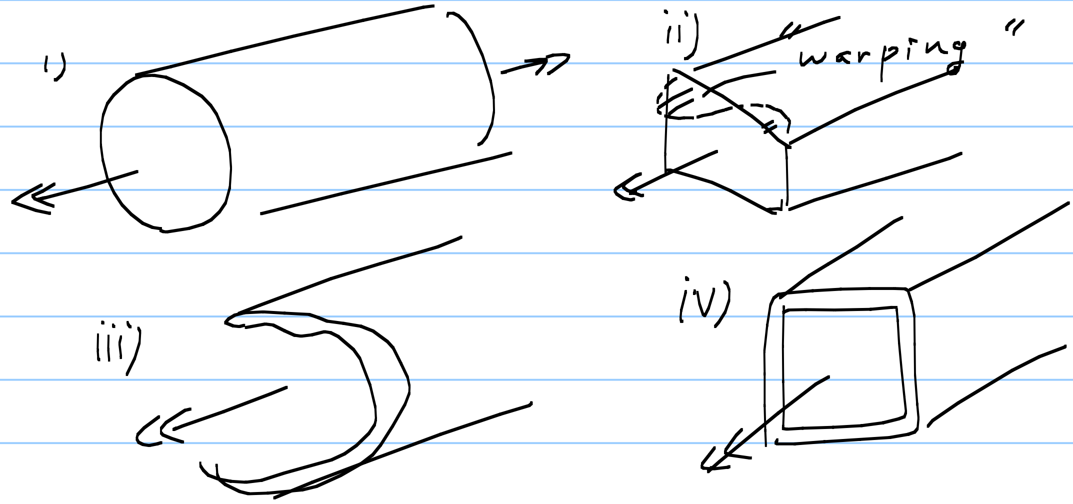
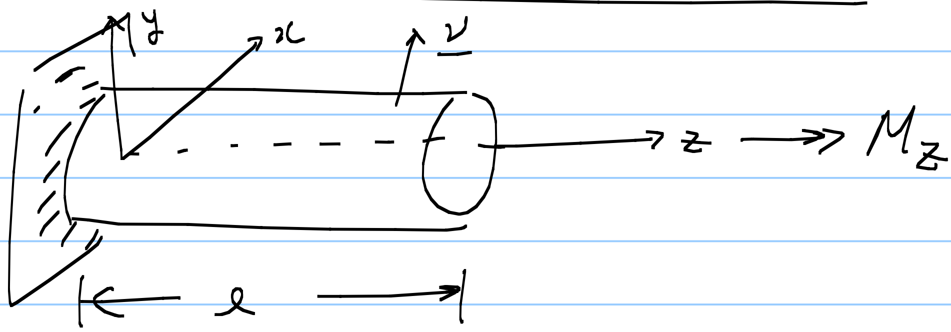
Lecture 3-1

< Torsion in a shaft >

노트 제목



< Torsion in a circular shaft >



under torsion

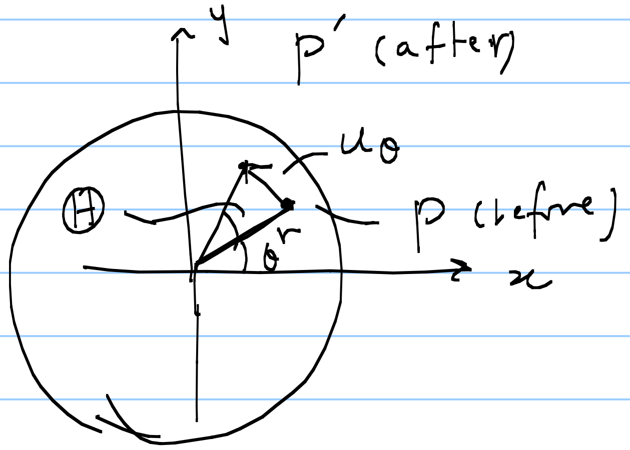
i) cross-section remain circular

ii) no warping

$\Longrightarrow$ i, ii) due to circular symmetry

(see strength of Materials books)

