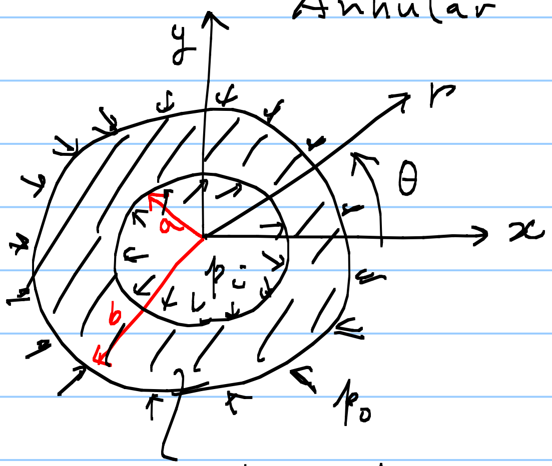
<Bring the textbook>

Plane Elasticity in Polar Coordinates

<Axisymmetric problems in Rings or Annular Bodies>



annular body or Ring

BC's:

$$\begin{cases} \text{at } r=a, & \sigma_{rr}=-p_i, \ \sigma_{r\theta}=0 \\ \text{at } r=b, & \sigma_{rr}=-p_o, \ \sigma_{r\theta}=0 \end{cases}$$

$$\nabla^4\phi(r,\theta) = \nabla^2\nabla^2\phi(r) \quad \text{axisymmetric case}$$

$$= \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\phi(r)$$

$$= \frac{d^4\phi}{dr^4}+\frac{2}{r}\frac{d^3\phi}{dr^3}-\frac{1}{r^2}\frac{d^2\phi}{dr^2}+\frac{1}{r^3}\frac{d\phi}{dr}$$

$$=0$$

Try $\phi(r) \sim r^k$

$$\rightarrow \quad k(k-1)(k-2)(k-3)$$
$$+2k(k-1)(k-2) - k(k-1) + k = 0$$

$$k\left\{(k-1)(k-2)(k-3) + 2(k-1)(k-2) \underbrace{-(k-1)+1}_{-(k-2)}\right\} = 0$$

$$k(k-2)\{\underbrace{(k-1)(k-3)+2(k-1)-1}_{\underbrace{k^2-4k+3+2k-2-1}_{\underbrace{k^2-2k}_{k(k-2)}}}\} = 0$$

$$\Rightarrow \quad k^2(k-2)^2 = 0$$

$$\therefore \quad k^2 = 0 \rightarrow \quad 1, \ \ln r$$
$$(k-2)^2 = 0 \rightarrow \quad r^2, \ r^2 \ln r$$

$$\therefore \boxed{\phi(r) = a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0}$$

Using $\quad \sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \quad \sigma_{r\theta} = 0$

$$\begin{cases} \sigma_{rr} = \frac{a_0}{r^2} + 2b_0 + c_0(1+2\ln r) \\ \sigma_{\theta\theta} = -\frac{a_0}{r^2} + 2b_0 + c_0(3+2\ln r) \\ \sigma_{r\theta} = 0 \end{cases}$$

NOTE:

Three unknowns a_0, b_0, c_0
2 boundary condition

$$\sigma_{rr}\Big|_{r=a} = -p_i \quad ; \quad \sigma_{rr}\Big|_{r=b} = -p_o$$

$\Rightarrow$ It appers to be inconsistent.

* Will be clear when Displacements are examined!!

$$(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}) \xrightarrow[\text{p-}\sigma]{} (\epsilon_{rr}, \epsilon_{\theta\theta}, \epsilon_{r\theta}) \xrightarrow{\text{Integrate}} (u_r, u_\theta)$$

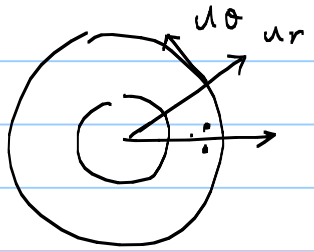
$$u_r = \frac{1}{E}\Big[-\frac{(1+\nu)}{r}a_0 + 2b_0(1-\nu)r + 2(1-\nu)C_0\, r\ln r - (1+\nu)C_0 r\Big] + h\sin\theta + k\cos\theta$$

$$u_\theta = \frac{4C_0 r\theta}{E} + h\cos\theta - k\sin\theta + fr$$

(h, k, f → rigid-body motion)

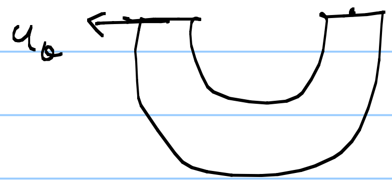
not single-valued
⇒ cannot be used in complete ring

⇒ $C_0 \equiv 0$ for complete Ring



$$\underbrace{\Big[u_\theta\Big|_{\theta=2\pi}\Big]}_{C_0\text{-term}} - \underbrace{\Big[u_\theta\Big|_{\theta=0}\Big]}_{C_0\text{-term}} = \frac{4C_0 r_1(2\pi)}{E} \neq 0$$

(But okay in



Incomplete ring)

