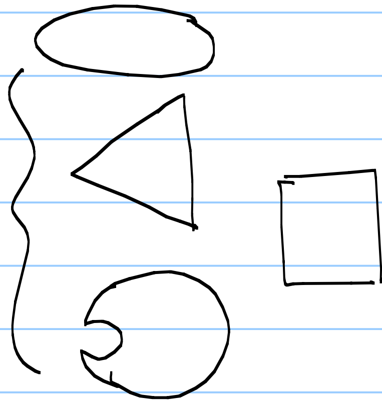
Lect 3.2 : Torsion for Typical Sections

Approach

i) Prandtl Stress Function Approach



Analogy!

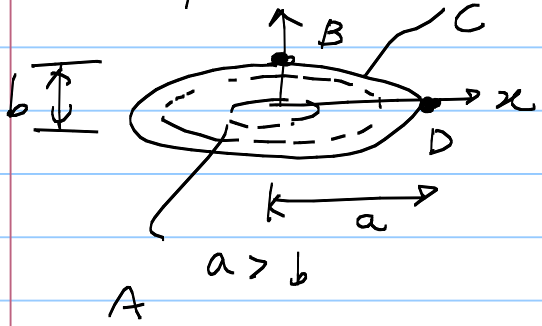
$$\begin{cases} \nabla^2 \psi = -2 & \text{in } R \quad (a) \\ \psi = 0 & \text{on } C \quad (b) \end{cases}$$

ii) Membrane problem

$$\begin{cases} \nabla^2 \eta = -p/T & \text{in } R \\ \eta = 0 & \text{on } C \end{cases}$$

$$\boxed{\begin{array}{l} \psi \sim \eta \\ \underset{\text{(stress vector)}}{\tau} \sim -\dfrac{d\eta}{d\nu}, \quad C_t = 2\mu \int \psi \, dA \sim \int \eta \, dA \end{array}}$$

< Elliptic cross section >



$C : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1 = 0$

Try : $\psi = m \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)$

→ satisfies (b)

and satisfies (a)

if $m = -\frac{a^2 b^2}{a^2 + b^2}$

τ_B : max

(membrane Analogy)

$$\begin{cases} \sigma_{zx} = \mu\alpha \dfrac{\partial \psi}{\partial y} = -2\mu\alpha \dfrac{a^2 y}{a^2+b^2} \\ \sigma_{zy} = -\mu\alpha \dfrac{\partial \psi}{\partial x} = \dfrac{2\mu\alpha a^2 x}{a^2+b^2} \end{cases}$$

$$C_t = 2\mu \int_R \psi \, dA = \frac{\pi\mu a^3 b^3}{a^2+b^2}$$

Q: $C_t|_{max} = ?$ for given $\pi a b = A_0$ (Area)

$$C_t = \frac{\pi\mu \left(\frac{A_0}{\pi}\right)^3}{a^2 + \left(\frac{A_0}{\pi a}\right)^2}$$

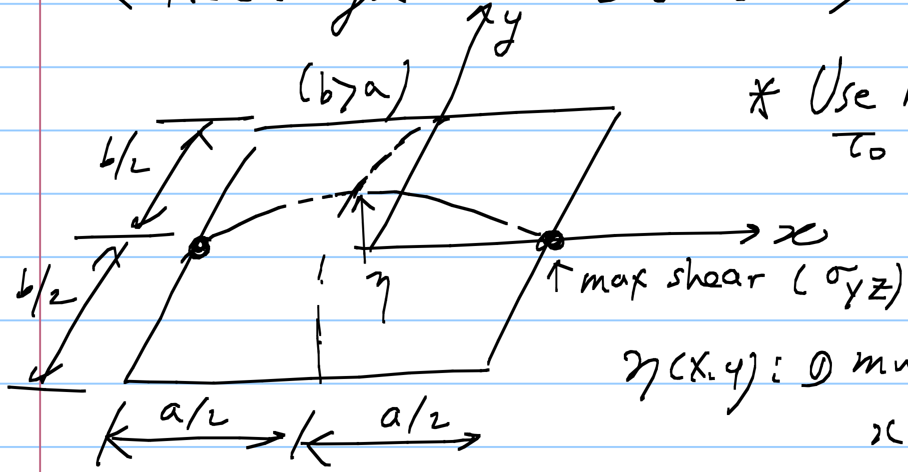
becomes max when

$$a^2 = \left(\frac{A_0}{\pi a}\right)^2$$

$$\pi a^2 = A_0$$

$\therefore$ Circular section has the largest torsional rigidity.