- look at the cross section



$$\begin{cases} u_r = 0 \\ u_z = 0 \\ u_\theta = r\,\Theta \\ \quad = r(\alpha z) \\ \quad = \alpha r z \end{cases}$$

$\boxed{\text{twist rate}}$ (twist angle / length)

In Cartesian coordinates

$$\begin{cases} u_x = -u_\theta \sin\theta = -\alpha z r \sin\theta \\ \quad = -\alpha z y \\ u_y = u_\theta \cos\theta = \alpha z x \end{cases}$$

- Stresses

1) polar coordinates

- $$\sigma_{\theta z} = \mu\left(\cancel{\frac{1}{r}\frac{\partial u_z}{\partial \theta}} + \frac{\partial u_\theta}{\partial z}\right)$$
$$= \mu \alpha r$$
$$\sigma_{\theta\theta} = \sigma_{rr} = \sigma_{zz} = \sigma_{rz} = \sigma_{r\theta} = 0$$

2) Cartesian coordinates

- $\sigma_{zx} = -\mu\alpha y$; $\sigma_{zy} = \mu\alpha x$

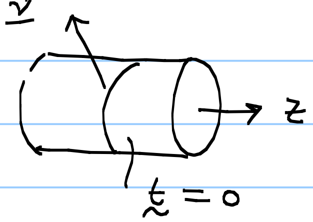
(other $\sigma_{ij} = 0$)

<check>

① satisfy eqm?

$$\left\{\begin{array}{l} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + \dfrac{\partial \sigma_{zx}}{\partial z} = 0 \quad \checkmark \\[2ex] \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{zy}}{\partial z} = 0 \quad \checkmark \\[2ex] \dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} = 0 \quad \checkmark \end{array}\right.$$

② traction BC's (on C)

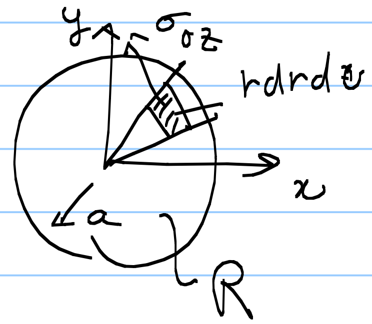


$\underset{\sim}{\nu} = \underset{\sim}{e}_r$; $\underset{\sim}{t} = \underset{\sim}{\nu} \cdot \underset{\sim}{\sigma}$

$$= \sigma_{rr} \underset{\sim}{e}_r + \sigma_{r\theta} \underset{\sim}{e}_\theta + \sigma_{rz} \underset{\sim}{e}_z$$
$$\equiv 0$$

- Applied Moment = Moment by internal stress

$$M_z = \int_R (\sigma_{\theta z} r\,dr\,d\theta) \times r$$



$$= \int_R \mu \alpha r^2 (r\,dr\,d\theta)$$

$$= \mu \alpha \int_R r^2 dA = \mu \alpha I_{zz}$$

↑ polar moment of inertia

(for circular section:

$$I_{zz} = \int_R r^2 dA = \int_0^{2\pi} \int_0^a r^2 r\,dr\,d\theta$$

$$= \frac{\pi}{2} a^4 \quad)$$

<u>Define "torsional rigidity C_t":</u>

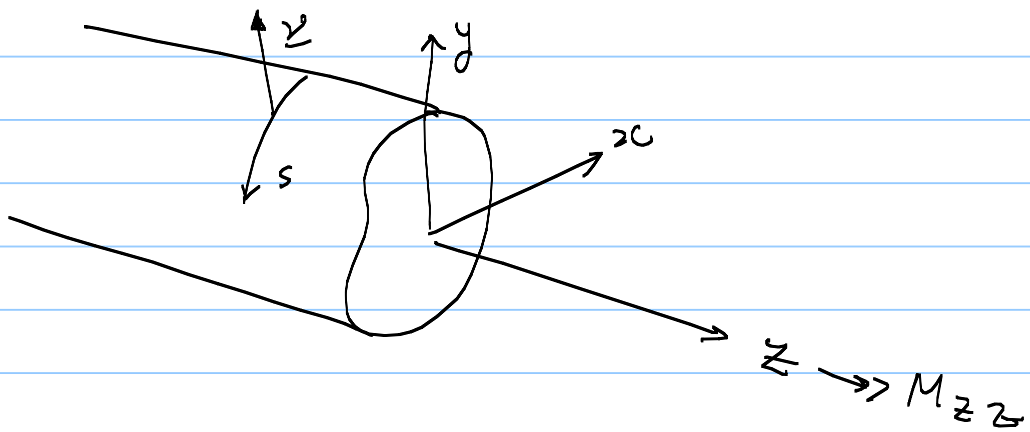
$$\underline{M_z = C_t \cdot \alpha}$$

In circular section:

$$C_t = \mu I_{zz} = \frac{\pi \mu a^4}{2} \simeq 0.159 \mu A^2$$

$$(A = \pi a^2)$$

⟨Torsion in noncircular shafts⟩



* non-circular section; no circular symmetry

⇒ warping can take place (i.e, $u_z \neq 0$)

As before $\left[\begin{array}{l} u_\theta = r\,\Theta = r\alpha z \\ u_r = 0 \end{array}\right.$

$$\Rightarrow \quad u_x = -\alpha z y$$
$$u_y = +\alpha z x$$

⊕ $u_z = \alpha\,\phi(x,y)$ ← unknown ftn yet

"Warping ftn"

not $\phi(x,y,z)$

⇒ To find governing eq. for $\phi(x,y)$,

consider eqm. equation.

So, stress calculation:

$$(A)\begin{cases} \sigma_{zx} = \mu\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \\ \quad = \mu\alpha\left(\frac{\partial \phi}{\partial x} - y\right) \\ \sigma_{zy} = \mu\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) \\ \quad = \mu\alpha\left(\frac{\partial \phi}{\partial y} + x\right) \end{cases}$$

Other $\sigma_{ij} = 0$

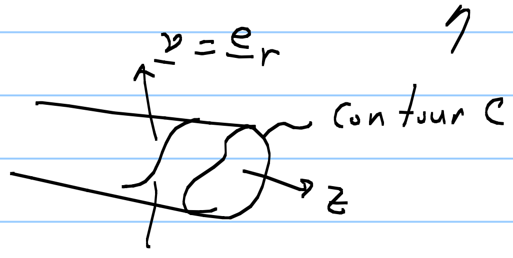
Check 1) Eqm

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}}^{0} = 0$$

$$\underset{(A)}{\Longrightarrow} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \; ; \quad \boxed{\nabla^2 \phi(x, y) = 0} \; ①$$

Laplace's equation

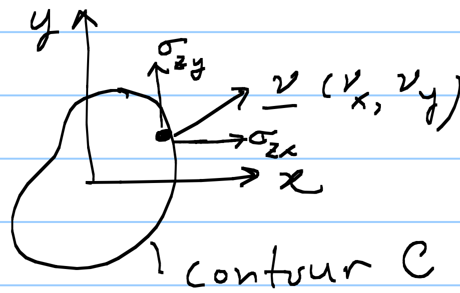
Check 2) traction BC's



$\tilde{t} = 0$

on C

$$\tilde{t} = \cancel{\sigma_{rr}}\, \underline{e}_r + \cancel{\sigma_{r\theta}}\, \underline{e}_\theta + \underbrace{\sigma_{zr}}_{t_z}\, \underline{e}_z \equiv 0$$