⇒ Now we have two unknowns a_0, b_0
and two boundary conditions:

$$\frac{a_0}{a^2} + 2b_0 = -p_i, \quad \frac{a_0}{b^2} + 2b_0 = -p_0$$

⇒
$$\begin{cases} \sigma_{rr} = \dfrac{p_i a^2 - p_0 b^2}{b^2 - a^2} + \dfrac{(p_0 - p_i)\, a^2 b^2}{b^2 - a^2} \dfrac{1}{r} \\[2ex] \sigma_{\theta\theta} = \dfrac{p_i a^2 - p_0 b^2}{b^2 - a^2} - \dfrac{(p_0 - p_i)\, a^2 b^2}{b^2 - a^2} \dfrac{1}{r} \\[2ex] \sigma_{r\theta} = 0 \end{cases}$$

<u>When $p_0 = 0$</u>

p_i

$$\sigma_{rr} = \frac{a^2 p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_{\theta\theta} = \frac{a^2 p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) > p_i$$

☆ always tensile hoop ($\sigma_{\theta\theta}$) stress under internal pressure

$$\sigma_{\theta\theta}\big|_{max} = \sigma_{\theta\theta}\big|_{r=a} = \frac{b^2+a^2}{b^2-a^2}\, p_i > p_i$$

- As $b \to \infty$ (or $a \to 0$)

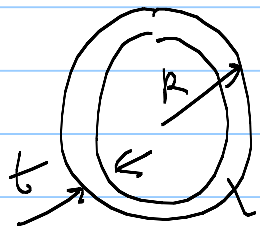
$$\sigma_{\theta\theta}\big|_{max} = p_i$$

- As $b - a = t \ll a = R$

$$\sigma_{\theta\theta}\big|_{max} = \frac{b^2+a^2}{(b-a)(b+a)}\, p_i$$

$$\simeq \frac{R^2}{2tR}\, p_i = \frac{1}{2}\left(\frac{R}{t}\right) p_i$$

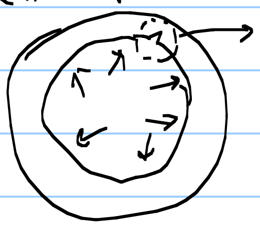
$$= \underbrace{O\left(\frac{R}{t}\right)}\, p_i$$



thin cylinder case

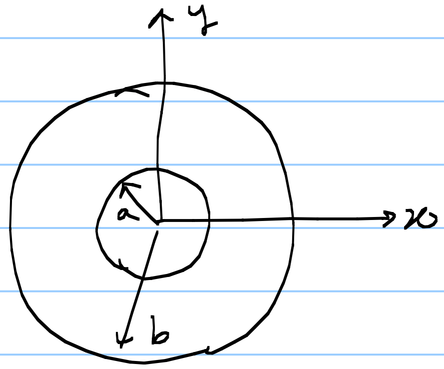
hoop stress Concentration of $O\left(\frac{R}{t}\right)$

Remark :



If there is a crack, positive hoop stress ($\sigma_{\theta\theta}$) may cause serious problems.

« Non-Axisymmetric Case »



※ Boundary conditions and stress/disp fields must be 2π-periodic in θ.

<u>BC</u>

$$\begin{cases} \sigma_{rr}\big|_{r=a} = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \\ \sigma_{r\theta}\big|_{r=a} = C_0 + \sum_{n=1}^{\infty} C_n \cos n\theta + \sum_{n=1}^{\infty} D_n \sin n\theta \end{cases}$$

$$\begin{cases} \sigma_{rr}\big|_{r=b} = A_0' + \sum_{n=1}^{\infty} A_n' \cos n\theta + \sum_{n=1}^{\infty} B_n' \sin n\theta \\ \sigma_{r\theta}\big|_{r=b} = C_0' + \sum_{n=1}^{\infty} C_n' \cos n\theta + \sum_{n=1}^{\infty} D_n' \sin n\theta \end{cases}$$

<u>Remark</u>: Boundary conditions must satisfy <u>global equilibrium</u> for traction boundary value problems

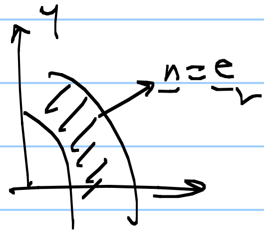
$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_z = 0$$

① $\sum F_x = 0$

$$\overbrace{\int_0^{2\pi} t_x\Big|_{r=b} (b\,d\theta)}^{F_x|_{r=b}} + \overbrace{\int_0^{2\pi} t_x\Big|_{r=a} (a\,d\theta)}^{F_x|_{r=a}} = 0$$

• $t_x\Big|_{r=b} = \underset{\sim}{t}\Big|_{r=b} \cdot \underline{e}_x = \left(\underline{e}_r \cdot \underset{\sim}{\sigma}\right)\Big|_{r=b} \cdot \underline{e}_x$

traction on $r=b$

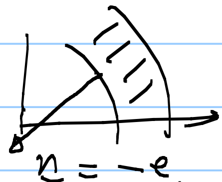


$$= \left(\sigma_{rr}\,\underline{e}_r + \sigma_{r\theta}\,\underline{e}_\theta\right)\Big|_{r=b} \cdot \underline{e}_x$$

$$= \left(\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\right)\Big|_{r=b}$$

• $t_x\Big|_{r=a} = \underset{\sim}{t}\Big|_{r=a} \cdot \underline{e}_x = \left(-\underline{e}_r \cdot \underset{\sim}{\sigma}\right) \cdot \underline{e}_x$

$$= -\left(\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\right)\Big|_{r=a}$$