< Rectangular Cross sections >



* Use Membrane Analogy To find Solution:

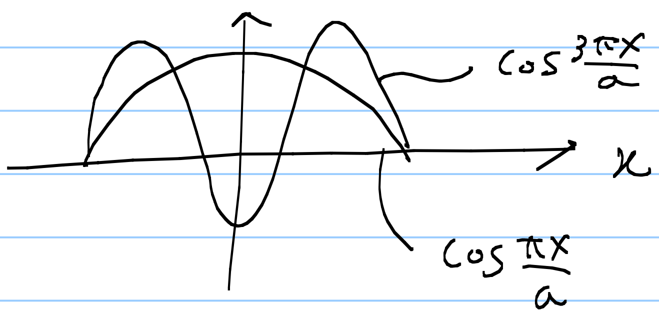
η(x,y) : ① must be sym wrt x and y axes

② η = 0 along the boundary

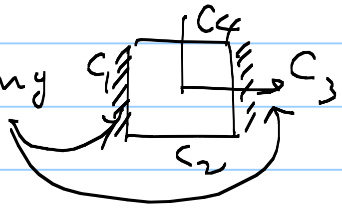
$$\boxed{\begin{aligned} &\nabla^2 \eta = -\frac{p}{T} \quad \text{in } R \quad ① \\ &\eta = 0 \quad \text{on } C \quad ② \end{aligned}}$$

$\therefore$ Let

$$\eta(x,y) = \sum_{n=0}^{\infty} \cos \frac{(2n+1)\pi}{a} x \; Y_n(y) \quad ③$$



i) ③ satisfies BC's along C_1, C_3



ii) Substitute ③ into ① and check BC's along other sides C_2, C_4

$$-\frac{p}{T} = \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}$$

$$= \sum_{n=0}^{\infty} \left\{ -\left[\left(\frac{2n+1}{a}\right)\pi\right]^2 Y_n(y) + Y_n''(y) \right\} \quad ④$$

$$\times \cos \frac{2n+1}{a} \pi x$$

Let $\gamma_n = \dfrac{(2n+1)\pi}{a}$,

Use

$$-\frac{p}{T} = \left(-\frac{p}{T}\right) \frac{4}{a} \sum_{n=0}^{\infty} \frac{(-1)^n}{\gamma_n} \cos \gamma_n x$$

Then ④ becomes

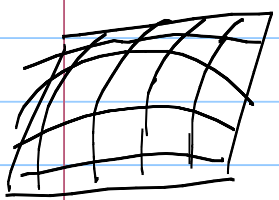
$$\boxed{\frac{d^2 Y_n(y)}{dy^2} - \gamma_n^2 Y_n(y) = -\frac{4p}{aT} \frac{(-1)^n}{\gamma_n}} \quad ⑤$$

for every n

Solution to ⑤

$$Y_n = Y_n^{c} + Y_n^{p}$$

$$= A_n \sinh \gamma_n y + B_n \cosh \gamma_n y$$



$$+ \frac{4p}{aT} \frac{(-1)^n}{\gamma_n^3}$$

• Using the solution symmetry wrt $y=0$,

$$A_n = 0$$

• Using the BC' at $y = \pm b/2$

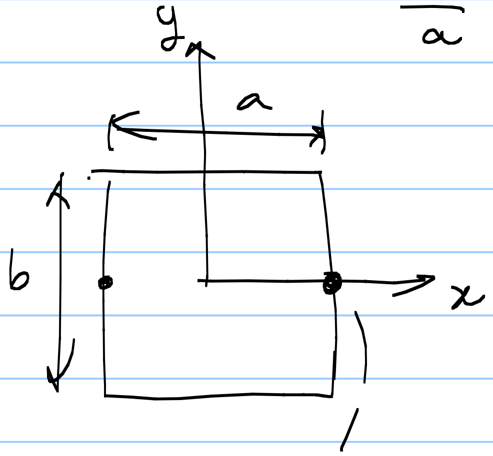
$$B_n \cosh \frac{b\gamma_n}{2} + \frac{4p}{aT} \frac{(-1)^n}{\gamma_n^3} = 0$$

$$\rightarrow B_n = -\frac{4p}{aT} \frac{(-1)^n}{\gamma_n^3} \cdot \frac{1}{\cosh b\gamma_n/2}$$