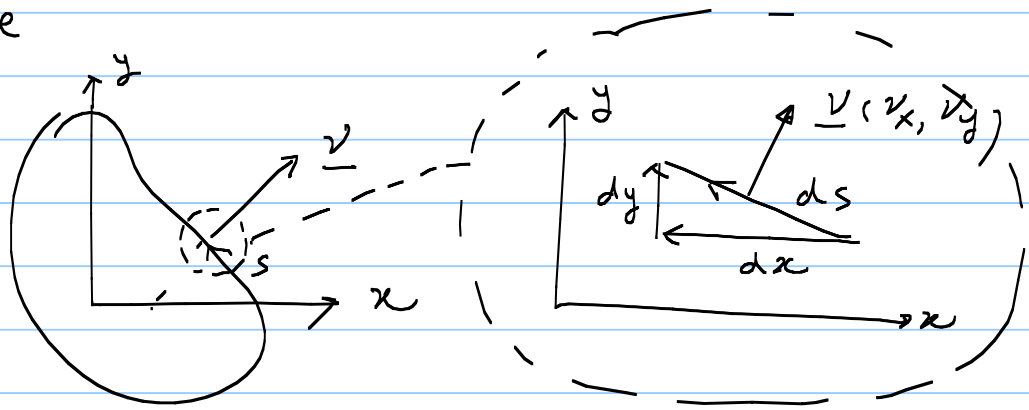
$$
\begin{aligned}
t_z &= \underline{\nu} \cdot \tilde{\sigma} \cdot \underline{e}_z \\
&= (\nu_x \underline{e}_x + \nu_y \underline{e}_y) \cdot (\sigma_{xz} \underline{e}_x + \sigma_{yz} \underline{e}_y) \\
&= \sigma_{xz} \nu_x + \sigma_{yz} \nu_y \\
&= \mu\alpha \left[\left(\frac{\partial \phi}{\partial x} - y \right) \nu_x + \left(\frac{\partial \phi}{\partial y} + x \right) \nu_y \right] = 0
\end{aligned}
$$

$$\therefore \quad \frac{\partial \phi}{\partial x} \nu_x + \frac{\partial \phi}{\partial y} \nu_y = y \nu_x - x \nu_y$$

OR $\boxed{\underline{\nabla} \phi \cdot \underline{\nu} = \frac{d\phi}{d\nu} = y \nu_x - x \nu_y}$ on C ②

①, ② ; Neumann-type BVP
(cf: Dirichlet-type BVP)

Note



$$\nu_x = \frac{dy}{ds}\,, \quad \nu_y = -\frac{dx}{ds}$$

Then

$$\text{RHS of } ② = y\,\nu_x - x\,\nu_y$$
$$= y\frac{dy}{ds} + x\frac{dx}{ds}$$
$$= \frac{1}{2}\frac{d}{ds}\left(x^2 + y^2\right)$$

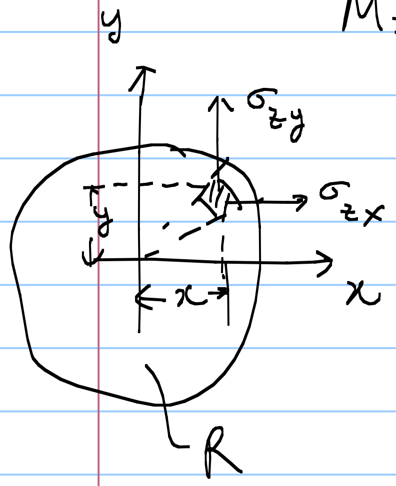
Because

$$\oint_C (y\,\nu_x - x\,\nu_y)\,ds = \oint_C \frac{1}{2}\frac{d}{ds}(x^2+y^2)\,ds = 0,$$

$$\oint_C \frac{d\phi}{d\nu}\,ds = 0 \leftarrow$$ "condition satisfied for Neumann-type problems"