- $$\int_0^{2\pi} t_x\Big|_{r=b} b\,d\theta = \int_0^{2\pi} \left(\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\right)\Big|_{r=b} b\,d\theta$$

$$= \int_0^{2\pi} \left(A_0' + \sum_{n=1}^{\infty} A_n'\cos n\theta + \sum_{n=1}^{\infty} B_n'\sin n\theta\right)\cos\theta\, b\,d\theta$$

$$- \int_0^{2\pi} \left(C_0' + \sum_{n=1}^{\infty} C_n'\cos n\theta + \sum_{n=1}^{\infty} D_n'\sin n\theta\right)\sin\theta\, b\,d\theta$$

$$\left(\text{Use } \int_0^{2\pi} \cos n\theta\cos\theta\,d\theta = \int_0^{2\pi} \sin n\theta\sin\theta\,d\theta = \pi\delta_{n1}\right)$$

$$= \pi b\,(A_1' - D_1') \leftarrow \Sigma F_x\big|_{r=b}$$

$$\boxed{\text{** only } \cos\theta, \sin\theta \text{ terms in Boundary Conditions contribute to non zero Resultant}}$$

$$\int_0^{2\pi} t_x\Big|_{r=a} a\,d\theta = \pi a\,(A_1 - D_1)$$

$\therefore$ $\Sigma F_x = 0$

$$\hookrightarrow \boxed{(A_1' - D_1')\,b = (A_1 - D_1)\,a}$$

Likewise,

$\sum F_y = 0$

$\hookrightarrow \boxed{(B_1' + C_1')\, b = (B_1 + C_1)\, a}$

$\sum M_z = 0$

$\hookrightarrow \boxed{C_0' b^2 = C_0 a^2}$

Remark: ① Coefficients in the above boxes cannot be prescribed arbitrarily.

② See Table 2.5.1 of Textbook for more details

Solutions to the above BVP?

Try

$$\phi(r,\theta) = \sum_{n=0}^{\infty} R_n(r) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}$$

Then,

$$\nabla^4 \phi(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} \right)^2 R_n(r) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}$$

$$= 0$$

$\therefore$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} \right)^2 R_n(r) = 0$$

Assuming $R_n(r) = r^k$

$$\Rightarrow \quad \left[(k-2)^2 - n^2 \right] \left(k^2 - n^2 \right) = 0$$

$\therefore \quad k = \pm n, \quad k = \pm n + 2$

$$R_0(r) \sim \{ r^2, r^2 \ln r, 1, \ln r \}$$

$$R_1(r) \sim \{ r^3, r, r \ln r \ \ r^{-1} \}$$

$$R_n(r) \sim \{ r^{n+2}, r^{-n+2}, r^n, r^{-n} \} \qquad n \geq 2$$

Airy Stress Function $\phi(r, \theta)$

$$\begin{aligned} \phi(r,\theta) = & \ a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0 \\ & + (a_1 r + b_1 r^3 + a_1' r^{-1} + b_1' r \ln r) \cos\theta \\ & + (c_1 r + d_1 r^3 + c_1' r^{-1} + d_1' r \ln r) \sin\theta \\ & + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2}) \cos n\theta \\ & + \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2}) \sin n\theta \end{aligned}$$

$\boxed{\text{Examine the characteristics of terms in } \phi}$

$\langle n=0 \rangle$ axisym

$\begin{cases} C_0 \to \text{multiply-valued} & C_0 \equiv 0 \text{ for complete ring} \\ d_0 \to \text{trivial} & \end{cases}$

(i.e., $\phi = d_0 \to$ no meaningful sol.)

⊙ no-solution for $\Sigma M_z \neq 0$ (cannot solve for $C_0 \neq 0$, $C_0' \neq 0$)

$\langle n=1 \rangle$ $\quad a_1 \to$ trivial

$\phi \sim \cos\theta$ $\quad \underline{b_1'} \to$ multiply-valued

(Keep it for the time-being)

$(p-\sigma)$ $\left(\begin{array}{l} E u_r = 2b_1' \theta \sin\theta, \\ E u_\theta = 2b_1' \theta \cos\theta \end{array} \right)$

$b_1, a_1', b_1' \Rightarrow \sum F_x = 0, \sum F_y = 0$

→ No term has net Resultant!

$\phi \sim \sin\theta$ $c_1 \to$ trivial

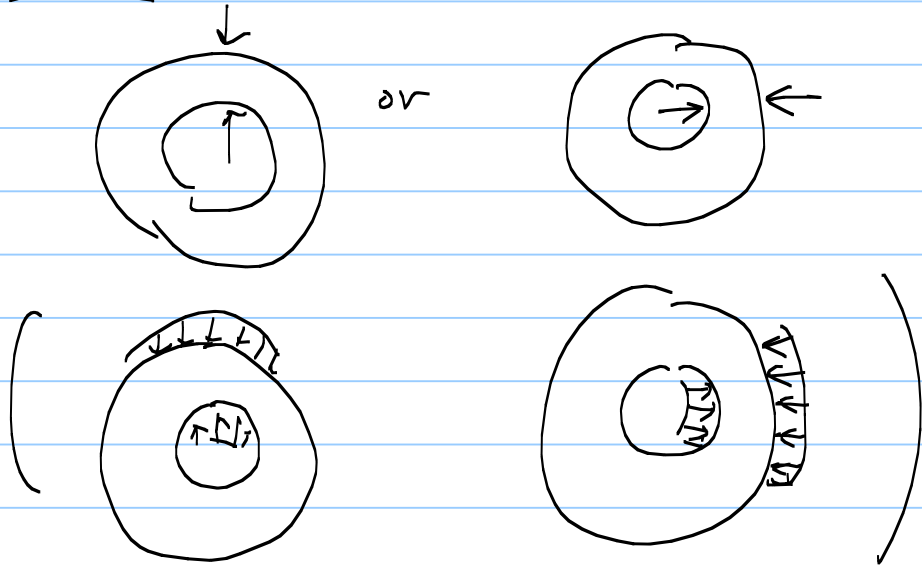
$d_1' \to$ multiply-valued
(keep it for the time-being)

