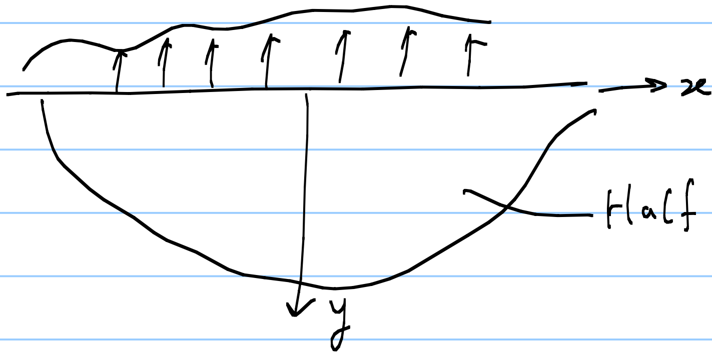
Plane Problems in a Half space



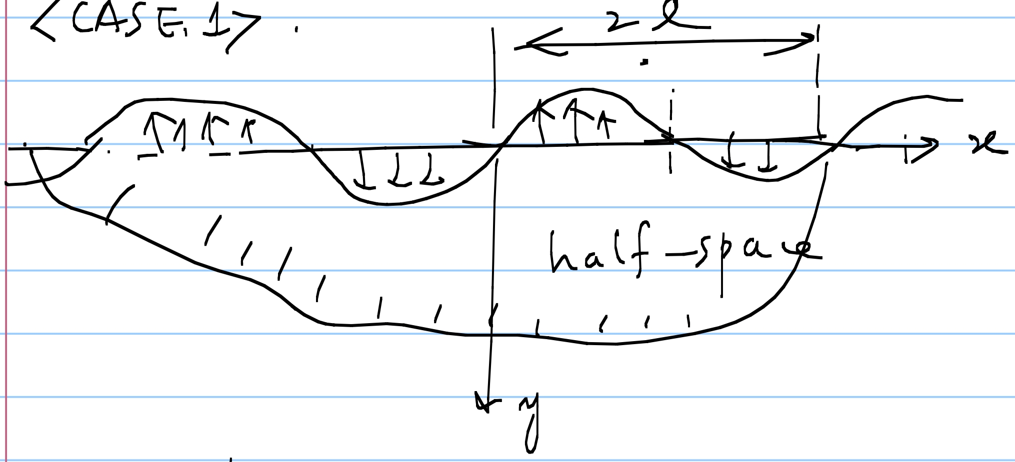
$$\begin{cases} |x| < \infty \\ y \geq 0 \end{cases}$$

Approach : i) Problems for periodic loads (period : L)
→ use Fourier series expansion
& solve Biharmonic eq for ϕ

ii) Problems for non-periodic loads
either ① Fourier transform
② take the limit of $L = \infty$
from solution (i)
(Remark : ①, ② : actually the same)

<< Fourier Series Solution >>

< CASE. 1 >



Boundary condition at $y=0$

$$\begin{cases} \sigma_{xy}(x,0) = 0 \\ \sigma_{yy}(x,y) = f(x) \end{cases}$$

↑ assume periodic with period 2ℓ

$f(x) = -f(-x)$ (odd ftn)

- One can expand $f(x)$:

$$f(x) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{\ell}$$

$$F_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin\frac{n\pi x}{\ell}\, dx$$

$\Rightarrow$ How to choose $\phi(x,y)$ that can solve this problem?

