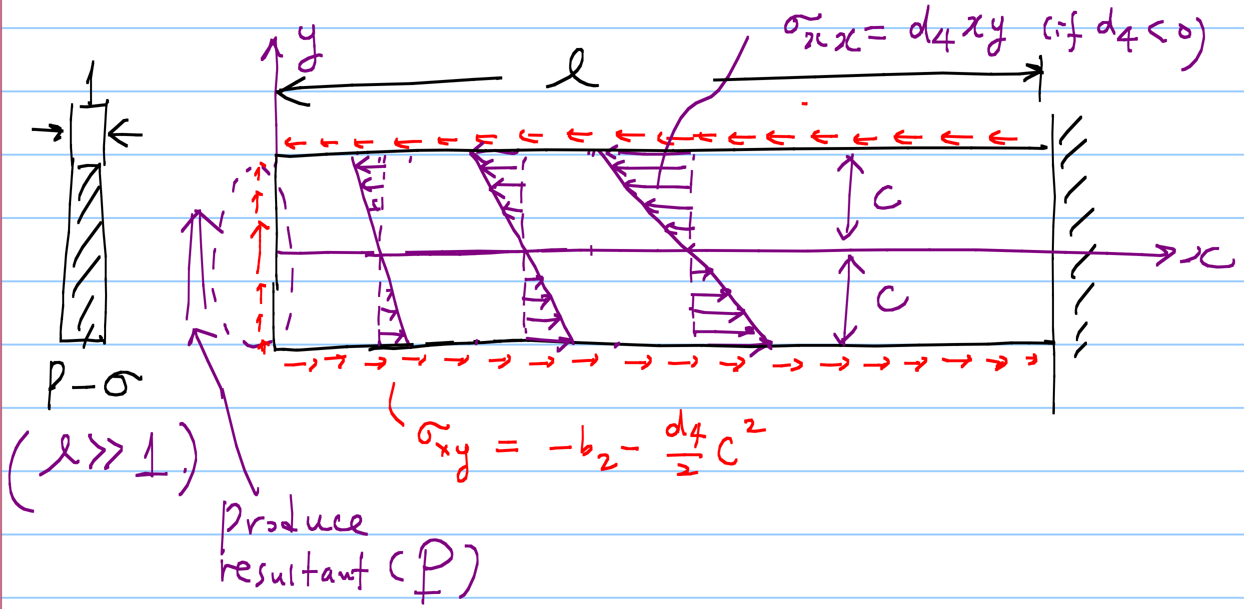
Among these, consider the stress field due to nonzero b_2 and d_4

$\boxed{} \ \phi = b_2 xy + \frac{d_4}{6} x y^3$

- $\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = d_4 xy$

- $\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2} = 0$
- $\sigma_{xy} = -b_2 - \dfrac{d_4}{2} y^2$

- See what problem can be solved by this distribution



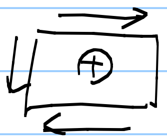
- Consider the following case → "Cantilever Beam"
  1) traction free on $y = \pm c$
  2) the sum of shear stress$\Big|_{x=0} = P$

  (iii) Let's not worry about the BC at $x = l$ in the meantime.)

$\Rightarrow$

i) $\sigma_{xy}\Big|_{y=\pm c} = -b_2 - \frac{d_4}{2}c^2 = 0$

$\therefore d_4 = -\frac{2b_2}{c^2}$

ii) $\underset{(\text{unward})}{P} = -\int_{-c}^{c} \sigma_{xy}\Big|_{x=0} dy$ (positive σ_{xy} )

$= \int_{-c}^{c} \left(b_2 - \frac{b^2}{c^2} y^2\right) dy$

$\Rightarrow b_2 = \frac{3}{4}\frac{P}{c}$ & $d_4 = -\frac{3P}{2c^3}$

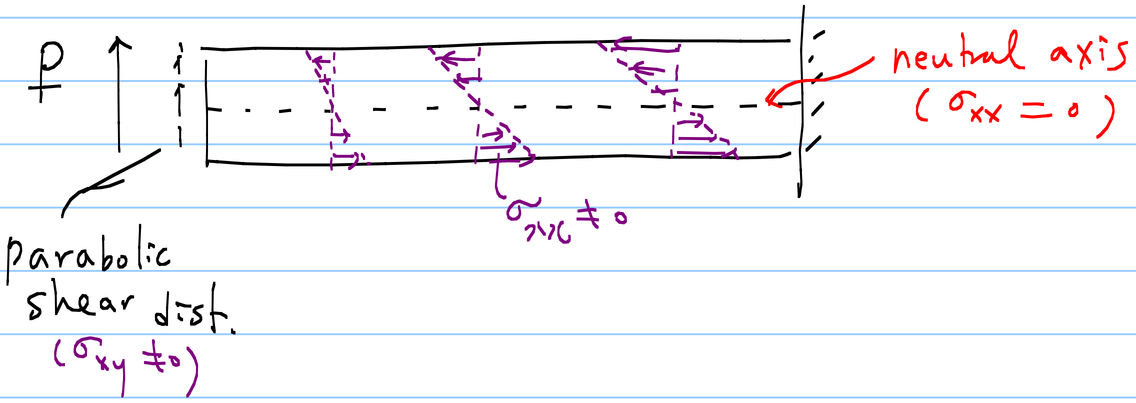
Thus
$$\begin{cases} \sigma_{xx} = -\frac{3}{2}\frac{P}{c^3}xy, \\ \sigma_{yy} = 0 \\ \sigma_{xy} = -\frac{3P}{4c}\left(1 - \frac{y^2}{c^2}\right) \end{cases}$$