$(p-\sigma)$ $\left(\begin{aligned} E u_r &= 2 d_1' \theta \cos\theta \\ E u_\theta &= 2 d_1' \theta \sin\theta \end{aligned} \right)$

$d_1, c_1', d_1' \Rightarrow \sum F_x = 0, \sum F_y = 0$

→ no term has net resultant!

So, With these solutions we cannot solve



$n \geq 2$

$\phi \sim \cos n\theta$ all self-equilibrated (i.e. $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M_z = 0$)

$\phi \sim \sin n\theta$ and single-valued

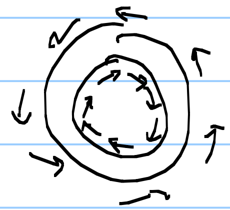
$\rightarrow$ Okay.

$\rightarrow$ Not all necessary solutions are in the form of $\phi(r,\theta) = \sum_{n=0}^{\infty} R_n(r) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}$.

(A) To be able to solve for (C_0, C_0')

$$\sigma_{r\theta}\big|_{r=a} = C_0, \quad \sigma_{r\theta}\big|_{r=b} = C_0'$$

$$\boxed{\sigma_{r\theta} \times r^2 = \text{CONST}} \leftarrow (C_0' b^2 = C_0 a^2)$$



$$\Rightarrow \boxed{\sigma_{r\theta} \sim \frac{1}{r^2} \text{ behavior}}$$

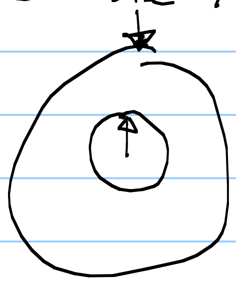
Recall

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$\boxed{\text{Add}}$ $\quad \underline{\phi(r, \theta) = a_0' \theta}$

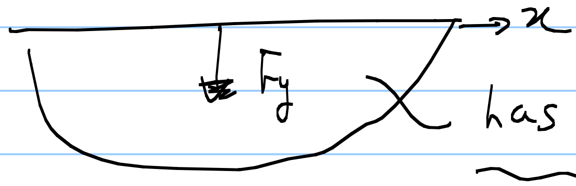
$$\rightarrow \sigma_{r\theta} = -a_0'$$

(B) To be able to solve for B_1', C_1', (B_1, C_1)



For $\sum F_y\big|_{r=a} = \sum F_y\big|_{r=b} \neq 0$

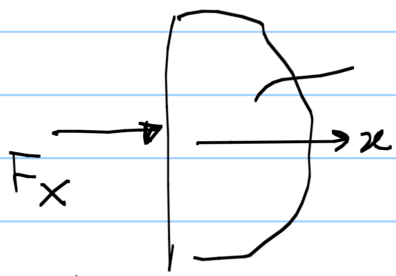
USE ① Solution to the half-space with $F_y \neq 0$



has $\sum F_y \neq 0$

$$\phi_{①} = -\frac{C_1}{2} r\theta \cos\theta$$

② Solution to the half-space with $F_x \neq 0$



has $\sum F_x \neq 0$

$$\phi_{②} = \frac{a_1}{2} r\theta \sin\theta$$

$$E u_r(r,\theta) = C_1 \ln r \sin\theta + \frac{1-\nu}{2} C_1 \theta\cos\theta \quad (*)$$

☒ Remark 1) $\phi_{①}, \phi_{②}$ has multiply-valued Displacements!

(Check: $\phi \to \sigma_{ij} \to \varepsilon_{ij} \to u_r, u_\theta$)

☒ Remark 2) : the multivaluedness of $\phi_{①}$ and $\phi_{②}$ can be corrected if b_1' and d_1'-terms are superposed.

Result

$$\begin{cases} b_1' = -a_1 \dfrac{(1-\nu)}{4}, \\ d_1' = -c_1 \dfrac{(1-\nu)}{4}. \end{cases} \quad (p-\sigma)$$

Thus, the final form of $\phi(r,\theta)$

$$\phi(r,\theta) = a_0 \ln r + b_0 r^2 + a_0'\theta$$
$$+\left[\frac{a_1}{2} r\theta\sin\theta + (b_1 r^3 + a_1' r^{-1} + b_1' r\ln r)\cos\theta\right]$$

$$+\left[-\frac{c_1}{2} r\theta\cos\theta + \left(d_1 r^3 + c_1' r^{-1} + d_1' r\ln r\right)\sin\theta\right]$$

$$+\sum_{n=2}^{\infty}\left(a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2}\right)\cos n\theta$$

with

$$b_1' = -a_1\frac{(1-\nu)}{4}, \quad d_1' = -c_1\frac{(1-\nu)}{4}.$$

Using $\phi \sim \sigma_{ij}$ relation

$\sigma_{rr}(r,\theta) = ---$ (2.5.32)