$$\therefore Y_n(y) = \frac{4p}{aT} \frac{(-1)^n}{\gamma_n^3} \left[1 - \frac{\cosh \gamma_n y}{\cosh b\gamma_n/2} \right]$$

$$\therefore \quad \psi(x,y) = \frac{2T}{\phi}\,\eta(x,y)$$

$$= \frac{8}{a}\sum_{n=0}^{\infty}\frac{(-1)^n}{\gamma_n^3}\left[1-\frac{\cosh\gamma_n y}{\cosh b\gamma_n/2}\right]\cos\gamma_n x$$



$b > a$

max shear

$\underbrace{\sigma_{zy}\left(x = \pm\frac{a}{2},\ y=0\right)}$

$$\sigma_{zy} = -\mu\alpha\frac{\partial\psi}{\partial x}$$

$$= \frac{8\mu\alpha}{a}\sum_{n=0}^{\infty}\frac{(-1)^n}{\gamma_n^2} \times \left[1-\frac{\cosh\gamma_n y}{\cosh b\gamma_n/2}\right]\sin\gamma_n x$$

$$\therefore \quad \tau_{max} = \sigma_{zy}\Big|_{\left(\frac{a}{2},0\right)}$$

$$= \frac{8\mu\alpha}{a}\sum_{n=0}^{\infty}\frac{(-1)^n}{\gamma_n^2}\left[1-\frac{1}{\cosh b\gamma_n/2}\right]\underbrace{\sin\left(n+\frac{\pi}{2}\right)}_{(-1)^n}$$

$$= \frac{8\mu\alpha}{a}\sum_{n=0}^{\infty}\left(\frac{1}{\gamma_n^2} - \frac{1}{\gamma_n^2\cosh b\gamma_n/2}\right)\Bigg]$$

Note
$$\left(\sum_{n=0}^{\infty}\frac{1}{\gamma_n^2} = \frac{a^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{a^2}{\pi^2}\times \underbrace{\frac{\pi^2}{8}}_{\text{sum of an infinite series}} \Rightarrow \frac{a^2}{8} \right)$$

$\therefore$

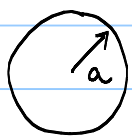
$$\tau_{max} = \mu\alpha a\left[1 - \frac{8}{a^2}\sum_{n=0}^{\infty}\frac{1}{\gamma_n^2\cosh(\gamma_n b/2)}\right]$$

$(b \geq a)$

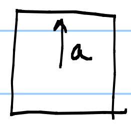
$$C_t = \frac{M_z}{\alpha} = 2\mu\int\psi\, dA$$

$$= \frac{\mu a^3 b}{3}\left[1 - \frac{192}{\pi^2}\left(\frac{a}{b}\right)\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\tanh \gamma_n b/2\right]$$

Compare



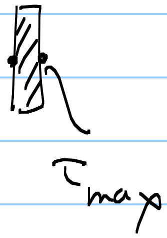
$\tau_{max} = \mu\alpha a$



$\tau_{max} = 0.675\mu\alpha a$

<Special Cases>

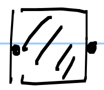
i) $b/a \to \infty$ $\quad \tau_{max} = \mu \alpha a$



$$C_t = \frac{1}{3}\mu a^3 b \qquad (\alpha = M_t / C_t)$$

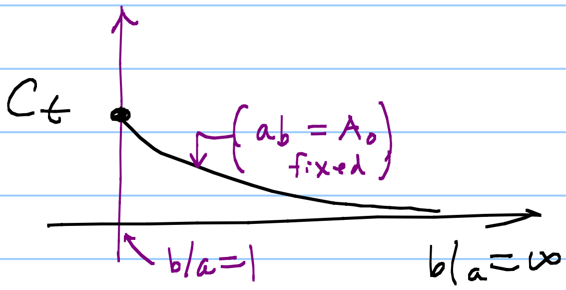
$$\left(= \left(\frac{a}{b}\right)\left[\frac{1}{3}\mu \underbrace{(ab)}_{\text{Area}}^2\right] \to 0 \right)$$

ii) $b = a$ $\quad \tau_{max} = 0.675\, \mu \alpha a$

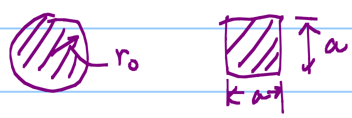


$$C_t = 0.422\left(\frac{1}{3}\mu a^3 b\right)$$

$$= 0.422 \underbrace{\left(\frac{a}{b}\right)}_{1}\left[\frac{1}{3}\mu (ab)^2\right]$$