Using the definition of moment of inertia I

$$I = \frac{(1)}{12}(2c)^3 = \frac{2}{3}c^3$$

The stress components can be written as

$$\boxed{\begin{aligned} \sigma_{xx} &= -\frac{P}{I}xy \\ \sigma_{yy} &= 0 \\ \sigma_{xy} &= -\frac{P}{I}\frac{1}{2}(c^2-y^2) \end{aligned}} \quad (*)$$



Observations

i) (*) $\Rightarrow$ the result by the elementary beam theory

ii) (*) is exact only if the shear force applied at both ends are distributed according to the parabolic law.

☑ Determine Disp Field due to (*) ?

$(\sigma_{xx}, \sigma_{xy}, \sigma_{yy})$

$\Downarrow$ ←---- constitutive relation

$(\epsilon_{xx}, \epsilon_{xy}, \epsilon_{yy})$

$\Downarrow$ ←---- Integration

(u_x, u_y)

<Analysis>

$\begin{pmatrix} u = u_x \\ v = u_y \end{pmatrix}$

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_{xx} - \nu \sigma_{yy}) = -\frac{Pxy}{EI} \quad (a)$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_{yy} - \nu \sigma_{xx}) = \frac{\nu Pxy}{EI} \quad (b)$$

$$2\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\sigma_{xy}}{G} = -\frac{P}{2GI}(c^2 - y^2) \quad (c)$$

Integrate (a, b):

$$u = -\frac{Px^2y}{2EI} + f(y) \quad (d)$$

$$v = \frac{\nu Pxy^2}{2EI} + f_1(x) \quad (e)$$

(d, e) → (c)

$$-\frac{Px^2}{2EI} + \frac{df(y)}{dy} + \frac{\nu P y^2}{2EI} + \frac{df_1(x)}{dx} = -\frac{P}{2GI}(c^2 - y^2)$$

OR

$$\underbrace{-\frac{Px^2}{2EI} + \frac{df_1(x)}{dx}}_{\text{ftn of } x} = \underbrace{-\frac{df(y)}{dy} - \frac{\nu P y^2}{2EI} + \frac{P}{2GI}(y^2 - c^2)}_{\text{ftn of } y}$$

$$\equiv \underset{\uparrow \text{ constant}}{d}$$

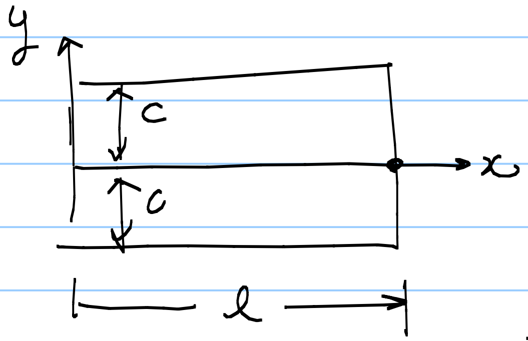
$$\Rightarrow \begin{cases} \dfrac{df_1(x)}{dx} = \dfrac{Px^2}{2EI} + d & \text{(f)} \\[2ex] \dfrac{df}{dy} = -\dfrac{\nu P y^2}{2EI} + \dfrac{P}{2GI}(y^2 - c^2) - d & \text{(g)} \end{cases}$$

Integrating (f, g) yields

$$\begin{cases} f_1(x) = \dfrac{Px^3}{6EI} + dx + h & \leftarrow \text{Integration const} \\[2ex] f(y) = \dfrac{-\nu P y^3}{6EI} + \dfrac{Py^3}{6GI} - \left(\dfrac{Pc^2}{2GI} + d\right)y + g \end{cases}$$

integration constant

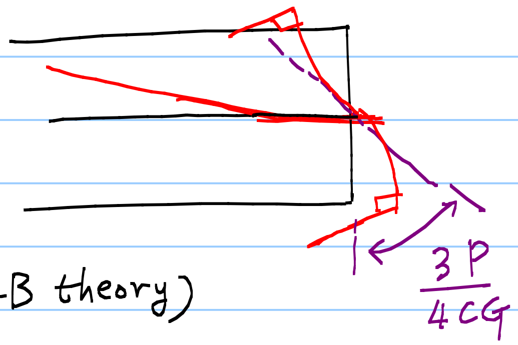
☑ Remark: to determine d, h, g
We need to provide three conditions
$\Rightarrow$ consider the disp condition $x = l$:



at $x = l$ and $y = 0$

① $u = 0$

② $v = 0$



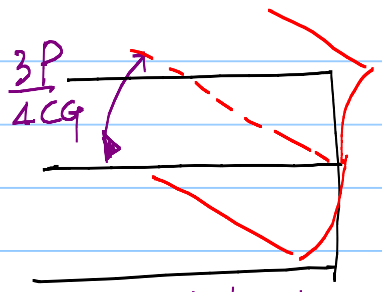
③ $\left.\dfrac{\partial v}{\partial x}\right|_{y=0} = 0$