$\sigma_{r\theta}(r,\theta) = --$ (2.5.33)

$\sigma_{\theta\theta}(r,\theta) = ---$ (2.5.34)

For $p-\sigma$

$E u_r(r,\theta) = ---$ (2.5.35)

$E u_\theta(r,\theta) = ---$ (2.5.36)

Using BC's in Fourier series
and general solutions in Eqs (2.5.32~36)

⟨n=0⟩

- Axisymmetric

$$\begin{bmatrix} \frac{1}{a^2} & 2 \\ \frac{1}{b^2} & 2 \end{bmatrix} \begin{Bmatrix} a_0 \\ b_0 \end{Bmatrix} = \begin{Bmatrix} A_0 \\ A_0' \end{Bmatrix}$$

- Torsional

$$\begin{Bmatrix} \frac{1}{a^2} \\ \frac{1}{b^2} \end{Bmatrix} a_0' = \begin{Bmatrix} C_0 \\ C_0' \end{Bmatrix}$$

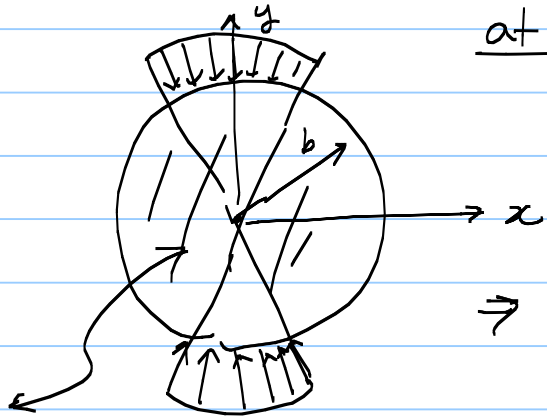
⟨n=1⟩

$$\begin{bmatrix} \frac{3+\nu}{4}\frac{1}{a} & 2a & -\frac{2}{a^3} \\ -\frac{1-\nu}{4}\frac{1}{a} & 2a & -\frac{2}{a^3} \\ \frac{3+\nu}{4}\frac{1}{b} & 2b & -\frac{2}{b^3} \end{bmatrix} \begin{Bmatrix} a_1 \\ b_1 \\ a_1' \end{Bmatrix} = \begin{Bmatrix} A_1 \\ D_1 \\ A_1' \end{Bmatrix}$$

similar equation for (c_1, d_1, c_1')

⟨n≥2⟩ 4×4 matrix equation
(2.5.40)

Example 2.5.1



Sol: must finite at $r=0$

at $r=b$

$$\begin{cases} \sigma_{rr} = -p & 45^\circ < |\theta| < 135^\circ \\ \sigma_{r\theta} = 0 \end{cases}$$

$\Rightarrow$ All coefficients are zero except A_n' terms

$$\sigma_{rr}\Big|_{r=b} = A_0' + \sum_{n=1}^{\infty} A_n' \cos n\theta$$

$(B_n' = 0)$

Step 1 : Compute A_n'

$$A_0' = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sigma_{rr}\Big|_{r=b} d\theta \quad \left(\text{or } \frac{1}{2\pi}\int_0^{2\pi} \sigma_{rr}\Big|_{r=b} d\theta \right)$$

$$= \frac{1}{2\pi}\left[\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}}\right](-p)\,d\theta$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)(-p) = -\frac{p}{2}$$

$$A_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} \sigma_{rr}\Big|_{r=b} \cos n\theta \, d\theta$$

$$= \frac{1}{\pi}\left(\int_{\pi/4}^{3\pi/4} (-p) \cos n\theta \, d\theta\right) \times 2$$

$$= -\frac{2p}{\pi}\left[\frac{\sin n\theta}{n}\right]_{\pi/4}^{3\pi/4}$$

$$\left(\begin{array}{l}
n=1; \quad \sin\theta\Big]_{\pi/4}^{3\pi/4} = \sin\frac{3\pi}{4} - \sin\frac{\pi}{4} = 0 \\
n=2; \quad \sin 2\theta\Big]_{\pi/4}^{3\pi/4} = \sin\frac{3\pi}{2} - \sin\frac{\pi}{2} = -2 \\
n=3; \quad \sin 3\theta\Big]_{\pi/4}^{3\pi/4} = \sin\frac{9\pi}{4} - \sin\frac{3\pi}{4} = 0 \\
n=4; \quad \sin 4\theta\Big]_{\pi/4}^{3\pi/4} = \sin 3\pi - \sin\pi = 0 \\
n=5; \quad \sin 5\theta\Big]_{\pi/4}^{3\pi/4} = \sin\frac{15}{4}\pi - \sin\frac{5\pi}{4} = 0 \\
n=6; \quad \sin 6\theta\Big]_{\pi/4}^{3\pi/4} = \sin\frac{9}{2}\pi - \sin\frac{3\pi}{2} = 2
\end{array}\right)$$

$$\vdots$$

$$= \begin{cases} (-1)^{\frac{n-2}{4}} \dfrac{4p}{n\pi} & (n=2, 6, 10, 14 \cdots \\ 0 & \text{else} \end{cases}$$

(1) For $n=0$

$$\begin{bmatrix} \frac{1}{a^2} & 2 \\ \frac{1}{b^2} & 2 \end{bmatrix} \begin{Bmatrix} a_0 \\ b_0 \end{Bmatrix} = \begin{Bmatrix} A_0 \\ A_0' \end{Bmatrix}$$