i) $\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$

ii) $\sigma_{yy}\Big|_{y=0} = \sum F_n \sin\frac{n\pi x}{l}$

$\Rightarrow$ <u>One may try</u>

$$\phi(x,y) = \sum_{n=1}^{\infty} g_n(y) \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} \phi_n(x,y) \qquad \text{—(A)}$$

put (A) into $\nabla^4 \phi(x,y)$

$$= \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)\phi(x,y) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[\left(\frac{n\pi}{l}\right)^4 g_n(y) - 2\left(\frac{n\pi}{l}\right)^2 g_n''(y) + g_n''''(y)\right] \sin\frac{n\pi x}{l} = 0$$

$$\rightarrow \quad g_n''''(y) - 2\gamma_n^2 g_n''(y) + \gamma_n^4 g_n(y) = 0 \quad (*)$$

where $\quad \gamma_n = \dfrac{n\pi}{\ell} > 0$

Let $\quad g_n(y) = A e^{sy}$,

Then

$$s^4 - 2\gamma_n^2 s^2 + \gamma_n^4 = (s^2 - \gamma_n^2)^2 = 0$$

$$\therefore \quad s = \pm\gamma_n \text{ (double root)}$$

$$\Rightarrow \quad g_n(y) = A_n e^{-\gamma_n y} + B_n e^{+\gamma_n y} + C_n y e^{-\gamma_n y} + D_n y e^{+\gamma_n y}$$

Because $\quad \sigma_{ij} \rightarrow$ finite as $y \rightarrow \infty$

$$B_n = D_n \equiv 0$$

$$\therefore \quad g_n = (A_n + C_n y) e^{-\gamma_n y}$$

$$\therefore\ \phi(x,y) = \sum \phi_n(x,y) = \sum g_n \sin\frac{n\pi x}{\ell}$$

$$= \sum_{n=1}^{\infty} (A_n + C_n y)\, e^{-\gamma_n y} \sin\frac{n\pi x}{\ell} \quad (**)$$

To determine A_n, C_n, use BC's on $y=0$

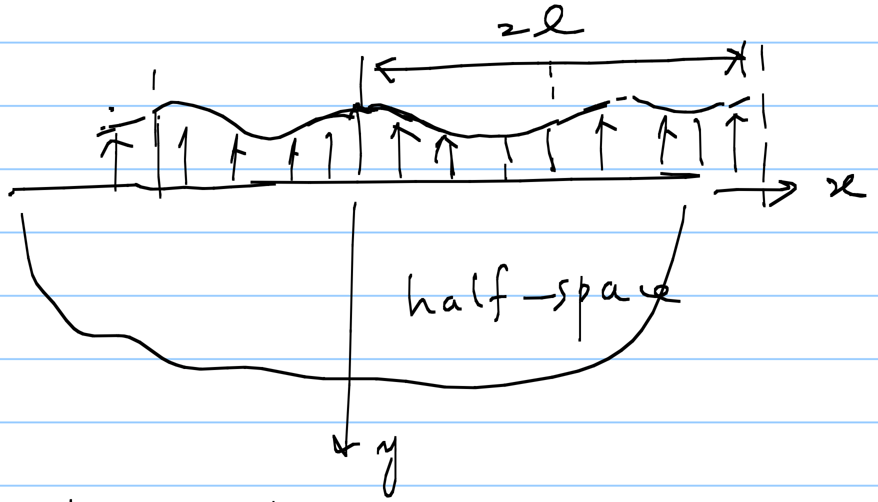
$$\begin{cases} \sigma_{xy}(x, y=0) = -\left.\dfrac{\partial^2\phi}{\partial x \partial y}\right|_{y=0} = 0 \\ \sigma_{yy}(x, y=0) = \left.\dfrac{\partial^2\phi}{\partial x^2}\right|_{y=0} = \displaystyle\sum_{n=1}^{\infty} F_n \sin\gamma_n x \end{cases}$$

$$\Rightarrow\ A_n = \frac{-F_n}{\gamma_n^2}\ ; \quad C_n = -\frac{F_n}{\gamma_n}$$

$$\boxed{\therefore\ \phi(x,y) = -\sum_{n=1}^{\infty} \frac{F_n}{\gamma_n^2}(1+\gamma_n y)\, e^{-\gamma_n y} \sin\gamma_n x \quad (y \geq 0)}$$

(solve problem 2.4.1 as a homework —)