To calculate $C_t = M_z/\alpha$,

determine M_z by



$$M_z = \int_R (x\,\sigma_{yz} - y\,\sigma_{xz})\,dA$$

$$= \mu\alpha \int_R \left[x\left(\frac{\partial\phi}{\partial y} + x\right) - y\left(\frac{\partial\phi}{\partial x} - y\right)\right] dA$$

$$= \mu\alpha \int_R \left(x^2 + y^2 + x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x}\right) dA$$

Check: $$C_t = \mu \int_R (x^2+y^2)\,dA - \mu \int_A \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] dA$$

$$\le \mu \int_R (x^2+y^2)\,dA = \mu I_{zz}$$

physics: Due to warping, C_t decreases.

< Prandtl Stress Function Formulation >

- Prandtl started with eqm equation and used the above result
  → what's advantge? "Membrane Analogy"
  ⇓
  easier analysis & experiment

1) • Eqm Eq

- $$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \quad \text{(non-trivial eqn)}$$

- Choose
$$\left\{\begin{aligned} \sigma_{zx} &= \mu\alpha \frac{\partial \psi}{\partial y} \\ \sigma_{zy} &= -\mu\alpha \frac{\partial \psi}{\partial x} \end{aligned}\right\} \quad (A)$$

- Use displacement-based result
$$\left\{\begin{aligned} \sigma_{zx} &= \mu\alpha \left(\frac{\partial \phi}{\partial x} - y \right) \\ \sigma_{zy} &= \mu\alpha \left(\frac{\partial \phi}{\partial y} + x \right) \end{aligned}\right\} \quad (B)$$

(A) ≡ (B) (meaning? "Compatability")

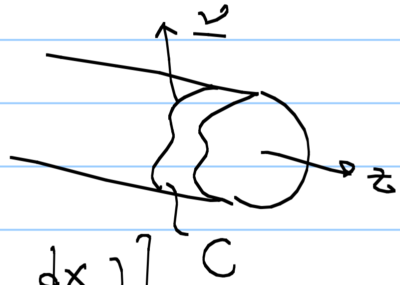
$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} - y \; ; \; -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} + x \quad (a,b)$$

Eliminate ϕ from (a, b)

$\Rightarrow \frac{\partial}{\partial y}(a) - \frac{\partial}{\partial x}(b)$

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2} \qquad ①$$

2) BC's on C



$$t_z = \sigma_{zx}\nu_x + \sigma_{zy}\nu_y$$

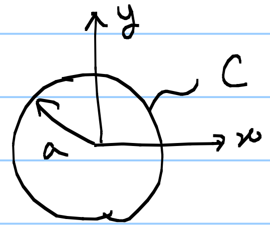
$$= \mu\alpha\left[\frac{\partial \psi}{\partial y}\frac{dy}{ds} + \left(-\frac{\partial \psi}{\partial x}\right)\left(-\frac{dx}{ds}\right)\right]$$

$$= \mu\alpha \frac{d\psi}{ds} \equiv 0 .$$

$\Rightarrow \quad \psi = k$ (constant) on C

$\Rightarrow \quad \boxed{\psi = 0 \text{ on } C}$ ($\sigma \sim$ derivative of ψ) ..... ②

①, ② : Dirichlet-type problem



Circular section

$C: x^2+y^2-a^2=0,$

Try $\psi = c_1(x^2+y^2-a^2)$

$\rightarrow$ satisfies ②

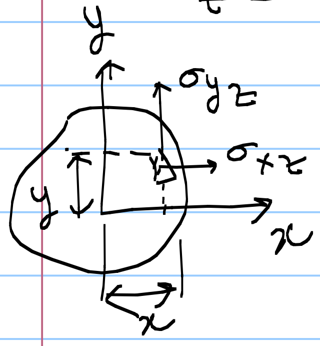
$\rightarrow$ satisfies ① if $c_1 = -\frac{1}{2}$