Because $\sigma_{rr} = \frac{a_0}{r^2} \to \infty$ as $r \to 0$, $a_0 \equiv 0$.

$$\to \boxed{2b_0 = A_0'} \equiv -\frac{p}{2}$$

$\therefore$ for $n=0$

$$\therefore \quad b_0 = -\frac{p}{4}$$

(2) For $n=1$, all BC's $=0$

(3) For $n \geq 2$

eliminate a_n', b_n'

because $\sigma_{rr} \sim \left\{ \begin{array}{l} a_n' r^{-n-2} \\ b_n' r^{-n} \end{array} \right. \to \infty$ as $r \to 0$

$\therefore$

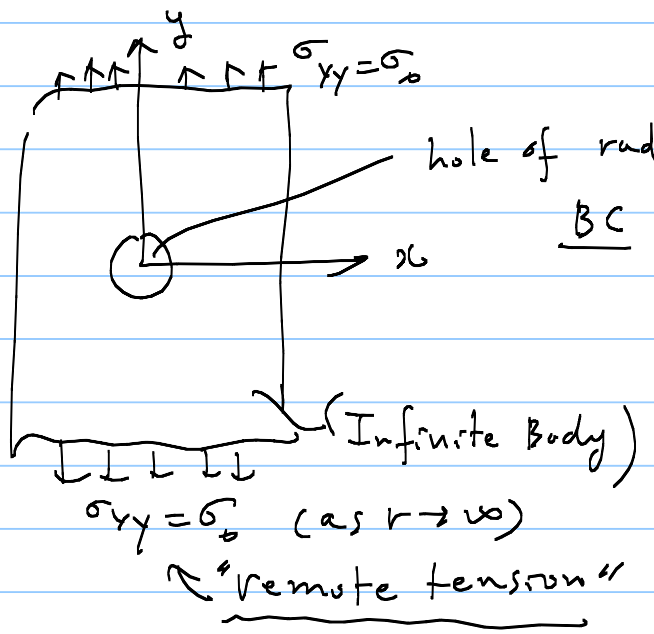
$n=2$ $$\begin{bmatrix} -2 & 0 \\ 2 & 6b^2 \end{bmatrix} \begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix} = \begin{Bmatrix} \frac{2p}{\pi} \\ 0 \end{Bmatrix}$$

$$\Rightarrow a_2 = -\frac{p}{\pi}, \quad b_2 = \frac{p}{3\pi b^2}$$

$n = 6 \ldots$, $\quad n = 10, \ldots$

$$\therefore \quad \sigma_{rr}(r,\theta) = 2b_0 - 2a_2\cos 2\theta + O(r^2)$$
$$= -\frac{p}{2} + \frac{2p}{\pi}\cos 2\theta + O(r^2)$$
$$\sigma_{r\theta}(r,\theta) = -\frac{2p}{\pi}\sin 2\theta + O(r^2)$$
$$\sigma_{\theta\theta}(r,\theta) = -\frac{p}{2} - \frac{2p}{\pi}\cos 2\theta + O(r^2)$$

Stress Concentration around a hole?



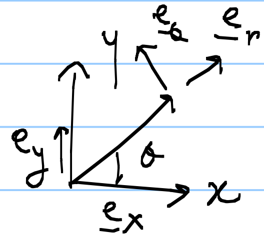
BC at $r = a$

$$\begin{cases} \sigma_{rr}|_{r=a} = 0 \\ \sigma_{r\theta}|_{r=a} = 0 \end{cases}$$

Given stress field at ∞

$$\sigma_{yy} = \sigma_0$$
$$\sigma_{xy} = \sigma_{xx} = 0 \quad \Rightarrow \quad \underset{\sim}{\sigma} = \sigma_0 \, \underline{e}_y \otimes \underline{e}_y$$

Let's convert in polar coordinates

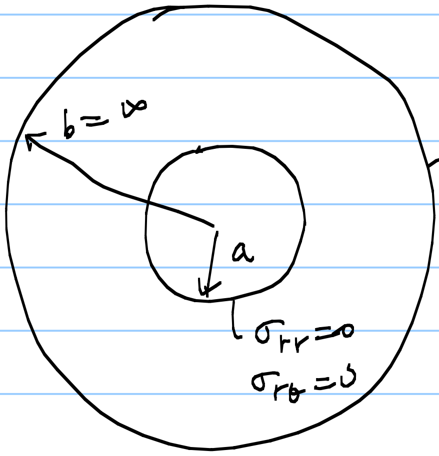


$$\sigma_{rr} = \underline{e}_r \cdot \underline{\sigma} \cdot \underline{e}_r$$
$$= \underline{e}_r \cdot (\sigma_0 \, \underline{e}_y \otimes \underline{e}_y) \cdot \underline{e}_r$$
$$= \sigma_0 \sin^2\theta = \frac{\sigma_0}{2}(1 - \cos 2\theta)$$

$$\sigma_{r\theta} = \underline{e}_r \cdot \underline{\sigma} \cdot \underline{e}_\theta$$
$$= \underline{e}_r \cdot (\sigma_0\, \underline{e}_y \otimes \underline{e}_y)\, \underline{e}_\theta$$
$$= \sigma_0 \sin\theta \cos\theta = \frac{\sigma}{2} \sin 2\theta$$

$$\sigma_{\theta\theta} = \underline{e}_\theta \cdot \underline{\sigma} \cdot \underline{e}_\theta$$
$$= \sigma_0 \cos^2\theta = \frac{\sigma}{2}(1 + \cos 2\theta)$$