<CASE 2>



Boundary condition

$$\begin{cases} \sigma_{xy} = 0 \\ \sigma_{yy} = h(x) \end{cases}$$

$$\begin{cases} 2l\text{-periodic} \\ h(x) = h(-x) \end{cases}$$

- One can expand $h(x)$ as

$$h(x) = \sum_{n=0}^{\infty} H_n \cos \frac{n\pi x}{l} = H_0 + \sum_{n=1}^{\infty} H_n \cos \frac{n\pi x}{l}$$

where
$$H_n = \frac{2}{l} \int_0^l h(x) \cos \frac{n\pi x}{l}\, dx \quad (n \neq 0)$$

$$H_0 = \frac{1}{l} \int_0^l h(x)\, dx$$

As before

" Try "

$$\phi(x,y) = \sum_{n=0}^{\infty} g_n(y) \cos\frac{n\pi x}{\ell}$$
$$\equiv \sum_{n=0}^{\infty} \phi_n(x,y) \qquad (*)$$

$(*) \rightarrow \nabla^4 \phi(x,y) = 0$

$$g_n''''(y) - 2\gamma_n^2 g_n''(y) + \gamma_n^4 = 0$$

$$(s^2 - \gamma_n^2)^2 = 0 \rightarrow s = \pm\gamma_n \text{ (double roots)}$$

i) For $n \neq 0$, same analysis as before

$$g_n(y) = A_n e^{-\gamma_n y} + C_n y^{-\gamma_n y}$$
$$(g_n(y) \rightarrow \text{finite as } y \rightarrow \infty)$$

ii) For $n = 0$,

$$s^4 = 0 \qquad (g_0''''(y) = 0)$$
$$g_0(y) = A_0 + B_0 y + C_0 y^2 + D_0 y^3$$
$$\phi_0(x,y) = g_0(y) \cos\frac{0\pi x}{\ell} = g_0(y)$$

$$\rightarrow \begin{cases} \sigma_{xx} = \dfrac{\partial^2 \phi}{\partial y^2} = 2C_0 + 6D_0 y \\ \sigma_{yy} = \dfrac{\partial^2 \phi}{\partial x^2} = 0 \\ \sigma_{xy} = 0 \end{cases} \Rightarrow \boxed{\text{cannot match "}\sigma_{yy} = H_0 \text{ term" on } y=0}$$

Observation:

i) Boundary condition $(\sigma_{yy}|_{y=0},\ \sigma_{xy}|_{y=0})$: periodic

ii) tried ϕ as a periodic ftn

iii) $\sigma_{xy}\big|_{y=0} = \dfrac{\partial^2 \phi}{\partial x^2}\Big|_{y=0}$ ← require 2nd derivative of $\phi(x,y)$

iv) special case: $\underbrace{\sigma_{xy}\big|_{y=0} = \text{const}}_{\text{viewed as a periodic ftn}}$

But periodic $\phi(x,y)$ cannot produce $\sigma_{xy}|_{y=0} = \text{const}$

v) ⇒ must look for other $\phi(x,y)$ yielding $\sigma_{xy}|_{y=0} = \text{const}$

Try $\phi(x,y) = \frac{A_0}{2} x^2$

$\uparrow$ polynomial sol.

$\Rightarrow$ $\sigma_{xx} = 0, \quad \sigma_{yy} = A_0, \quad \sigma_{xy} = 0$

$$\therefore \phi(x,y) = \frac{A_0}{2} x^2 + \sum_{n=1}^{\infty} \left(A_n e^{-\gamma_n y} + C_n y e^{-\gamma_n y} \right) \cos \frac{n\pi x}{l}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