$$\therefore \boxed{\psi = -\frac{1}{2}(x^2+y^2-a^2)}$$

$\hookrightarrow$ exact sol. for circular sections

Torsional Rigidity C_t in terms of ψ

$$C_t = M_z/\alpha = \left[\int_R (x\,\sigma_{yz} - y\,\sigma_{xz})\,dA\right]/\alpha$$



$$= -\mu \int_R \left[x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y}\right] dA$$

$$= -\mu\left[\int_R \left[\frac{\partial}{\partial x}(x\psi) + \frac{\partial}{\partial y}(y\psi) - 2\psi\right] dA\right.$$

$$= -\mu \int_R \left[\frac{\partial}{\partial x}(x\psi) + \frac{\partial}{\partial}(y\psi)\right] dA$$

$$+ 2\mu \int_R \psi \, dA$$

Check: $\int_R \underbrace{\left[\frac{\partial}{\partial x}(x\psi) + \frac{\partial}{\partial y}(y\psi)\right]}_{= \nabla \cdot (x\psi,\, y\psi)} dA$

Div. Th.: $\overset{?}{=} \oint_C \underline{\nu} \cdot (x\psi, y\psi) \, dA$

$$= \oint_C (x\nu_x + y\nu_y)\psi \, dA$$

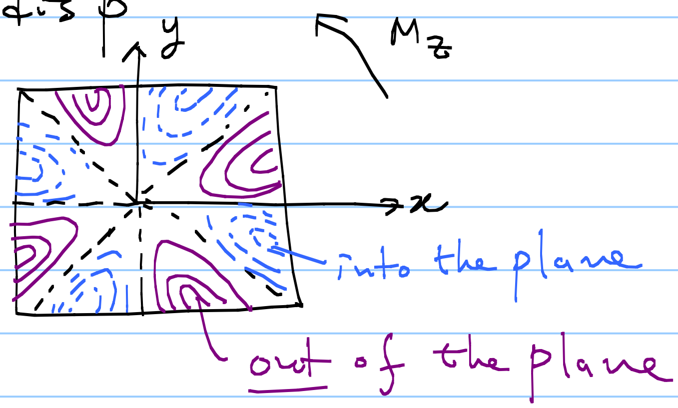
$= 0$ Because $\psi = 0$ on C

$$\boxed{\therefore C_t = 2\mu \int_R \psi \, dA}$$

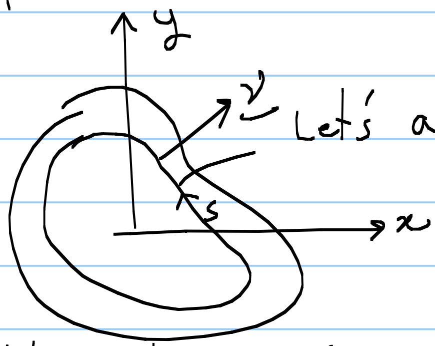
Significance of $\begin{cases} \phi = \text{const line} \\ \psi = \text{const line} \end{cases}$

1) ϕ = const line

$\phi \to$ warping, out-of-of plane disp



2) ψ = const line



Let's assume ψ = const here

• Along the ψ = const

$$0 = \frac{d\psi}{ds} = \frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds}$$

$$= \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right) \cdot \left(\underbrace{\frac{dy}{ds}}_{\nu_x}, \underbrace{-\frac{dx}{ds}}_{\nu_y} \right)$$