BVP in Polar Coordinates



$$\sigma_{rr} = \frac{1}{2}\sigma_0 - \frac{1}{2}\sigma_0 \cos 2\theta$$
$$\sigma_{r\theta} = \frac{1}{2}\sigma_0 \sin 2\theta$$

$$\underset{\sim}{\sigma_{ij}} = \underset{\sim}{\sigma_{ij}}^{(0)} + \underset{\sim}{\sigma_{ij}}^{(2)}$$

$\uparrow$ Zeroth Harmonic $\qquad \uparrow$ 2nd Harmonic

① For axisym case (self-eqm)

$$\sigma_{rr}^{(0)}(r) = \frac{a_0}{r^2} + 2b_0$$

$$\left(\sigma_{r\theta}^{(0)}(r) = 0, \quad \sigma_{\theta\theta} = -\frac{a_0}{r^2} + 2b_0 \right)$$

$$\begin{cases} \sigma_{rr}^{(0)}(a) = \dfrac{a_0}{a^2} + 2b_0 = 0 \\ \sigma_{rr}^{(0)}(r)\Big|_{r\to\infty} = 2b_0 = \frac{1}{2}\sigma_0 \end{cases}$$

$$\rightarrow b_0 = \frac{\sigma_0}{4} \; ; \; a_0 = -2a^2 b_0 = -\frac{a^2}{2}\sigma_0$$

$$\begin{cases} \sigma_{rr}^{(0)}(r) = \dfrac{\sigma_0}{2}\left[1 - \left(\dfrac{a}{r}\right)^2\right] \\ \sigma_{\theta\theta}^{(0)}(r) = \dfrac{\sigma_0}{2}\left[1 + \left(\dfrac{a}{r}\right)^2\right] \end{cases}$$

② For cos 2θ case (self-eqm)

$$\begin{cases} \sigma_{rr}^{(2)}(r,\theta) = \left(-a_2 - \dfrac{6a_2'}{r^4} - \dfrac{4b_2'}{r^2}\right)\cos 2\theta \\ \sigma_{r\theta}^{(2)}(r,\theta) = \left(a_2 + \cancel{6b_2 r^2} - \dfrac{6a_2'}{r^4} - \dfrac{2b_2'}{r^2}\right)\sin 2\theta \\ \sigma_{\theta\theta}^{(2)}(r,\theta) = \left(a_2 + \cancel{12b_2 r^2} + \dfrac{6a_2'}{r^4}\right)\cos 2\theta \end{cases}$$

must be finite at $r \to \infty$

Match BC's

① at $r = \infty$

$$\begin{cases} \sigma_{rr}^{(2)}\Big|_{r=\infty} = -a_2 = -\dfrac{\sigma_0}{2} \\ \sigma_{r\theta}^{(2)}\Big|_{r=\infty} = a_2 = \dfrac{1}{2}\sigma_0 \end{cases}$$

$$\Longrightarrow a_2 = \sigma_0/2$$

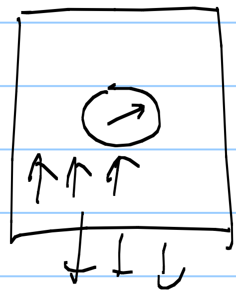
② at $r = a$

$$\begin{cases} \sigma_{rr}^{(2)}\Big|_{r=a} = -\dfrac{\sigma_0}{2} - \dfrac{6a_2'}{a^4} - \dfrac{4b_2'}{a^2} = 0 \\ \sigma_{r\theta}^{(2)}\Big|_{r=a} = \dfrac{\sigma_0}{2} - \dfrac{6a_2'}{a^4} - \dfrac{2b_2'}{a^2} = 0 \end{cases}$$

$$\Longrightarrow \boxed{a_2' = \frac{\sigma_0}{4}a^4, \quad b_2' = -\frac{\sigma_0}{2}a^2}$$

$$\therefore \begin{cases} \sigma_{rr}^{(2)}(r,\theta) = \left[-\dfrac{\sigma_0}{2} - \dfrac{3}{2}\sigma_0 \left(\dfrac{a}{r}\right)^4 + 2\sigma_0 \left(\dfrac{a}{r}\right)^2 \right] \cos 2\theta \\ \sigma_{r\theta}^{(2)}(r,\theta) = \left[\dfrac{\sigma_0}{2} - \dfrac{3}{2}\sigma_0 \left(\dfrac{a}{r}\right)^4 + \sigma_0 \left(\dfrac{a}{r}\right)^2 \right] \sin 2\theta \\ \sigma_{\theta\theta}^{(2)}(r,\theta) = \left[\dfrac{\sigma_0}{2} + \dfrac{3}{2}\sigma_0 \left(\dfrac{a}{r}\right)^4 \right] \cos 2\theta \end{cases}$$

$$\sigma_{ij}^{\text{Total}} = \sigma_{ij}^{(0)} + \sigma_{ij}^{(2)}$$



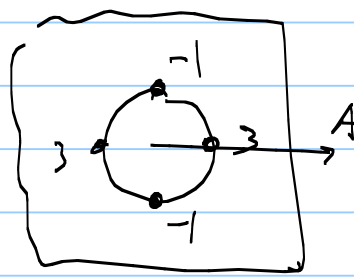
<u>Check $\sigma_{\theta\theta}$</u>

$$\sigma_{\theta\theta} = \frac{\sigma_0}{2}\left[1 + \left(\frac{a}{r}\right)^2\right] + \frac{\sigma_0}{2}\left[1 + 3\left(\frac{a}{r}\right)^4\right] \cos 2\theta$$