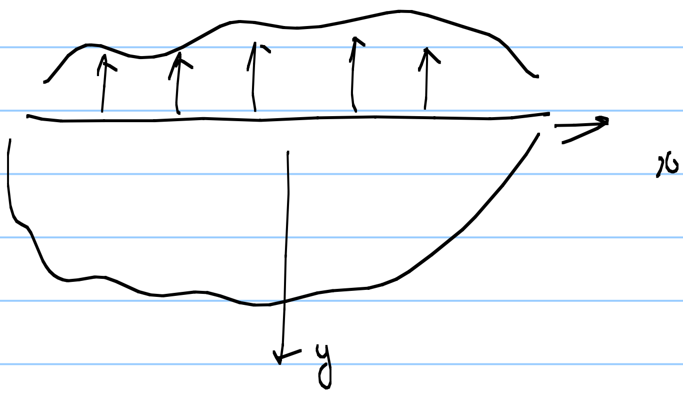
$$\begin{cases} \sigma_{yy}\big|_{y=0} = H_0 + \sum_{n=1}^{\infty} H_n \cos \frac{n\pi x}{l} \\ \sigma_{xy}\big|_{y=0} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = \sum_{n=1}^{\infty} H_n (1 - \gamma_n y) e^{-\gamma_n y} \cos \frac{n\pi x}{l} \\ \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = H_0 + \sum_{n=1}^{\infty} H_n (1 + \gamma_n y) e^{-\gamma_n y} \cos \frac{n\pi x}{l} \\ \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = \sum_{n=1}^{\infty} H_n \gamma_n y e^{-\gamma_n y} \sin \frac{n\pi x}{l} \end{cases}$$

« Fourier Transform »

↑ applies to non-periodic ftn

$(-\infty < x < \infty)$



$$\begin{cases} \sigma_{xy} = 0 \\ \sigma_{yy} = f(x) \end{cases}$$ ← non-periodic ftn

$$\left(\left| \int_{-\infty}^{\infty} f(x)\, dx \right| < \infty \right)$$

Fourier transform pairs

"Forward"

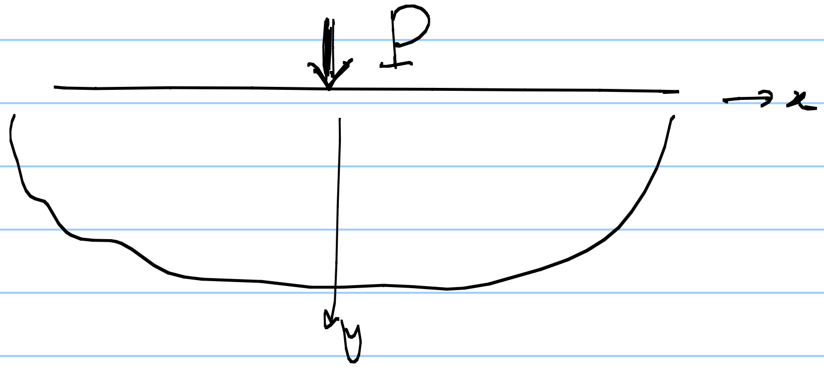
$$G(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{+i\gamma x}\, dx$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\gamma)\, e^{-i\gamma x}\, d\gamma$$

"Inverse"

(Detailed procedure : skipped
$\Rightarrow$ see 2.4.2 and Appendix A-1)

For $f(x) = -P_y \delta(x)$



One can show

$$\boxed{\left\{\begin{aligned} &\phi(r,\theta) = \frac{P_y}{\pi} r\theta\cos\theta \\ &\boxed{\begin{aligned}&\sigma_{rr}(r,\theta) = -\frac{2P_y}{\pi}\frac{\sin\theta}{r} \\ &\sigma_{\theta\theta}(r,\theta) = \sigma_{r\theta} = 0\end{aligned}}\end{aligned}\right.}$$