$\sigma_{zx} = \mu\alpha \frac{\partial \psi}{\partial y}$

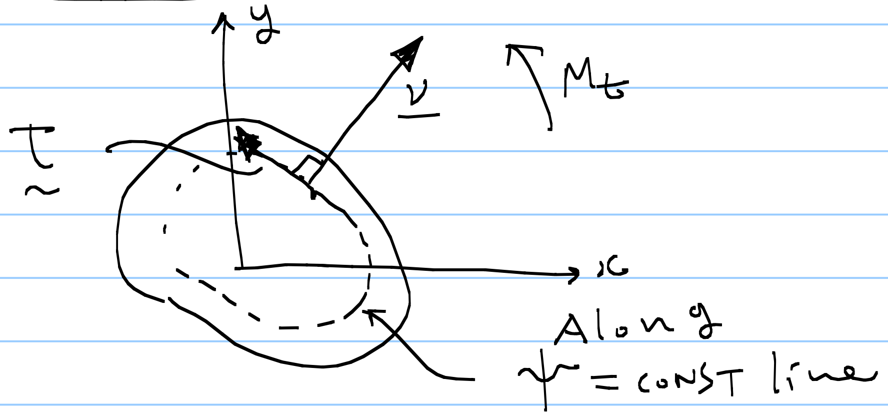
$\sigma_{zy} = -\mu\alpha \frac{\partial \psi}{\partial x}$

$$= \frac{1}{\mu\alpha} (\sigma_{zx}, \sigma_{zy}) \cdot \underline{\nu}$$

$$0 \triangleq \frac{1}{\mu\alpha} \underset{\sim}{\tau} \cdot \underset{\sim}{\nu} \quad (\text{along } \psi = \text{CONST line}) \quad (*)$$

τ called traction vector

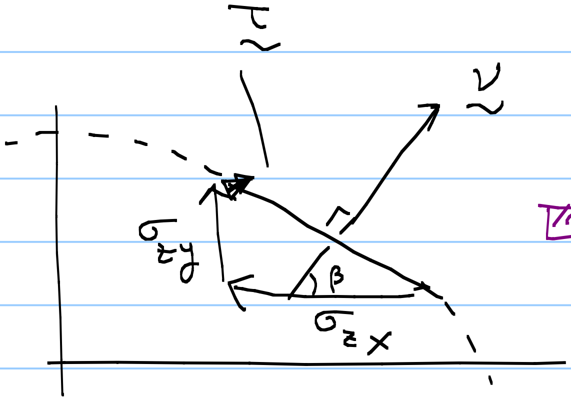
Meaning of (*)



the resultant vector $\underset{\sim}{\tau}$ of the shear stress must be orthogonal to $\underline{\nu}$

Thus

$\left(\begin{array}{l} \cos\beta = \nu_x \\ \sin\beta = \nu_y \end{array} \right)$



$\Rightarrow$ Ratio of σ_{zy} to σ_{zx} is determined.

$$\tau = |\underset{\sim}{\tau}| = \left(\sigma_{zx}^2 + \sigma_{zy}^2\right)^{\frac{1}{2}}$$

$$\underset{\text{or}}{=} -\sigma_{zx}\sin\beta + \sigma_{zy}\cos\beta$$

$$= -\sigma_{zx}\nu_y + \sigma_{zy}\nu_x$$

$$= -\left(\mu\alpha\frac{\partial\psi}{\partial y}\right)\nu_y + \left(-\mu\alpha\frac{\partial\psi}{\partial x}\right)\nu_x$$

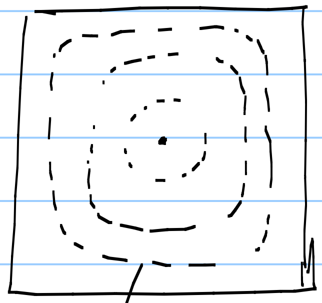
$$= -\mu\alpha\left(\frac{\partial\psi}{\partial y}\nu_y + \frac{\partial\psi}{\partial x}\nu_x\right)$$

$$= -\mu\alpha\frac{d\psi}{d\nu}$$

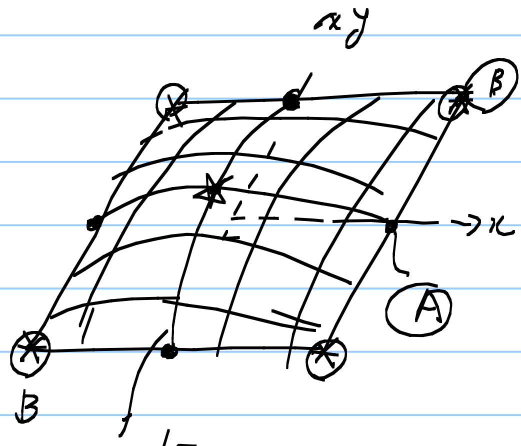
$\Rightarrow$ shear stress τ // $\dfrac{d\psi}{d\nu}$