<u>at $r=a$ (around the hole)</u>

$$\sigma_{\theta\theta} = \sigma_0 (1 + 2\cos 2\theta)$$

$$= \begin{cases} \boxed{3\sigma_0} & \text{at } \theta = 0, \pi \\ -\sigma_0 & \text{at } \theta = \frac{\pi}{2}, \frac{3\pi}{2} \end{cases}$$



Along A $(\theta = 0)$

$$\sigma_{\theta\theta}(r,\theta) = \frac{\sigma_0}{2}\left[2 + \left(\frac{a}{r}\right)^2 + 3\left(\frac{a}{r}\right)^4\right]$$

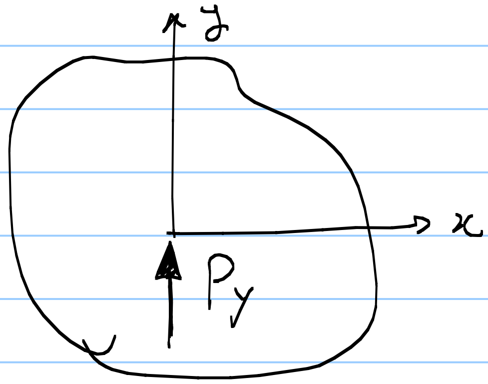
$\Rightarrow$ Algebraic decay as $r \to \infty$

$$\begin{cases} \sigma_{\theta\theta}(r=a, \theta=0) = 3\sigma_0 \\ \sigma_{\theta\theta}(r=3a, \theta=a) \approx 1.07\sigma_0 \end{cases}$$

Point Force in a 2D Full Space

((See Sec 2.3

↑ In books, the $p-\varepsilon$ condition was used)



Solution for a point Force in a 2-D full space (use $p-\sigma$ condition)

- Conditions
  1. has a force resultant component in y
  2. displacement must be single-valued

⇨ **Use the solution developed in a ring**

$$\phi = \underbrace{-\frac{c_1}{2} r\theta\cos\theta}_{\cdot\ \Sigma F_y \neq 0} + \underbrace{d_1' r \ln r \sin\theta}_{\cdot\ \Sigma F_y = 0}$$

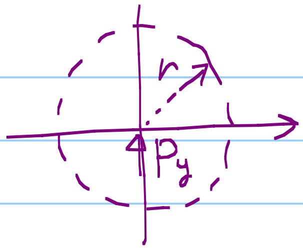
$$\cdot \begin{cases} u_r = \dfrac{1-\nu}{2} c_1 \theta \cos\theta \\ u_\theta = \dfrac{1-\nu}{2} c_1 \theta \sin\theta \end{cases} \qquad \cdot \begin{cases} u_r = 2 d_1' \theta \cos\theta \\ u_\theta = 2 d_1' \theta \sin\theta \end{cases}$$

multiply-valued   multiply-valued

Can be cancelled if $d_1' = -c_1 \dfrac{(1-\nu)}{4}$

$$\begin{cases} \sigma_{rr} = \dfrac{c_1 + d_1'}{r} \sin\theta \\ \sigma_{r\theta} = \dfrac{d_1'}{r} \cos\theta \\ \sigma_{\theta\theta} = \dfrac{d_1'}{r} \sin\theta \end{cases}$$

Resultant $\Sigma F_y = 0$



$$0 = P_y + \int_0^{2\pi} t_y \, r d\theta$$

$$= P_y + \int_0^{2\pi} (\sigma_{rr} \sin\theta + \sigma_{r\theta} \cos\theta)\, r d\theta$$

$$= P_y + c_1 \pi$$

$$\therefore \quad c_1 = -\frac{P_y}{\pi}$$

$$\therefore \quad \phi = \frac{P_y}{2\pi}\left(r\theta\cos\theta + \frac{1-\nu}{2} r \ln r \sin\theta \right)$$

$$\left\{ \begin{aligned} \sigma_{rr} &= -\frac{3+\nu}{4\pi} P_y \frac{\sin\theta}{r} \\ \sigma_{\theta\theta} &= \frac{1-\nu}{4\pi} P_y \frac{\sin\theta}{r} \\ \sigma_{r\theta} &= \frac{1-\nu}{4\pi} P_y \frac{\cos\theta}{r} \end{aligned} \right\} \quad (*)$$

(※ $p\text{-}\sigma \to p \to \varepsilon$: ν in $(*) \to \frac{\nu}{1-\nu}$ see Textbook (2.3.13)

$$\sigma_{rr} = -\frac{(3-2\nu)}{4\pi(1-\nu)} P_y \frac{\sin\theta}{r}$$

$$\sigma_{\theta\theta} = \frac{1-2\nu}{4\pi(1-\nu)} P_y \frac{\sin\theta}{r}, \quad \sigma_{r\theta} = -\frac{(1-2\nu)}{4\pi(1-\nu)} P_y \frac{\cos\theta}{r}\Bigg)$$