$\rightarrow$ $\sigma = O\left(\frac{1}{r}\right)$

or

$$\phi(x,y) = \frac{P_y}{\pi}\,\theta x \qquad (\theta = \tan^{-1} y/x)$$

$$\sigma_{xx}(x,y) = -\frac{2P_y x^2 y}{\pi (x^2+y^2)^2}$$

$$\sigma_{yy}(x,y) = -\frac{2P_y y^3}{\pi (x^2+y^2)^2}$$

$$\sigma_{xy}(x,y) = -\frac{2P_y x y^2}{\pi (x^2+y^2)^2}$$

▫ Displacement field (p−σ)

$$(\sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta}) \xrightarrow[\text{condition}]{p-\sigma} (\varepsilon_{rr}, \varepsilon_{r\theta}, \varepsilon_{\theta\theta}) \longrightarrow (u_r, u_\theta)$$

(see Textbook for details, pp. 192−195)

$$u_r(r,\theta) = -\frac{2P_y}{\pi E}\ln r\sin\theta + \frac{(1-\nu)P_y}{\pi E}\theta\cos\theta + h_1\cos\theta + h_2\sin\theta \quad (*)$$

$$u_\theta(r,\theta) = -\frac{2\nu P_y}{\pi E}\cos\theta - \frac{2P_y}{\pi E}\ln r\cos\theta - \frac{(1-\nu)P_y}{\pi E}(\theta\sin\theta + \cos\theta) - h_1\sin\theta + h_2\cos\theta + kr \quad (**)$$

Note: h_1 ← rigid-body translation in x
h_2 ← " " in y
K ← rigid-body rotation

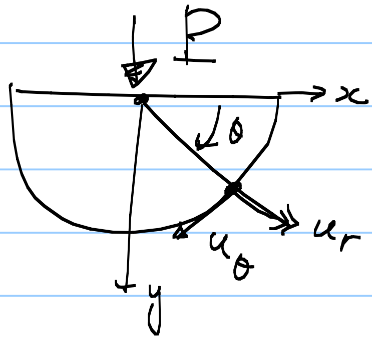
(check: h_1, h_2-terms

$$u_x = u_r\cos\theta - u_\theta\sin\theta$$
$$= (h_1\cos\theta + h_2\sin\theta)\cos\theta - (-h_1\sin\theta + h_2\cos\theta)\sin\theta$$
$$= h_1(\cos^2\theta + \sin^2\theta) = h_1$$

$$u_y = u_r \sin\theta + u_\theta \cos\theta$$
$$= (h_1 \cos\theta + h_2 \sin\theta)\sin\theta$$
$$+ (-h_1 \sin\theta + h_2 \cos\theta)\cos\theta$$
$$= h_2(\sin^2\theta + \cos^2\theta) = h_2$$

h_1, h_2, k : arbitrary,

but we ensure that



$$u_\theta\Big|_{x=0} \equiv 0 :$$

or
$$u_\theta\Big|_{\theta=\pi/2} = 0$$

$$u_\theta\left(r, \theta = \tfrac{\pi}{2}\right) \underset{(*)}{=} \underbrace{-\frac{(1-\nu)P_y}{2E}}_{\text{const}} - h_1 + \underbrace{kr}_{\text{ftn of } r} \equiv 0$$

$$\Rightarrow \begin{cases} \underline{k = 0} \\ \underline{h_1 = -\dfrac{(1-\nu)P_y}{2E}} \end{cases}$$

Since there is no condition for Rigid-body translation in y (i.e, for h_2)