$\uparrow$ the amount of change on ψ along the normal direction $\underset{\sim}{\nu}$

example



lines of $\psi = $ CONST

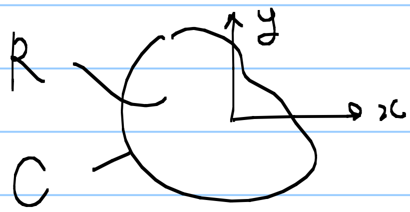


$\bullet$: Max shear stress $\rightarrow d\psi/d\nu = \max$

$\ast$, $\circledast$: no shear stress $\rightarrow d\psi/d\nu = 0$

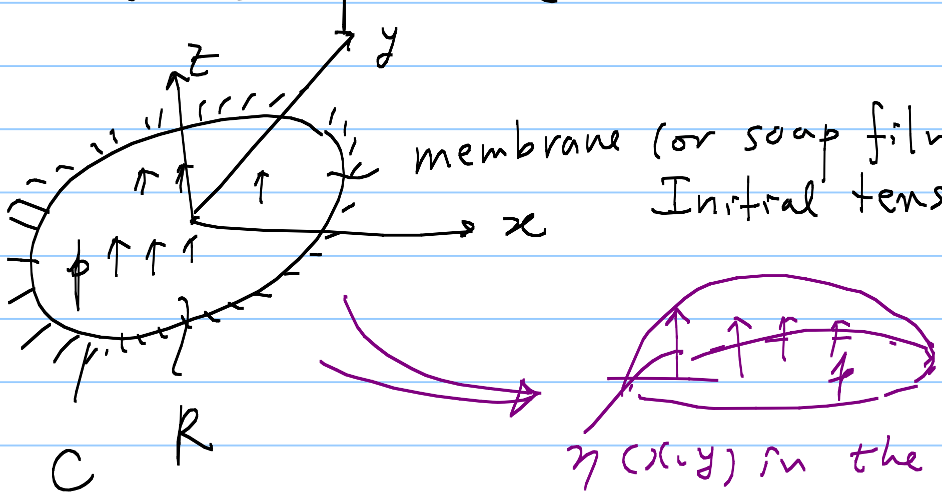
Membrane Analogy (Prandtl Function)

- Prandtl Stress Function: ψ



$$\nabla^2 \psi = -2 \text{ in } R$$
$$\psi = 0 \text{ on } C$$

- Equation for thin Membrane Deflection under pressure



membrane (or soap film)
Initial tension: T

$\eta(x,y)$ in the z-direction under pressure p in R

$$\begin{cases} \nabla^2 \eta(x,y) = -\dfrac{p}{T} & \text{in } R \\ \eta = 0 & \text{on } C \end{cases}$$

Analogy

① $\eta = \frac{p}{2T} \psi$

② $\tau = -\mu\alpha \frac{d\psi}{d\nu} = \frac{-2\mu\alpha}{T} \frac{d\eta}{d\nu}$

shear stress $\Rightarrow \tau \propto$ change rate of membrane deflection

③ $C_t = 2\mu \int_R \psi \, dA$

$= \frac{4\mu T}{p} \underbrace{\int_R \eta \, dA}$

volume enclosed by the η surface and the x-y plane

** Remark 1) : Complicated physics on torsion can be easily understood or visualized by membrane analogy in terms of membrane deflection

⇒ Key Advantage of using Prandtl stress function

Remark 2) : Experiment on torsion
⇒ done with membrane