Comparison : $A_0 = \pi r_0^2 = a^2 \qquad (a = \sqrt{\pi}\, r_0)$



circular cylinder

$$C_t = \frac{\mu A^2}{2\pi} \simeq 0.159\, \mu A_0^2$$

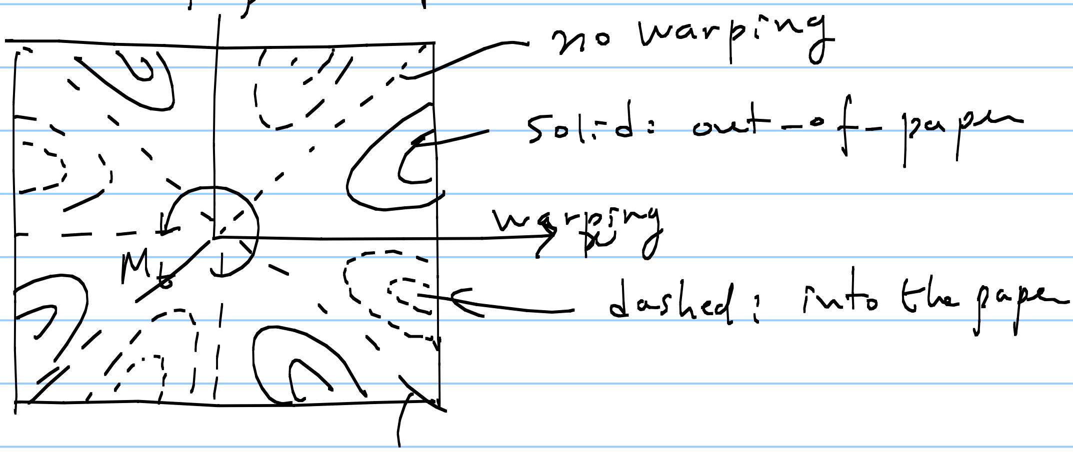
$$\tau_{max} = 0.637 \frac{M_z}{r_0^3}$$

square cylinder

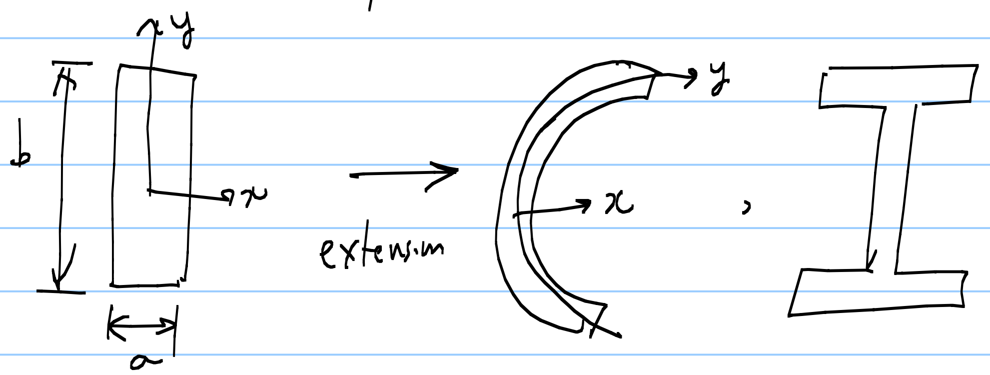
$$C_t = 0.141\, \mu A_0^2 \quad (89\%)$$

$$\tau_{max} = 0.863 \frac{M_z}{r_0^3} \quad (135\%)$$

Check the warping ftn ϕ



<Narrow Rectangular Sections>



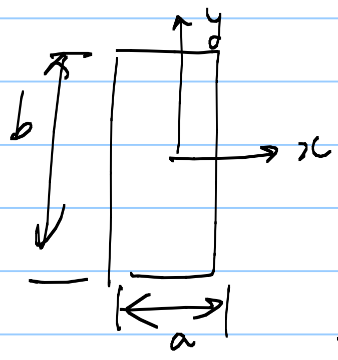
etc.