set $\underline{h_2 = 0}$

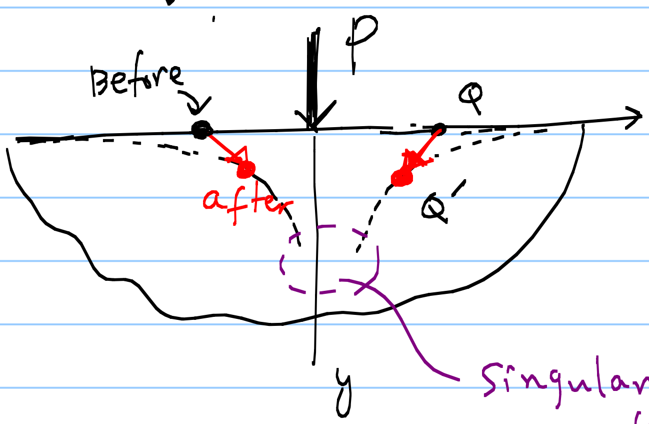
Final result

$$\begin{cases} u_r(r,\theta) = -\dfrac{2P_y}{\pi E}\ln r\sin\theta + \dfrac{(1-\nu)P_y}{\pi E}\left(\theta-\dfrac{\pi}{2}\right)\cos\theta \\ u_\theta(r,\theta) = -\dfrac{2\nu P_y}{\pi E}\cos\theta - \dfrac{2P_y}{\pi E}\ln r\cos\theta \\ \qquad\qquad - \dfrac{(1-\nu)P_y}{\pi E}\left(\theta-\dfrac{\pi}{2}\right)\sin\theta \\ \qquad\qquad - \dfrac{(1-\nu)P_y}{\pi E}\cos\theta \end{cases}$$

Then

$$\begin{cases} u_r(r,\theta=0) = -\dfrac{(1-\nu)P_y}{2E} \\ u_r(r,\theta=\pi) = -\dfrac{(1-\nu)P_y}{2E} \end{cases}$$

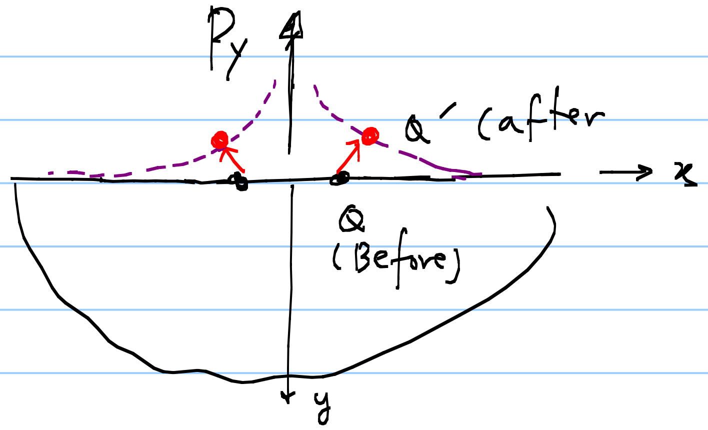
☆ ** Physics



Displacement field

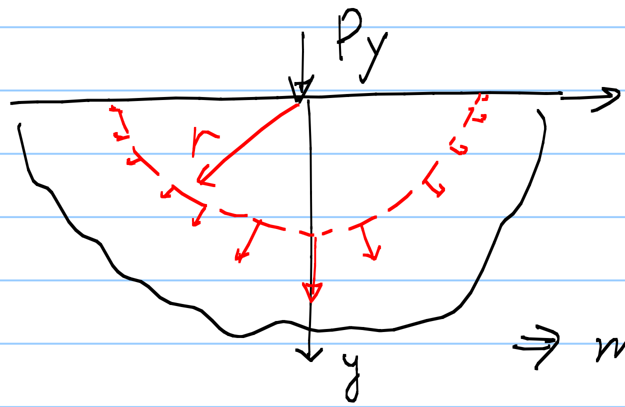
Singular Behavior $[u_y = u_\theta]_{x=0} \sim \ln r$

Compare:



☑ Review of Stress

$$\sigma_{rr} = -\frac{2P_y}{\pi}\frac{\sin\theta}{r}, \quad \sigma_{r\theta} = \sigma_{\theta\theta} = 0 \Rightarrow \underline{\sigma = O\left(\frac{1}{r}\right)}$$



☑ Eqm. Condition

$$\underline{P_y + \int_0^{\pi} t_y \, r d\theta = 0}$$

$\Rightarrow$ must valid for any r

$\therefore$ $\underline{t_y \sim O\left(\frac{1}{r}\right)}$

Stress