Slope of neutral axis $= 0$
(usual condition in E-B theory)

OR

③′ $\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = 0$

Slope of normal $= 0$



$\left.\dfrac{\partial v}{\partial x}\right|_{x=l,\,y=0} \neq 0$
(Due to shear angle)

Case I: Solving for 1, 2, 3

$$d = -\frac{Pl^2}{2EI}, \quad h = \frac{Pl^3}{3EI}, \quad g = 0$$

$$u = -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6GI} + \left(\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}\right)y$$

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI}$$

1) Along $y=0$

- $$v\Big|_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI}$$

  (Deflection of neutral axis)

- $$\therefore \; v\Big|_{\substack{x=0\\y=0}} = \frac{Pl^3}{3EI}$$

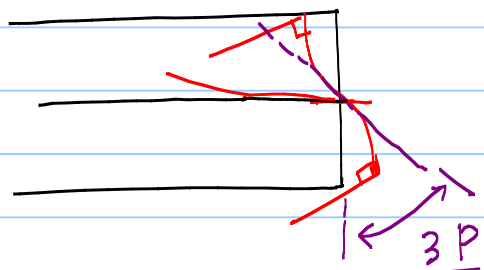
  (tip displacement)

2) Chek u displacement at $x=l$

$$u\Big|_{x=l} = -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6GI} - \frac{Pc^2y}{2GI}$$

$$\left.\frac{\partial u}{\partial y}\right|_{x=l} = \frac{-\nu P y^2}{2EI} + \frac{P y^2}{2GI} - \frac{P c^2}{2GI}$$

$$\therefore \left.\frac{\partial u}{\partial y}\right|_{\substack{x=l \\ y=0}} = -\frac{P c^3}{2GI} = -\frac{3}{4}\frac{P}{cG}$$



$\frac{3P}{4CG}$ ← due to shear stress $\left.\sigma_{xy}\right|_{y=0} = -\frac{3P}{4c}$

→ Normals do not remain normal; "distortion" occurs.

(II) Case II: Solving for ①, ②, ③'

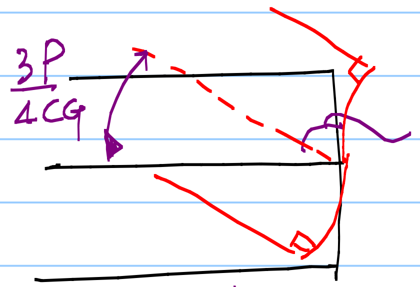
$$\left.v\right|_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2 x}{2EI} + \frac{Pl^3}{3EI} + \frac{Pc^2}{2GI}(l-x)$$

← additional term

shear modulus

$$\left.\frac{\partial v}{\partial x}\right|_{y=0} = \frac{Px^2}{2EI} - \frac{Pl^2}{2EI} - \frac{Pc^2}{2GI}$$

$$\left.\frac{\partial v}{\partial x}\right|_{\substack{y=0\\x=l}} = -\frac{Pc^2}{2GI} = -\frac{3P}{4CG}$$



Rotation of normal

not 90° because $\sigma_{xy} \neq 0$ here

$$\left.\frac{\partial v}{\partial x}\right|_{x=l,\, y=0} \neq 0$$

(Due to shear angle)

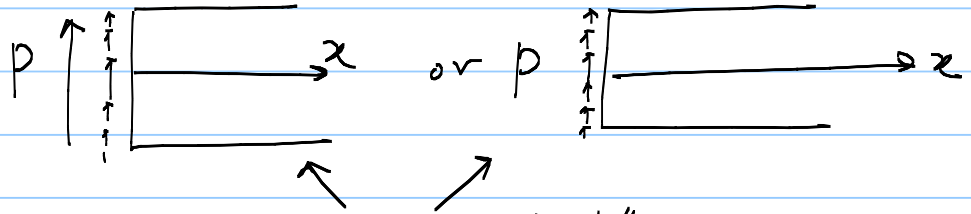
※ Observation from CASE I & CASE II

① $$\underset{\text{CASE II}}{\left. v \right|_{y=0}} = \underset{\text{CASE I}}{\left. v \right|_{y=0}} + \underbrace{\frac{Pc^2}{2IG}(l-x)}_{\text{positive term}}$$

additional deflection due to rotation of the neutral axis

$\therefore$ $\dfrac{Pc^2}{2IG}(l-x)$: gives an estimate of the effect of shearing force on the deflection

(2) What if the applied load is not distributed as a power law?



- same "Resultant" but different distribution

$\Rightarrow$ Solutions (stress field) away from the loaded end are virtually same for the two loads as long as the resultant is the same

"St. Venant Principle"

- Stress difference in two solution;
  → decays rapidly (usally exponentially)

These local solutions : "End-effects" (solution)