$b \gg a$

$$C_t = \frac{1}{3}\mu a^3 b$$

$$\tau_{max} = \mu \alpha a = M_z \Big/ \frac{1}{3}a^2 b$$

Analysis for Narrow Rectangular Case



$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = -\frac{p}{T}$$

($\eta = 0$ along boundary)

$b \gg a$

Introduce Dimensionless Coord:

$$X = \frac{x}{a}, \quad Y = \frac{y}{b} \quad |X|, |Y| \le \frac{1}{2}$$

Then,

$$\frac{\partial^2 \eta}{\partial X^2} + \frac{a^2}{b^2}\frac{\partial^2 \eta}{\partial Y^2} = -\frac{a^2 p}{T} \quad (*)$$

as $b/a \to \infty$, $(*)$ becomes

$$\frac{\partial^2 \eta}{\partial X^2} = -\frac{a^2 p}{T}$$

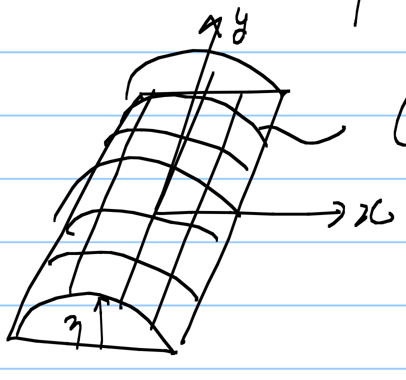
With $X = ax$

$$\frac{\partial^2 \eta}{\partial x^2} = -\frac{p}{T}.$$

Using $\eta = 0$ at $x = \pm a/2$

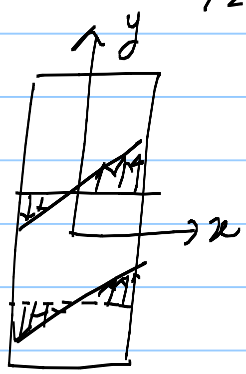
$$\eta = -\frac{p}{2T}\left(x^2 - \left(\frac{a}{2}\right)^2\right)$$

i.e. $\psi = \frac{2T}{p}\eta = \left(\frac{a}{2}\right)^2 - x^2$



$\left(\frac{a}{2}\right)^2 - x^2$

$*$ not valid near $y = \pm b/2$



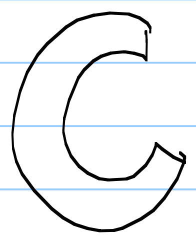
- Stress:
$$\sigma_{yz} = -\mu\alpha \frac{\partial\psi}{\partial x} = 2\mu\alpha x$$
$$\sigma_{xz} = \mu\alpha \frac{\partial\psi}{\partial y} = 0$$

$$\therefore \begin{cases} \tau_{max} = \sigma_{yz}\Big|_{x=\pm a/2} = \mu\alpha a \\ \sigma_{yz}\Big|_{x=0} = 0 \quad \text{(along the centerline)} \end{cases}$$

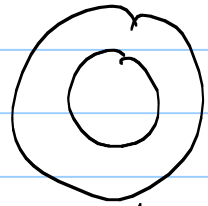
- Torsional rigidity
$$C_t = 2\mu \int \psi \, dA$$
$$= 2\mu \int_{-a/2}^{a/2} \left[\left(\frac{a}{2}\right)^2 - x^2 \right] L\, dx$$
$$= \frac{1}{3}\mu a^3 b$$

Extension to thin-walled open sections

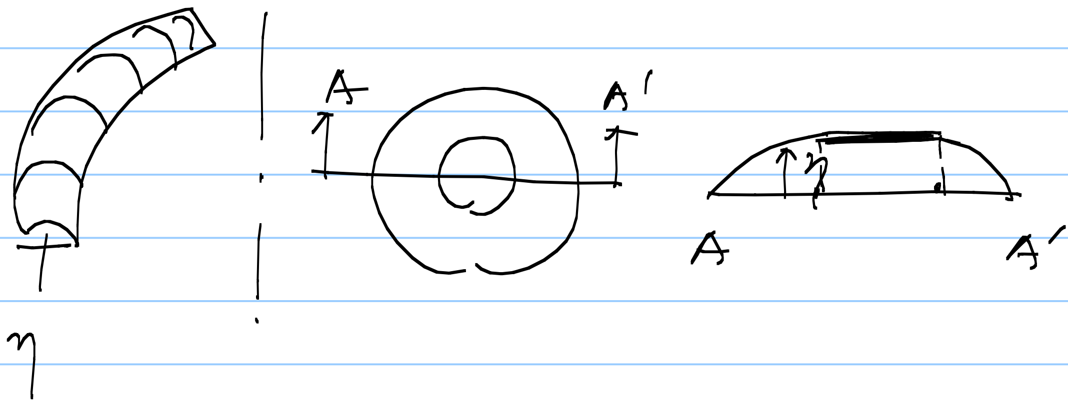
open



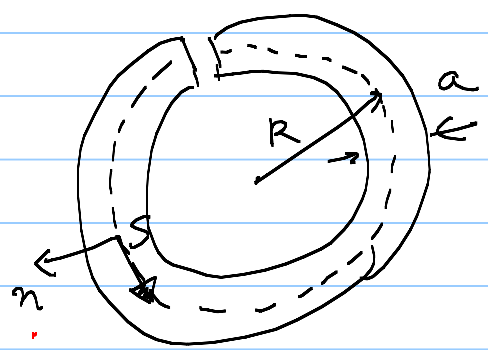
closed



Behavior ~ completely different !!



< Example >



$R \gg a$

$$\begin{cases} \sigma_{zy} = 2\mu\alpha x \\ \text{But } x \leftrightarrow n,\ y \leftrightarrow s \end{cases}$$

Thus

$$\therefore \begin{cases} \sigma_{sz} = 2\mu\alpha n \\ \sigma_{sz}(n=0) = 0 \quad \left(\sigma^{0}_{sz} = 0\right) \end{cases}$$

↑centerline

$$\Rightarrow \boxed{\begin{array}{c}\varepsilon^{o}_{sz}(s,z) = \varepsilon_{sz}(n=0,s,z) = 0 \\ \text{along the middle line}\end{array}}$$

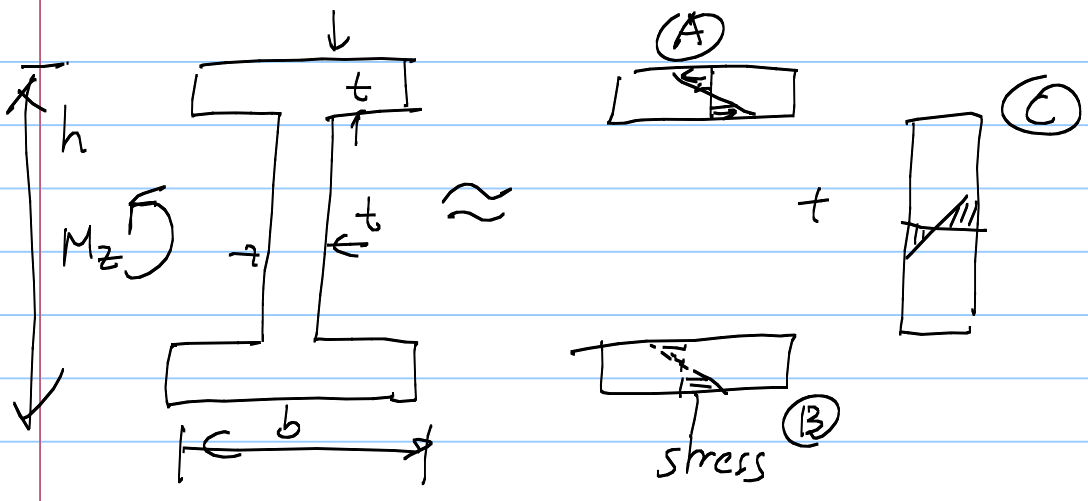
- $C_t = \frac{1}{3}\mu a^3 b$

$$= \frac{1}{3}\mu a^3 (2\pi R)$$

$b = 2\pi R$

< Another Example>

↳ visualize the membrane deformation



$h, b \gg t$

$$C_t = C_t^{Ⓐ} + C_t^{Ⓑ} + C_t^{Ⓒ}$$
$$= \frac{1}{3}\mu b t^3 + \frac{1}{3}\mu b t^3 + \frac{1}{3}\mu h t^3$$
$$= \frac{1}{3}\mu (2b + h) t^3$$

$$\tau_{max} = 2\mu\alpha n$$
$$= 2\mu\alpha (t/2) = \mu\alpha t$$
$$= \mu t \cdot \frac{M_z}{C_t} = \frac{3M_z}{(2b+h)t^2}$$

(In this analysis, the detailed stress distribution near the corner is not considered).